Resolving the singularity by looking at the dot and demonstrating the undecidability of the continuum hypothesis

Abhishek Majhi*

Indian Statistical Institute,
Plot No. 203, Barrackpore, Trunk Road,
Baranagar, Kolkata 700108, West Bengal, India

Abstract

Einsteinian gravity, of which Newtonian gravity is a part, is fraught with the problem of singularity that has been established as a theorem by Hawking and Penrose. The *hypothesis* that founds the basis of both Einsteinian and Newtonian theories of gravity is that bodies with unequal magnitudes of masses fall with the same acceleration under the gravity of a source object. Since, the Einstein's equations is one of the assumptions that underlies the proof of the singularity theorem, therefore, the above hypothesis is implicitly one of the founding pillars of the same.

In this work, I demonstrate how one can possibly write a non-singular theory of gravity which manifests that the above mentioned hypothesis is only valid in an approximate sense in the "large distance" scenario. To mention a specific instance, under the gravity of the earth, a 5 kg and a 500 kg fall with accelerations which differ by approximately 113.148×10^{-32} meter/sec^2 and the more massive object falls with less acceleration. Further, I demonstrate why the concept of gravitational field is not definable in the "small distance" regime which automatically justifies why the Einstein's and Newton's theories fail to provide any "small distance" analysis. In course of writing down this theory, I demonstrate why the continuum hypothesis as spelled out by Goedel, is undecidable. The theory has several aspects which provide the following realizations: (i) Descartes' self-skepticism concerning exact representation of numbers by drawing lines (ii) Born's wish of taking into account "natural uncertainty in all observations" while describing "a physical situation" by means of "real numbers" (iii) Klein's vision of having "a fusion of arithmetic and geometry" where "a point is replaced by a small spot" (iv) Goedel's assertion about "non-standard analysis, in some version" being "the analysis of the future".

Keywords: Axiom of point; Logic, operation and intuition; Continuum hypothesis; Semantics of physics; Singularity resolution in gravity; Natural Uncertainty

*abhishek.majhi@gmail.com

Contents

1 Introduction: a "philosophical nonsense" and a legitimate dilemma – in the middle of being "operational" and "logical"

One of the most critical open problems in modern theoretical physics is to present a resolution of the singularity problem in the theories of gravity[1, 2]. The problem has given birth to several directions of research under the name "quantum gravity" e.g. see ref.[3] and the references therein. While such investigations may be interesting and novel in their own rights, however, none of them has been able to present a unanimously acceptable solution. Rather, routes to "quantum gravity" have been known to be plagued with contradictions[4, 5]. In this work, I discuss a possible solution to the singularity problem based on direct demonstrative reasoning that, in the process, connects to certain foundational questions of mathematical science.

To understand the solution it is necessary to understand the origin of the problem at first. The root of the singularity problem goes back to the antiquity when Newton used the axioms of geometry to write Principia[6, 7]. The axiom which founds the problem of singularity is the first axiom of geometry and it is common to both Euclidean[8] and non-Euclidean geometries[10, 11]:

$$\text{“}\textit{A point is that which has no part.}\text{”} \tag{1}$$

– see page 153 of ref.[8]. Such notion of "a point" is taken for granted with so much belief that Hilbert did not even bother to state it as an axiom while formalizing the foundations of geometry[9]. It is no surprise that the singularity problem plagues both Newton's and Einstein's theories of gravity[2]. I may explain this as follows.

Since physics is intended to explain our experiences (experimentally observed phenomena), the expression of the physicist must be truthful to his perception. However, this has not necessarily been the case during the development of the subject[40]. For example, Einstein used the word "experience" on several occasions in the introductory sections of his famous paper on special relativity to explain simultaneous events, clock synchronization and especially to put forward his second postulate[12]. Such explication of concepts based on practical experience, is what Bridgman called being "operational" [14][1]. Now, if I take this lesson from Einstein to become operational and judge myself on how I express my experience of seeing "a point", then the first axiom of geometry appears to be a false statement due to the following reason. Whenever I demonstrate to someone what "a point" is, I put a dot on the paper with my pencil and the dot must be visible for the demonstration process to proceed[2]. If I can see the dot, it definitely has some extension. Then, it is a straightforward lie to say that "a dot has no extension". And, whatever has extension, certainly has parts. So, it is also a straightforward lie to say that "a dot has no part". However, it is such "a dot" that I call "a point". A direct switch of words shows that, it is a straightforward lie to say that "a point is that which has no part". In order to make it true, I must stop the demonstration process because I can not put "a dot" on the paper to demonstrate what "a point" is. So, once I start the demonstration process, the situation becomes such that, although "I can see the dot" I must believe that "I can not see the dot" so that the first axiom of geometry holds true. For me, it is a direct denial of an empirically demonstrated truth i.e.

[1] "Operational" way of explaining concepts was advocated by Mach in ref.[15] which especially becomes clear from his use of words "experimental propositions" among other writings. It is not unknown today that Mach was one of those who had a profound influence on Einstein e.g. see pp. 141-145 of ref.[16].

[2] Such explanations are necessary alongside the equations of theoretical physics.

I am denying my experience of vision of the dot. In other words, I am not being operational while expressing the perception of the dot. Bridgman might have written that the first axiom of geometry is "anti-operational" (see Appendix (A)). However, I need to emphasize that this operational viewpoint is directed towards maintaining a truthful connection between what I can speak/write and what I can draw i.e. a dot on the paper is a method of demonstration by which I express what I mean by "a point"[3]. This is different from writing about my experience regarding the measurement process while doing an experiment in the laboratory. It was such experimental experience that the great scientists like Mach[15], Einstein[12] and Bridgman[14] focused on, but not on the experience of their own expressions, while being operational. The truthfulness of our own expressions is also judged by ourselves through our own perceptions. For example – when I write, I need to keep my eyes open to be able to see whether I am writing correctly; when I speak, I need to keep my ears open so that I can listen whether I am speaking correctly. One generally takes such processes as granted, and become unaware of one's reliance on such perceptions, unless the person looses one or more of the modes of perceptions.

In spite of the simplicity and direct realizability of the above example concerning the demonstration of "a point", the modern physicist may just dismiss the above arguments as "philosophical nonsense" or just "simply nonsense"[4]. So, I provide now a different argument to explain the problem at hand, that concerns the most cherished tool of the physicist to check the consistency of any expression involving quantities, namely, dimensional analysis[17]. With a dot, I generally represent the statement "$r = 0$", rooted to which is the singularity problem, as I draw the following conclusion from Newton's law of gravitation:

$$\lim_{r\to 0} F = \lim_{r\to 0} G\frac{m_1 m_2}{r^2} = \infty \tag{2}$$

where m_1 and m_2 are the two masses involved and "r" is said to be the distance between the two "point" masses[6, 7]. It is easy to see that on dimensional grounds[17], "$r \to 0$" is an incorrect statement because "r" has length dimension, but "0" is a number i.e. without any physical dimension! An immediate counter argument from the modern physicist may be that one can easily resolve this problem by simply writing the chosen unit of measurement in the expression e.g. "$r = 0\lambda_0$ where λ_0 stands for the chosen unit of measurement like metre, kilometre, etc." However, this counter argument is anti-operational because no human being has ever experienced or experimentally measured "length of zero unit" e.g. "0 metre", "0 kilometre", etc. So, any operational notion of length, or experimentally measured (experienced) length, can at most be negligible with respect to the chosen unit of length, but not exactly zero. This is why any length must always be written in terms of the chosen length unit if I stick to operational perspective and my expression is truthful to my experience. Thus, any experienced length is relational length i.e. the corresponding expression should be a relation between two lengths – the experienced length (L) and the chosen length unit (λ_0). Such a relational expression must look like

$$L = n_L^{\lambda_0}\lambda_0, \qquad \text{where } n_L^{\lambda_0} \text{ is some real positive number.} \tag{3}$$

The indices of "n" signify the relation between the two lengths L and λ_0. Now, if $n_L^{\lambda_0} \lll 1$, or equivalently $L \lll \lambda_0$, then the experimenter can say "the experienced length is negligibly small compared to chosen length unit". But, if one allows $n_L^{\lambda_0} = 0$ on theoretical grounds (which one can always do

[3]Poincare pointed towards a "star" to demonstrate what a "point" means – see page no. 38 of ref.[30]. As Born noted, Klein indeed envisioned to replace "a point" with "a small spot"; in Klein's words, there needs to be something "concrete" to explicate the "abstract" – see Appendix (C).

[4]The skeptic reader should necessarily consult refs.[40, 41]

by choice), can the experimenter verbally express the situation? I believe the experimenter can not, because this "absolute zero" can not be experienced so that a corresponding expression can be given verbally[5] and allowing this in theory renders it anti-operational (nevertheless, not incorrect).

Therefore, the modern physicist arrives at a dilemma – either the physicist has to let go of the operational viewpoint on which Einstein founded special relativity[6] (as Bridgman explained in ref.[14]), or the physicist has to admit of the above arguments of mine to be reasonable. This, of course, brings in the question that even if my arguments are considered to be reasonable and such objections are considered as valid, is there a better alternative that can overcome these obstacles of reason? According to me, the answer is in the affirmative and my motto is to explore this alternative in what follows – it is technically quite simple, especially when compared to any of the modern existing theories in the literature that intend to resolve the problem of singularity.

2 "Truth" of the first axiom of geometry, visual experience of the dot and representation of an object as a whole

In view of what I have discussed in the previous section, the singularity problem can be resolved if I can overcome two obstacles of reason – firstly, there is a conflict between the logical truth of the first axiom of geometry regarding "a point" and the empirical truth of the vision of a dot that is indispensable for demonstrating what "a point" is; secondly, the statement "$r = 0$" is incorrect on dimensional ground and the statement "$r = 0\lambda_0$ (length of zero unit)" is not a truthful expression of experience. Interestingly, none of the above objections arise if I write – *a dot is a negligible extension compared to any other extension that is called a line*, where "a dot" can be replaced with "a point" i.e. instead of the first axiom of geometry, I consider the following statement or proposition:

A point is that which has negligible extension compared to any other extension which I call "a line".

Now, it seems consistent with the immediate experience of seeing a dot and the corresponding realization of the smallness of its extension with respect to any extension that I call "a line". With such a line I may represent the chosen length unit, on paper. Also, I may point out a crucial difference between the first axiom of geometry and the statement that I consider instead. The "truth" of the first axiom of geometry is a logical truth, as Einstein had stated with emphasis in Chapter 1 of ref.[20], and hence, such a truth is universal or absolute. In contrast to that, my statement is only a relational truth because of the comparison of the extension of "a dot" or "a point" with "a line"[7] (consult Appendix (B) for further discussions).

[5]The experimenter may say "I have measured zero unit of length", but the experimenter can not point out "what" has been measured because there is nothing to be measured in the first place. Thus the experience is empty i.e. it is not an experience at all. So the expression becomes meaningless. Consult refs.[40, 41] for further reading.

[6]Einstein relied on logical truths of the axioms of geometry to formulate the theories of relativity. Bridgman analyzed only the operational aspect of Einstein's formulation and did not question the truth of the axioms of geometry from an operational viewpoint. See Appendix (A) for a discussion.

[7]My statement guarantees that I need to put the dot in such a way that I can slide my pencil over the paper to demonstrate what "a line" is. This is not possible if, instead of a pencil, if I use a very thick paint-brush and slide it over the same paper. This is because on touching the paper with the brush, I get "a patch" rather than "a dot" and on sliding the brush I get "an extended patch" rather than "a line". This is an intuitive refinement of an axiom in the pursuit of truth, as Brouwer asserted, "*intuititon subtilizes logic*" and "*denounces logic as the source of truth*"[39]. Such Brouwerian viewpoint has recently been discussed in ref.[37, 38, 4] as well. The significance and impact of such a viewpoint, on the foundations of calculus, becomes manifest from ref.[4].

2.1 Representing "an object as a whole" with "a dot" and a comparison with Newton's reliance on the first axiom of geometry

Newton represented an object, considered as a whole, by putting a dot on the paper (see relevant diagrams in ref.[6, 7]), and called it "body or mass" right at the very beginning of Principia; see page no. 1 of ref.[6]. This is what we refer today as "point mass"[18]. The depiction of any geometric curve with that dot represents the motion of the object from the observer's perspective. The subject concerning such study is what we call "point (or classical) mechanics"[18]. Using the first axiom of geometry Newton denied the visibility of the dot that he put on paper. Due to such denial, he could not represent the actual object of experiment ("mass" in his own words), with the extension of the dot itself. So, he had to consider "mass" as a physical dimension other than length, in order to represent in theory the actual object of experiment. Since he considered the object as a whole (which has no part), the words "point mass" stand justified. Einstein called it "material point" e.g. see ref.[12, 20]. I may emphasize that by "body or mass" Newton did not refer to the type, the shape or the external appearance of the particular object of investigation, rather he expressed the thought of the object irrespective of its type or shape or external features.

In the present scenario, I represent an object as a whole, by a dot which has an extension that is negligible compared to any other extension that I can call a line. With such a line I represent the chosen unit of length in terms of which experimental measurement is performed. Generally, the external shape of the object has the characteristic extension comparable to the length unit. Thus, what Newton called "mass", in the present scenario, is not an independent physical dimension other than length. Rather it is understood as an extension in relation to the chosen length unit. Here, "point mass" is written as follows:

$$s_i \lll \lambda_0 \quad \Leftrightarrow \quad s_i = \epsilon_{s_i}^{\lambda_0} \lambda_0 \quad : 0 < \epsilon_{s_i}^{\lambda_0} \lll 1, \quad (\text{“:” means “such that”}) \tag{4}$$

where s_i is the characteristic extension of the i-th dot representing the i-th object in case there are more than one object in consideration and λ_0 stands for length unit (like meter, kilometer, etc.) which I choose according to requirement, but within my restricted ability. A visual demonstration of the relation between s_i and λ_0 shall reveal shortly why the upper bound of $\epsilon_{s_i}^{\lambda_0}$ is actually 1/2 and not 1. Before that, I may discuss some further motivations behind proposition (4) through an explanation of the interrelation between two different ways of writing physics – one with mass as an independent physical dimension and the other without mass as an independent physical dimension.

2.2 Motivating the refinement of the first axiom of geometry by understanding "strong gravity" and "weak gravity" in terms of "force"

According to Newton's hypothesis, the force of interaction between two point masses m_1 and m_2, separated by a distance r in accord with Euclid's axioms of geometry, is given by

$$F \propto \frac{m_1 m_2}{r^2}.$$

The proportionality constant is determined through the torsion balance experiments[45, 46, 48, 49, 47], named as "Newton's gravitational constant", symbolized by "G". Consequently, we write Newton's law of gravitation as

$$F = G\frac{m_1 m_2}{r^2}.$$

This "G" enters Einstein's equations through a choice that is made in such a way that the general relativity produces the relevant equations of Newtonian gravity[8] in the "weak gravity" and "low-velocities/non-relativistic" limit[55](also see pp. 81-82 of ref.[16] to check how Einstein fitted G into his equations).

The "low-velocities" correspond to the condition $v \lll c$ where "v" represents the characteristic velocities of the objects in motion under gravity. The comparison between v and c justifies "velocities are low compared to the velocity of light in vacuum" i.e. low compared to "what" gets specified. Such limit is studied through the post-Newtonian approximations[55, 28].

The "weak gravity" and "strong gravity" are characterized by the conditions $Gm/c^2r \lll 1$ and $Gm/c^2r \sim 1$ respectively, where m and r are the characteristic mass scale and characteristic distance scale involved in the physical phenomena under study e.g. see refs.[28, 53, 54]. I may note that, although we generally use the concept of "force" to express words "weak" and "strong", here it is expressed in terms of "length" that seems to call for some connection of the conditions $Gm/c^2r \lll 1$ and $Gm/c^2r \sim 1$ with conditions like $F/F_0 \lll 1$ ("weak gravity") and $F/F_0 \sim 1$ ("strong gravity") respectively. The call for such a connection can be motivated from the experimental point of view as well as follows.

I may note that the torsion balance experiments determine G through the use of the concept of "force". Otherwise, Newton's hypothesis can not be used for such experiments. Therefore, it seems quite feasible to grasp the notion of "weak gravity" and "strong gravity" through the conditions $F/F_0 \lll 1$ and $F/F_0 \sim 1$ respectively, where F_0 is some characteristic upper bound on the two body gravitational interaction force F. That is, the strength of F is now judged with some reference F_0 such that weak or strong with respect to "what" is specified. It now seems reasonable to wonder whether there is a connection between $Gm/c^2r \lll 1$ and $F/F_0 \lll 1$, and whether Newton's law of gravitation follows only in an approximate sense under such conditions. Then, the question arises that how the two body interaction look like for $Gm/c^2r \sim 1$ and $F/F_0 \sim 1$.

I try to address the above emerging speculations by 'playing' with c and Newton's law of gravitation as follows:

$$F = G\frac{m_1 m_2}{r^2} = \frac{c^4}{G}\frac{(Gm_1/c^2)(Gm_2/c^2)}{r^2} = F_0\frac{s_1 s_2}{r^2} \quad : \; F_0 = c^4/G, \; s_i = Gm_i/c^2 \; \forall i \in [1,2].$$

$$\therefore \; \frac{F}{F_0} = \frac{s_1 s_2}{r^2}.$$

Using $G = 6.674 \times 10^{-11}$ meter^3 kg^{-1} sec^{-2} and $c = 2.997 \times 10^8$ meter sec^{-1}, as per currently accepted CODATA values[50], I obtain $F_0 = 12.088 \times 10^{43}$ kg meter sec^{-2} (Newton) and for 1 kg, $s_{kg} = \frac{G}{c^2} \times 1$ kg $=$ 7.430×10^{-28} meter. It becomes evident that the two body gravitational interaction forces F that are involved in torsion balance experiments are much smaller than F_0 i.e. $F \lll F_0$. However, instead of masses m_1, m_2, there are lengths s_1, s_2 that appear in the expression which are extremely small compared to the conventionally chosen unit of length, namely, meter. This motivates proposition (4) i.e. instead

[8] I may point out that the relevant equation that is taken as the "Newtonian limit" of general relativity is not Newton's law of gravitation regarding two body interaction force, but the Poisson equation which is written in terms of gravitational potential[55](also see pp. 81-82 of ref.[16] for Einstein's original writings). It is crucial to note that starting from the Newton's law of gravity, to introduce the gravitational potential, one needs to use differential calculus. Therefore, more subtleties arise due to the involvement of the concept of limit, which however motivates a relation between mass and length like $s = Gm/c^2$, as I have explained in Appendix (F).

of using the first axiom of geometry concerning "point" and an independent physical dimension called "mass" to explain "point mass", I may use "an intrinsic length that is extremely small compared to the conventionally chosen unit of length" to explain "point mass". The difference is startling because dimensional analysis that lies at the base of physics (e.g. see ref.[17]) needs to be modified. The notion of "mass measurement" needs to be revisited. It is the language of physics that becomes different. So, the relations $F_0 = c^4/G, s_i = Gm_i/c^2$ provide a bridge between two different languages of physics, alongside the fact that $s_i = Gm_i/c^2$ automatically motivates proposition (4). For further discussions on the relation $s_i = Gm_i/c^2$ the reader may consult Appendix (F).

Now, there is a severe problem i.e. the symbol "r" looses its meaning because the first axiom of geometry can not be used anymore. To be more explicit, if the notion of "point" (that is stated in the first axiom) is now replaced by some length, then the usual interpretation of "r" as "distance between two points" becomes meaningless. What needs to be investigated is how to give meaning to "distance between two lengths s_1, s_2". Therefore, the notion of "distance" itself needs to be refined and this is what I discuss next.

3 The dot, the void and distance: How many points are there on a line?

When I see the dot (s_i), I also have the realization of the surrounding void which is *not* the dot. The realization of the one depends on the other. Without the realization of the dot, the realization of the void is not possible. Also, without the realization of the void, the realization of the dot is not possible. The realizations of the two opposite categories occur in a dependent way and not in isolation i.e. the dot can not be realized without the realization of the void and vice versa. This is what I have demonstrated in fig.(1).

Further, the totality of the perception of the dot and the void provides me with a realization of distance (d), in any particular direction through the void but founded on the dot. That is, the thought of distance in any direction along the surface of the paper, originates from the realization of the dot. Thus, I can realize neither the dot nor the distance in isolation. Rather, I can realize both, in association with each other – an association that I can not dissociate – as if the two seemingly opposite categorical realizations originate in an interdependent way. So, the notion of distance must include the dot. To demonstrate this, I have provided a slightly enlarged view of the dot in fig.(2) so that it becomes clearly visible that the brackets, which are meant for the expressions of the thought of "distance" in different directions, include the dot. In order to provide reference for the reader's trust on an authority, I may mention that the dot represents what Einstein may have called "body of reference" based on which the thought of "distance" is perceived of [20]. Nevertheless, this body of reference did not play a role in the associated calculations of Einstein because he denied the truth of the dot owing to his reliance on the "truth" of the first axiom of geometry. However, in the present scenario, the dot is the indispensable foundation for the thought of distance and therefore, of any further demonstration.

Since the thought of distance originates in association with, and hence founded on, the visual experience of the dot owing to its characteristic extension, then certainly, now I can *imagine* several dots are needed to fill up the distance i.e. a comparison between the dot and the distance arises. The demonstration of such an imagination, with a zoomed dot is *expressed* through a drawing in the fig.(3). I continue to repeatedly iterate with the dot along the direction in which the distance is thought, until I exhaust the required distance.It is this direction that I call "straight" and, the line that I draw by

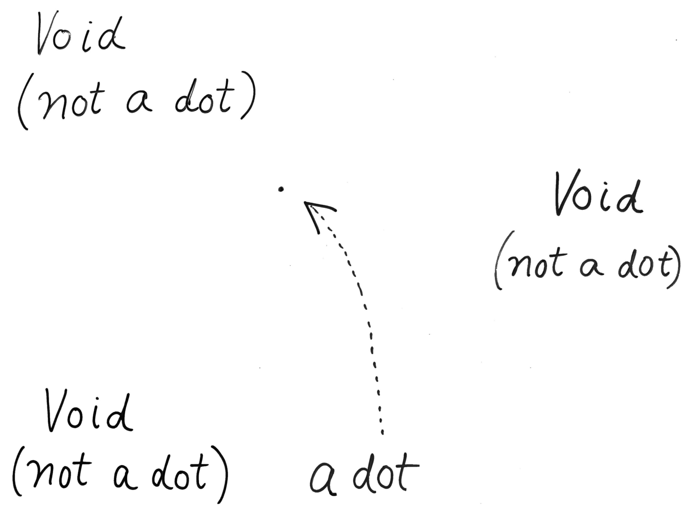


Figure 1: I see a dot and also the void which is not a dot. Realizations of the dot and the void, which are seemingly of opposite categories, happen in relation to each other. Without the realization of the dot it is not possible for me to realize what is not the dot, i.e. the void, and without the realization of the void it is not possible for me to realize what a dot is.

filling the distance with such dots, a straight line. However, after each iteration I put a cut in order to memorize the mark where the last iteration finished. These "cuts" are like the cross-wire of a microscope or the marks on a scale, which introduce an irremovable error in the measurement process. Undeniably, the removal of the cross-wire or the marks on a scale, makes measurement impossible. Likewise, removal of the "cuts" renders my *expression* through drawing, and hence any demonstration, impossible. Thus, the cuts are indispensable for the demonstration procedure. And, since the cuts are visible, then each cut has certain amount of thickness. Consequently there is some error involved due to these cuts that needs to be taken into account if I want to express my experience of the cuts in a truthful manner, or in Bridgman's words, in an operational manner.

So, I write

$$d > s_i \;\Leftrightarrow\; d = (N_d^{s_i} + 1 + \delta_d^{s_i})s_i = r_i + s_i \;: r_i := (N_d^{s_i} + \delta_d^{s_i})s_i,\; 0 < \delta_d^{s_i} < 1,\; N_d^{s_i} = 0, 1, 2, \cdots . \quad (5)$$

The symbol ":=" stands for "defined as"[9]. Here, $N_d^{s_i}$ is the number of iterations that I need to make in order to fill the associated distance, "1" signifies that the dot itself can not be removed as it is the foundation of this whole demonstration process, $\delta_d^{s_i} s_i$ is the collective thickness of all the cuts. Since the cuts can not be removed for the counting process to be demonstrated, therefore, the number of dots can not be exactly counted i.e. an element of doubt always accompanies in the demonstration process. This is the essence of having $\delta_d^{s_i} \neq 0$. Considering the scenario altogether, the dot s_i and the associated distance d are interdependent realizations and the cuts are the premise based on which the

[9] Here, the words "defined as" carry the meaning "abbreviation for" in symbolic terms i.e. a short hand for typographical purpose.

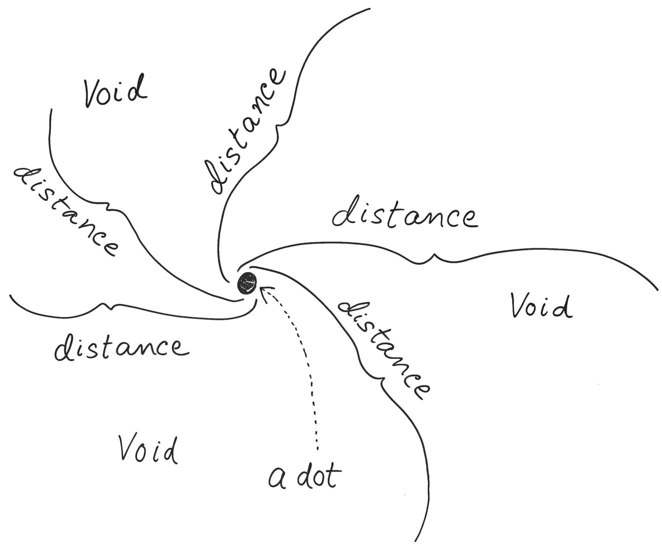


Figure 2: "Distance" is a thought that arises in association with the dot. So, the thought of distance, in any direction, includes the dot itself. This is why the brackets include the dot irrespective of the direction in which the distance is thought. So, the dot is the foundation of any following demonstration like drawing a line with more dots. I may emphasize that, in order to demonstrate this inclusion of the dot, I have shown the dot of fig.(1) with an enlarged view in this present diagram.

measurement of d in terms of s_i can be demonstrated. Hence, the thickness of the cuts (i.e. $\delta_d^{s_i} s_i$), itself, can not be measured within this process of demonstration. Thus, $\delta_d^{s_i} s_i$ represents the incompleteness of this measurement process. Consequently, the number of dots required to fill the distance is neither completely countable nor it is completely not countable. Rather the number of dots required to fill the distance can be known approximately for serving a practical purpose, while the whole truth of the demonstration process always contains an element of doubt owing to the mode of demonstration itself. In other words, the thought of exact numbers, which are expressed through the numerals like $0, 1, 2, ...$ etc., can not be demonstrated with complete perfection because the mode of demonstration itself is the imperfection[10].

Now, the collective thickness of the cuts is demonstrable in terms of the individual ones in a straightforward manner as I can write the following:

$$\delta_d^{s_i} s_i = (N_d^{s_i} + 2)\delta_I^{s_i} s_i < s_i \qquad : \delta_I^{s_i} s_i \text{ is the thickness of an individual cut.} \tag{6}$$

[10] Possibly due to such reasons Descartes wrote "as closely as possible" and not "exactly" while representing the concept of "number" through "geometry" on page no. 2 of ref.[19]: "*..... taking one line which I shall call unity to relate it as closely as possible to numbers.*". If I may use an everyday example to demonstrate this fact, then I must write that any reasonably honest person should admit that when he says that he has five fingers in one hand, then certainly he makes an approximation because every finger is different from the other. So, it is just a working convention to write "exactly five fingers", which is nevertheless extremely useful for daily purpose and considered to be logical and therefore, assumed to be exact. In this regard, the reader may consult Appendix (C) for some relevant statements by Born and Klein.

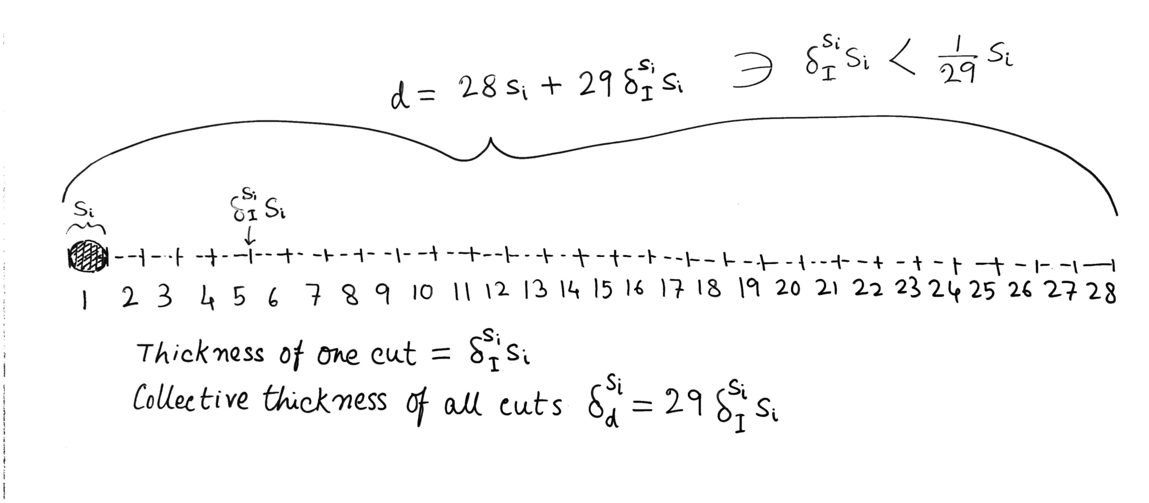


Figure 3: This figure is a demonstration of the notion of a distance d measured in terms of the dot s_i. The cuts are necessary as the mode of demonstration. Here, $d = 28s_i + 29\delta_I^{s_i}s_i : \delta_I^{s_i}s_i < \frac{1}{29}s_i$. Thickness of one cut is $\delta_I^{s_i}s_i$. Collective thickness of all the cuts is $\delta_d^{s_i}s_i = 29\delta_I^{s_i}s_i$.

The subscript "I" in $\delta_I^{s_i}$ stands for "individual". In view of this, (5) can be rewritten as follows:

$$\begin{aligned}
d &= (N_d^{s_i} + 1 + \delta_d^{s_i})s_i \qquad : \delta_d^{s_i}s_i < s_i \\
&= [N_d^{s_i} + 1 + (N_d^{s_i}+2)\delta_I^{s_i}]s_i \qquad : (N_d^{s_i}+2)\delta_I^{s_i}s_i < s_i \\
&= [N_d^{s_i} + (N_d^{s_i}+2)\delta_I^{s_i}]s_i + s_i \qquad : \delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i}+2} \\
\therefore \quad r_i &= [N_d^{s_i} + (N_d^{s_i}+2)\delta_I^{s_i}]s_i \qquad : \delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i}+2},\ N_d^{s_i} = 0, 1, 2, \cdots .
\end{aligned} \tag{7}$$

Now, I write down the following cases explicitly.

- For $N_d^{s_i} = 0, \quad r_i = (0 + 2\delta_I^{s_i})s_i \quad : \delta_I^{s_i}s_i < \frac{s_i}{0+2}. \quad \therefore r_i < s_i,\ d < 2s_i.$
- For $N_d^{s_i} = 1, \quad r_i = (1 + 3\delta_I^{s_i})s_i \quad : \delta_I^{s_i}s_i < \frac{s_i}{1+2}. \quad \therefore r_i > s_i,\ d > 2s_i.$
- For $N_d^{s_i} = 2, \quad r_i = (2 + 4\delta_I^{s_i})s_i \quad : \delta_I^{s_i}s_i < \frac{s_i}{2+2}. \quad \therefore r_i > s_i,\ d > 2s_i.$
- For $N_d^{s_i} = 3, \quad r_i = (3 + 5\delta_I^{s_i})s_i \quad : \delta_I^{s_i}s_i < \frac{s_i}{3+2}. \quad \therefore r_i > s_i,\ d > 2s_i$ and so on.

So, I can write that for $N_d^{s_i} > 2, r_i = [N_d^{s_i} + (N_d^{s_i}+2)\delta_I^{s_i}]s_i$ such that $\delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1} \simeq \frac{s_i}{N_d^{s_i}}$.

3.1 Explication of Klein's envisioned distinction between "naive intuition" and "refined intuition (logic)"

I may note that the condition $\delta_I^{s_i}s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$ has been imposed in order to refine the demonstration process of counting dots by minimizing the errors incorporated due to the cuts. It does not look illogical or like a mistake if I omit s_i from both sides and write $\delta_I^{s_i} < \frac{1}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$, because

I just "cancel" the same physical dimension (of length) from both sides as if s_i represents some non-zero number[40]. Then, for $N_d^{s_i} \to \infty$, I obtain $\delta_I^{s_i} < 0$. While mathematically there is no problem with such a conclusion, this particular step of calculation is not demonstrable. That is, the relation "$\delta_I^{s_i} s_i < 0 s_i$"(written by choice) has no corresponding demonstration in terms of the cuts i.e. it is not a relation that is explicable, if a relation at all. Therefore, I may conclude that infinite measurement and arbitrary refinement of the demonstration process is not possible. This raises the doubt whether exact "zero" and exact "infinity" are demonstrable at all. In view of this and ref.[36] (see Appendix(C) for specific quotes), Klein could have explicated the present situation as follows. The expression "$\delta_I^{s_i} s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$" represents "*naive intuition*" because the "*concrete*" dot is taken into account as s_i and then complete refinement is not possible as "$\delta_I^{s_i} s_i < 0 s_i$" is not demonstrable. The expression "$\delta_I^{s_i} < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$" allows complete refinement and therefore, it represents completely "*refined intuition*" and "*logical* ", but it is "*not properly intuition at all* " because of the loss of "*concrete*"-ness owing to the absence of s_i. Although the "*exact mathematician*" may not find a problem with the expression devoid of s_i, I must "*maintain*" my stance in using the expression with s_i that is demonstrable and useful for practical purpose for explaining phenomena observed in "*ordinary life*", which is the aim of the present theoretical discussion. Henceforth, I shall call the expressions involving physical dimensions, like $\delta_I^{s_i} s_i < \frac{s_i}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$, as *physico-mathematical* expressions in order to make a distinction from mathematical expressions, like $\delta_I^{s_i} < \frac{1}{N_d^{s_i}}(1 + \frac{2}{N_d^{s_i}})^{-1}$, which are devoid of physical dimensions[40].

Now, I must provide the following clarification regarding fig.(3). It is important to note that I have drawn a zoomed version of the dot so as to make the situation understandable for the reader. If it were the original dot of fig.(1), then the cuts would not have been visible. Therefore, in this sense, I may write that I have demonstrated a "hypothetical" measurement of the distance in terms of the associated dot. It is just a visual demonstration of how I think of the distance on the basis of my realization of the dot and due to the association of the dot how its characteristic extension plays the role in realization of the distance in relation to the dot.

3.2 Meaning of "large distance" and "small distance" in relation to the dot

What I have already demonstrated is the fact that the realizations of the dot and the associated distance occur interdependently and then the estimate of the distance in terms of numbers is given by the characteristic extension of the dot, albeit always with the error due to the mode of demonstration – the cuts. Therefore, it is now easy to understand that the words "large distance" and "small distance" are simply meaningless or rather carry incomplete sense because there need to be a mention of "compared to what". This completion is now automatic as the dot is taken as the foundation of any demonstration and it is with respect to the characteristic extension of the dot that the largeness of the associated distance (along any direction) can be specified.

So, from (5), I write down the two following cases which I call "small distance" and "large distance" respectively as follows:

- **Small Distance:** $N_d^{s_i} = 0 \quad \Leftrightarrow \quad r_i < s_i \quad \Leftrightarrow \quad s_i < d < 2s_i$
- **Large Distance:** $N_d^{s_i} = 1, 2, \cdots \quad \Leftrightarrow \quad r_i > s_i \quad \Leftrightarrow \quad d > 2s_i$

where "⇔" stands for "equivalently". In order to showcase explicitly how s_i/d behaves differently in the above conditions and to justify the distinction of the two cases as "small" and "large", I write the following.

- **Small Distance Expansion (SDE):**

  For $r_i < s_i \quad \Leftrightarrow \quad s_i < d < 2s_i$ I can write

$$\frac{1}{2} \;<\; \frac{s_i}{d} = \frac{s_i}{r_i + s_i} = \sum_{n=0}^{\infty} \left(-\frac{r_i}{s_i}\right)^n \;<\; 1. \tag{8}$$

- **Large Distance Expansion (LDE):**

  For $r_i > s_i \quad \Leftrightarrow \quad d > 2s_i$ I can write

$$0 \;<\; \frac{s_i}{d} = \frac{s_i}{r_i + s_i} = \frac{s_i}{r_i} \sum_{n=0}^{\infty} \left(-\frac{s_i}{r_i}\right)^n \;<\; \frac{1}{2}. \tag{9}$$

Here, the words "large" and "small" carry a clear and obvious comparative sense because "compared to what" is now declared[11]. So, I may emphasize at this point that if I need to draw a line to demonstrate some experimental phenomenon where an object is observed as a whole, then I may call it "large distance physics" and condition (13) needs to be satisfied. If it happens that the theory written down with such a condition fails to explain the observed phenomenon, then it means that the assumption of an object as a whole fails or not suitable for the description of that particular observed phenomenon. I believe that such assertions will appear to be more meaningful as I proceed.

3.3 Realization of Descartes' doubt and Born's wish: the "natural uncertainty" in the physical act of explicating numbers

Now, I introduce the role of λ_0 in the context so as to explain the role of the length unit in terms of which actual measurement is to be done. As Einstein did specify in ref.[20] (specific quote to be given in the next subsection), the chosen standard of length is also a "distance"[12]. Thus, I can write

$$\lambda_0 = (N_{\lambda_0}^{s_i} + 1 + \delta_{\lambda_0}^{s_i})s_i : N_{\lambda_0}^{s_i} \ggg 1,\; \delta_{\lambda_0}^{s_i} s_i = (N_{\lambda_0}^{s_i} + 2)\delta_I^{s_i} s_i < s_i, \tag{10}$$

which can be recast as

$$s_i = \epsilon_{s_i}^{\lambda_0} \lambda_0 \qquad \text{where} \;\; \epsilon_{s_i}^{\lambda_0} := (N_{\lambda_0}^{s_i} + 1 + \delta_{\lambda_0}^{s_i})^{-1} \tag{11}$$

and it is easy to see that

$$0 < \epsilon_{s_i}^{\lambda_0} \lll \frac{1}{2} \quad \text{as} \;\; N_{\lambda_0}^{s_i} \ggg 1. \tag{12}$$

[11] This is not a mere useless nitpicking of words. In case such a thought arises in the reader's mind, I suggest a consultation of Appendix(D) and of refs.[40, 41].

[12] Cantor might call λ_0 "unit distance"[44].

Any distance d that is to be measured in terms of λ_0 must be greater than λ_0 so that at least one iteration is possible. So, to represent the length unit λ_0 by a line on the paper and to demonstrate the measurement of any distance in terms of this line, I consider following condition to hold:

$$d > \lambda_0 \ggg 2s_i. \tag{13}$$

This is what I have tried to demonstrate in fig.(4).

Therefore, considering such explications, the distance d founded on the dot s_i, measured in terms of the length unit λ_0, can be expressed from (5) as follows:

$$\begin{aligned} d &= (N_d^{s_i} + 1 + \delta_d^{s_i})s_i \\ &= (N_d^{s_i} + 1 + \delta_d^{s_i})\epsilon_{s_i}^{\lambda_0}\lambda_0 \\ &= [(N_d^{s_i} + \delta_d^{s_i})\epsilon_{s_i}^{\lambda_0} + \epsilon_{s_i}^{\lambda_0}]\lambda_0. \end{aligned} \tag{14}$$

Now, (14) represents the act of putting a dot, the subsequent operations of drawing a line to represent d and the iterations by λ_0 along with the errors introduced by the cuts, each of which has the same thickness as that of the dot by *assumption* (a *choice* of construction). This can be represented as

$$d = (N_d^{\lambda_0} + 1)\lambda_0 + (N_d^{\lambda_0} + 2)s_i = [(N_d^{\lambda_0} + 1) + (N_d^{\lambda_0} + 2)\epsilon_{s_i}^{\lambda_0}]\lambda_0 \tag{15}$$

and then the following relation needs to hold:

$$(N_d^{s_i} + \delta_d^{s_i})\epsilon_{s_i}^{\lambda_0}\lambda_0 \;=\; (N_d^{\lambda_0} + 1)(1 + \epsilon_{s_i}^{\lambda_0})\lambda_0 \qquad : N_d^{\lambda_0} = 0, 1, 2, 3 \cdots . \tag{16}$$

Then, I may recast (15) as follows:

$$d = [(N_d^{\lambda_0} + 1)(1 + \epsilon_{s_i}^{\lambda_0}) + \epsilon_{s_i}^{\lambda_0}]\lambda_0 \tag{17}$$

Considering such a construction of the iteration process, it provides a good estimate of the distance (d) measurement with the length unit (λ_0) until the following condition is satisfied: $(N_d^{\lambda_0} + 2)\epsilon_{s_i}^{\lambda_0}\lambda_0 < \lambda_0 \;\Leftrightarrow\; (N_d^{\lambda_0} + 2)s_i < \lambda_0$; otherwise this will lead to mistake. I shall designate "$(N_d^{\lambda_0} + 1)(1 + \epsilon_{s_i}^{\lambda_0})$" as "$x_{s_i}^{\lambda_0}$" which is an integer but always associated with an error of at least $\epsilon_{s_i}^{\lambda_0}$. With such clarifications of the symbols, I may now write

$$d = r_i + s_i = (x_{s_i}^{\lambda_0} + \epsilon_{s_i}^{\lambda_0})\lambda_0. \tag{18}$$

I may note that, for $N^{\lambda_0} = 0$, $x_{s_i}^{\lambda_0} = 1 + \epsilon_{s_i}^{\lambda_0}$ and therefore, $d = (1 + 2\epsilon_{s_i}^{\lambda_0})\lambda_0$. This situation is visually demonstrated in fig.(4), which justifies why I began with the condition (13) i.e. $d > \lambda \ggg 2s_i$.

This is a possible explanation of why Descartes doubted whether his own demonstration[13] of "numbers" by drawing line was exact in ref.[19] (see footnote (10)). Also, this is a possible explanation of how one can demonstrate, as Born would write[34] (see Appendix (C)), the "*natural uncertainty*" of physical observations by writing real numbers in such a way, where the physical aspect is expressed by the physical dimension of s_i, d, λ_0 which justifies my use of the word "physico-mathematical". As far as the issue of "*natural uncertainty*" is concerned, similar issues have been discussed recently in ref.[37, 38].

[13] I consider Descartes' act of casting and analyzing doubts of his own reasoning as an example of self-inquiry.

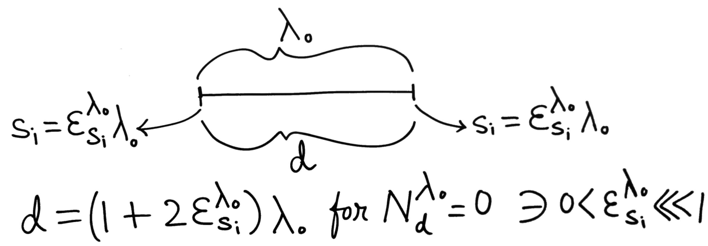


Figure 4: The thought of distance d, while demonstrated by drawing a line and then measured with an iteration of λ_0, the cuts at the two ends of the line that designate the iteration process ("mark off" in Einstein's words, see the next subsection), result in an error of $2s_i = 2\epsilon_{s_i}^{\lambda_0}\lambda_0$. This is the least possible measurable d in terms of λ_0 i.e. $N_d^{\lambda_0} = 0$.

3.4 Einstein's "mark off" for simple length measurement and his ignorance of the experienced truth of the marks

In view of what I have discussed regarding the "natural uncertainty" (in Born's words) in the physical act of distance measurement to explicate the concept of numbers, it should be now of importance to have a comparison with Einstein's views regarding the same so as to understand more clearly in what sense the present scenario is different from that of Einstein which is however the prevalent and accepted viewpoint in the standard literature of physics. Although Einstein did not consider the extension of the dot and never tried to answer the question regarding the number of dots on a line , however he explained distance measurement in the same way i.e. by repeated iterations with a chosen "distance" (length) which is considered as standard. On p 4 of ref.[20], Einstein wrote:

"*On the basis of the physical interpretation of distance which has been indicated, we are also in a position to establish the distance between two points on a rigid body by means of measurements. For this purpose we require a "distance" (rod S) which is to be used once and for all, and which we employ as a standard measure. If, now, A and B are two points on a rigid body, we can construct the line joining them according to the rules of geometry; then, starting from A, we can mark off the distance S time after time until we reach B. The number of these operations required is the numerical measure of the distance AB. This is the basis of all measurement of length. Here we have assumed that there is nothing left over, i.e. that the measurement gives a whole number. This difficulty is got over by the use of divided measuring-rods, the introduction of which does not demand any fundamentally new method.*"

Certainly here Einstein, as I have emphasized earlier also, did base his explanations on "the rules of geometry". Therefore, not only he did not take into consideration the dots which he needed to put in order to demonstrate the points A and B, but he also did not consider the thickness of the marks so as to keep track of how many times he had to "mark off" the standard length. Due to such ignorance of direct experiences, Einstein concluded with confidence that "measurement gives a whole number" without mentioning the irremovable error due to the marks that he needed to put so as to make the measurement process possible. Even though there is "nothing left over", there is the collective thickness of the marks which, itself, can not be measured within this measurement process and appear as, as Born might say (see Appendix(C)), "*natural uncertainty*" in the process . In view of this I may write that in order to take into account such errors it "does not demand any fundamentally new method" but it demands a fundamentally new attitude – an attitude of truthfulness of expressions of experience along with the admittance of the incompleteness of the measurement process. Ignorance of the experience of such errors is a mistake on the scientific grounds in pursuit of truth.

3.5 Goedel's translation of Cantor's "continuum problem" without any demonstration of "a point" and "a line"

In relation to what I have discussed in the introduction of Section(3), I may bring to the attention of the reader the following. It was Goedel who posed the following question on page no. 58 of ref.[31]:

"*How many points are there on a straight line on a Euclidean space?*"

This was his translation of Cantor's continuum problem[32]. Now, first and foremost, the reader may wonder why I should indulge in such a question at all while the physicist can just comfortably ignore it and remain content with writing theories which satisfactorily explain experimental observations. For example, Einstein explained, in the *Abstract* of ref.[29], his theory of relativity as "*evolution of the notion of space and time into that of the continuum with metric structure.*", without worrying about the word "continuum". Such ignorance of the physicist might have been justified by Einstein in the following manner (in the same *Abstract*):

"*Physics constitutes a logical system of thought.... The justification (truth content) of the system rests in the proof of usefulness of the resulting theorems on the basis of sense experiences, where the relations of the latter to the former can only be comprehended intuitively.*"

While I strongly believe that the essence of ideas lie in their practical use, however, an inquiry regarding the "truth content" of the parts of the "logical system", i.e. the axioms, becomes necessary when "a logical system of thought" is pushed to its limits leading to a logical catastrophe like the singularity theorem[1, 2]. If I call Einstein's 'intuitive comprehension of the relations among the resulting theorems with sense experiences' as "outer intuition", then I may call 'intuitive comprehension of the relations among the starting axioms with sense experiences' as "inner intuition". Brouwer might have written in this present context that such inner intuition "subtilizes logic" and possibly he could have called such an act of inward directed inquiry as "inner inquiry"[39], which I may call "self-inquiry"[40]. Such self-inquiry was not a matter of concern for Goedel, like Einstein, which I may elucidate as follows. On p 1 of ref.[33], Goedel adopted "*the system* Σ *of axioms for set theory*" and wrote about "*theorems demonstrable in* Σ" to deal with the continuum problem. However, he did not bother to demonstrate the axioms themselves i.e. the intent was to make outward inquiry and not inward inquiry. Such intent becomes even more vivid from the fact that Goedel did not demonstrate his translation of the continuum problem through a drawing, neither did Cantor do himself [44]. Such a demonstration process is necessary because, as far as I understand, without putting a dot on a paper, one can not possibly convey the meaning of the word "point" to somebody else. Consequently, the words "point" and "line" become empty of any essence without such demonstration. That is, one could have simply asked Goedel (or Cantor) the following question: *Can you explain what you mean by the words "point" and "line"?* If this were the case, I believe, Goedel would have picked up a pencil and put a dot on a paper to start his demonstration in order to intuitively comprehend the relations among the starting axioms with his own sense experiences. As Newton wrote, in the Preface to the First Edition, on page no. - xvii, of ref.[6], that "*Geometry does not teach us to draw these lines, but requires them to be drawn;..*". Such an act of self-inquiry is the subject matter of the present discussion so as to elucidate why the singularity problem is a logical catastrophe and to discuss how the solution can possibly be found in the middle of being logical and operational.

Now, it does not take a super-intelligence to sense an association of the present discussion with the foundations of set theory[25]. However, I must refrain myself from bringing into this discussion any such abstractions which are nevertheless plagued with several antinomies[25]. Due to such reasons any discussion regarding set theory will lead to further discussions concerning philosophical, logical and linguistic issues which will add unnecessary complications to the present context. Most importantly, such abstractions do not have any immediate connection with the experimental works which support the theories of gravity because the experimental physicists are only worried about the equations provided by the theoretical physicist and the corresponding interpretations[28].

4 The two body interaction with each of the bodies considered as a whole

Having the notion of "distance" clarified, now I discuss how the two body gravitational interaction can be analyzed both in the large distance and the small distance scenarios starting with a proposition concerning the ratio F/F_0 motivated by the arguments provided in section (2.2).

In terms of measurement with λ_0 i.e. from the observer's perspective, taking into account what I have explained regarding (18), I write the following:

$$\mathbf{O}_{\lambda_0}\left[\frac{s_i}{d}\right] := \frac{s_i}{r_i + s_i} \quad : \quad s_i = \epsilon_{s_i}^{\lambda_0}\lambda_0, \quad r_i = x_{s_i}^{\lambda_0}\lambda_0, \quad 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2, \tag{19}$$

where "$\mathbf{O}$" stands for "observed". Here, I have used the term "observed" in the sense of "measured in terms of λ_0", but without explicating who/what an observer is[40]. I may note that

$$\begin{array}{rcccccc} 0 < \mathbf{O}_{\lambda_0}\left[\frac{s_i}{d}\right] & < & \frac{1}{2} & \text{for} & d > 2s_i & \Leftrightarrow & r_i > s_i \\ \frac{1}{2} < \mathbf{O}_{\lambda_0}\left[\frac{s_i}{d}\right] & < & 1 & \text{for} & s_i < d < 2s_i & \Leftrightarrow & r_i < s_i \end{array}$$

Now, I consider two dots s_1 and s_2 to represent two objects (each considered as a whole) in order to analyze the gravitational two body interaction and, motivated by the discussion in section (2.2), begin with a proposition concerning F/F_0 as follows:

$$\begin{aligned} \frac{F}{F_0} &= \mathbf{O}_{\lambda_0}\left[\frac{s_1}{d}\right] \cdot \mathbf{O}_{\lambda_0}\left[\frac{s_2}{d}\right] \\ &= \frac{s_1}{r_1 + s_1} \cdot \frac{s_2}{r_2 + s_2} \quad : \quad s_i = \epsilon_{s_i}^{\lambda_0}\lambda_0, \quad r_i = x_{s_i}^{\lambda_0}\lambda_0, \quad 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2, \quad \forall i \in [1,2]. \end{aligned} \tag{20}$$

The symbols "$\forall$", "$\in$" stand for "for all", "belongs to" respectively. Since $d > s_i$, each of the series involved converges to some value between 0 and 1. Consequently, $0 < F/F_0 < 1$ i.e. F_0 is some upper bound on F. Eq.(20) can be analyzed for various cases depending on the relation among d, s_1, s_2. Here, I discuss the two most relevant cases in what follows, namely, the large distance scenario that corresponds to $0 < \frac{F}{F_0} < \frac{1}{4}$, may be termed as "weak gravity", and the small distance scenario that corresponds to $\frac{1}{4} < \frac{F}{F_0} < 1$, may be termed as "strong gravity", where the nomenclatures have clear comparative sense. However, I may note that the proposition (20) is useless because it is impossible to carry out computations with two different notion of "units", namely, s_1 and s_2 because each leads to a different "r" i.e. r_1 and r_2 with which any computation is impossible[14]. So, a relation between s_1 and s_2 needs to be assumed so as to perform computations and deem the proposition (20) useful. The worth of such assumption can only be justified by the significance its consequence.

4.1 "Geometric results" in the large distance approximation and for measurable distance

Considering what I have already discussed up to here, the condition that I call "large distance" and "measurable distance" is written as

$$d > \lambda_0 \ggg 2s_i \quad \forall i \in [1,2]. \tag{21}$$

Owing to the reasons discussed in the previous section, I *assume*: $s_2 = n_0 s_1 \;\; : n_0 > 0$ and n_0 is a constant[15]. From this assumption I may write the following:

$$\frac{s_2}{d} = n_0 \frac{s_1}{d}. \qquad \therefore \mathbf{O}_{\lambda_0}\left[\frac{s_2}{d}\right] = n_0 \mathbf{O}_{\lambda_0}\left[\frac{s_1}{d}\right]. \tag{22}$$

Using (22), I may recast (20) as follows:

$$\begin{aligned}
\frac{F}{F_0} &= \mathbf{O}_{\lambda_0}\left[\frac{s_1}{d}\right] \cdot \mathbf{O}_{\lambda_0}\left[\frac{s_2}{d}\right] \;=\; n_0 \; \mathbf{O}_{\lambda_0}\left[\frac{s_1}{d}\right] \cdot \mathbf{O}_{\lambda_0}\left[\frac{s_1}{d}\right] \\
&= n_0 \cdot \frac{s_1}{(r_1+s_1)} \cdot \frac{s_1}{(r_1+s_1)} \quad : \; s_i = \epsilon_{s_i}^{\lambda_0}\lambda_0, \;\; r_i = x_{s_i}^{\lambda_0}\lambda_0, \;\; 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2 \;\; \forall i \in [1,2] \\
&= \frac{s_2 s_1}{(r_1+s_1)^2} \quad : \; s_i = \epsilon_{s_i}^{\lambda_0}\lambda_0, \;\; r_i = x_{s_i}^{\lambda_0}\lambda_0, \;\; 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2 \;\; \forall i \in [1,2] \; ; \; s_2 = n_0 s_1 \\
&= \frac{s_2 s_1}{r_1^2}\left(1 + \frac{s_1}{r_1}\right)^{-2} \quad : \; s_i = \epsilon_{s_i}^{\lambda_0}\lambda_0, \;\; r_i = x_{s_i}^{\lambda_0}\lambda_0, \;\; 0 < \epsilon_{s_i}^{\lambda_0} \lll 1/2 \;\; \forall i \in [1,2] \; ; \; s_2 = n_0 s_1.
\end{aligned} \tag{23}$$

Owing to the condition (21), which leads to the condition $r_1 \ggg s_1$, I write the above expression in the following form:

$$\frac{F}{F_0} = \frac{s_2 s_1}{r_1^2}\left[1 - \frac{2s_1}{r_1} + \frac{3s_1^2}{r_1^2} - \cdots\right] \tag{24}$$

It is important to note that from the conditions (21) and (9) it becomes clear that this present scenario is indeed the situation where

$$0 < \frac{F}{F_0} \lll \frac{1}{4}. \tag{25}$$

i.e. the correspondence between "large distance" and "weak gravity" becomes apparent, where both the terms are valid in a comparative sense.

At this point it may seem that only $d > \lambda_0 \ggg 2s_1$ (hence, $r_1 \ggg s_1$) has played the role in doing the computation leading to eq.(24), and $d > \lambda_0 \ggg 2s_2$ is redundant. However, given the symmetry

[14] It is as nonsense as trying to do arithmetic operations with two different unities, say, 1_1 and 1_2. Such concern was aptly raised by Frege on page no. 57 of [51]: *"But the mere existence of the difference is already enough,...which is utterly incompatible with the existence of arithmetic"*.

[15] In general, s_2 may depend on s_1 and d as well in a nontrivial manner.

of the Newton's law of gravitation in m_1 and m_2, and owing to the present motive of manifesting its approximate nature, it is expected that the above steps of calculation go through even if I interchange the role of s_1 and s_2, leading to

$$\frac{F}{F_0} = \frac{s_1 s_2}{r_2^2}\left[1 - \frac{2s_2}{r_2} + \frac{3s_2^2}{r_2^2} - \cdots\right], \tag{26}$$

when $d > \lambda_0 \ggg 2s_2$, and hence, $r_2 \ggg s_2$. This expectation is fulfilled if the above analysis can be done by interchanging the role of s_1 and s_2, which happens if and only if the condition (21) holds. Consequently, as $r_i \ggg s_i \; \forall i \in [1,2]$: (i) firstly, the sub-leading terms can be neglected in eq.(24) and eq.(26); (ii) secondly, I can write $d = r_i + s_i \simeq r_i \; \forall i \in [1,2]$ and, hence, $r_1 \simeq r_2 \simeq r$ (say). Therefore, under the validity of such conditions (24) and (26) can be approximated to

$$\frac{F}{F_0} \simeq \frac{s_1 s_2}{r^2}. \tag{27}$$

It may be noted that the above expression is valid for $F \lll F_0/4$ (condition (25), "weak gravity").

Certainly it is possible to reconstruct theoretical physics, from scratch, by attaching the words "point mass" with "$2s \lll \lambda_0$" (see Appendix (E) for a glimpse). In that case "mass" is not an independent physical dimension and issues as basic as dimensional analysis (from theoretical viewpoint) and "mass measurement" (from experimental/operational viewpoint) need to be revisited. However, I do not wish to explore such possibility in this article which will give rise to a different course of discussion than what is necessary in the present context. Rather I shall take the opportunity to showcase how this singularity-free analysis consistently demonstrates the presently accepted scenario where "mass" is an independent physical dimension by writing $s_i := Gm_i/c^2$ and $F_0 := c^4/G$, that I have already motivated through the discussion in section(2.2). The first consequence is to see that (27) immediately leads to

$$F = G\frac{m_1 m_2}{r^2}, \tag{28}$$

where now it is understood that it is "weak gravity", i.e. $F \lll F_0/4$ (condition (25)), and corresponds to $r \lll Gm_i/c^2 \; \forall i \in [1,2]$.

Here, I intend to explicate the situation where one of the two objects is considered as a test object and the other one is considered as the source object e.g. the scenario where the experimenter drops objects on the earth surface to analyze gravitational phenomena. This is motivated by the fact that our observation of the test object's motion only lets us understand a phenomenon and therefore, its further study. So I consider s_1 as the test object and use $s_1 = Gm_1/c^2, F_0 = c^4/G$ and eq.(24) takes the following form:

$$F = \frac{Gm_2 m_1}{r_1^2}\left[1 - \frac{G}{c^2}\frac{2m_1}{r_1} + \frac{G^2}{c^4}\frac{3m_1^2}{r_1^2} - \cdots\right] \tag{29}$$

where obviously $m_2 = n_0 m_1$.

Now, let me consider the quantity F/m_1 and call it "the gravitational field due to m_2 as felt by the test object with mass m_1", which I designate as $g_{2\to 1}$. Here, I do not distinguish between inertial mass and gravitational mass i.e. I assume the equivalence of inertial mass and gravitational mass. Then, from eq.(29) I have

$$g_{2\to 1} := \frac{F}{m_1} = \frac{Gm_2}{r_1^2}\left[1 - \frac{G}{c^2}\frac{2m_1}{r_1} + \frac{G^2}{c^4}\frac{3m_1^2}{r_1^2} - \cdots\right] \tag{30}$$

The sub-leading m_1-dependent terms signify the back-reaction of the test object. The leading back-reaction term, namely $-2Gm_1m_2/c^2r_1^3$, contributes as a negative acceleration, but suppressed by G/c^2 factor.

Now, at this point, applying a bit of intuition, if I consider all such test masses for which $d \ggg 2s_1 = 2Gm_1/c^2$, then I can write $d = r_1 + s_1 \simeq r_1$ for all such test masses i.e. the subscript "1" of "r_1" can now be erased and I can write simply "r" in place of "r_1". This can be visualized as follows – two dots of different sizes look approximately the same if the lines drawn with them are very large compared to their characteristic extensions. In such a scenario, I can rewrite eq.(29) as follows:

$$F \quad = \quad \frac{Gm_2m_1}{r^2}\left[1 - \frac{G}{c^2}\frac{2m_1}{r} + \frac{G^2}{c^4}\frac{3m_1^2}{r^2} - \cdots\right] \tag{31}$$

and eq.(30) as follows:

$$g_{2\to1} := \frac{F}{m_1} \quad = \quad \frac{Gm_2}{r^2}\left[1 - \frac{G}{c^2}\frac{2m_1}{r} + \frac{G^2}{c^4}\frac{3m_1^2}{r^2} - \cdots\right] \tag{32}$$

If I consider the first term of eq.(32), then I obtain what we know today as the gravitational field due to a source of mass m_2, which appears to be independent of the test mass m_1 from and this apparently is supported by the experimentally verified fact that all masses fall with same acceleration. Certainly, the full expression on the right hand side of eq.(32) is dependent on the test mass m_1. Consequently the question arises whether all masses fall with same acceleration.

4.2 Do different magnitudes of masses with same composition fall with same acceleration?

Certainly the answer to the above question is in the affirmative according to Einsteinian and Newtonian theories of gravity simply because the mass of the test object does not play any role in the geodesic equation in general relativity or the equation of motion in Newtonian gravity (where I have used the word "mass" only in one sense as the equivalence between inertial mass and gravitational mass is assumed in both Einsteinian and Newtonian scenarios. However, the present analysis provides a ground to realize how such a hypothesis is only valid in an approximate limit of large distance physics. Now, as far as the experimental verification of such a hypothesis is concerned, the results of experimental measurements are always subject to refinement due to availability of means to make more precise measurements. Therefore, as it stands, if the above discussion of mine can be considered as valid, then the difference in accelerations of two different masses must be beyond current experimental precision. To verify whether this is the case, I consider only up to the first sub-leading order term of eq.(32) to do calculations, that is,

$$g_{2\to1} := \frac{F}{m_1} \quad \simeq \quad \frac{Gm_2}{r^2}\left[1 - \frac{G}{c^2}\frac{2m_1}{r}\right]. \tag{33}$$

From eq.(33) it is easy to obtain the following:

$$g_{2\to1'} - g_{2\to1''} \quad \simeq \quad -\frac{2G^2}{c^2}\frac{m_2(m_{1'} - m_{1''})}{r^3}. \tag{34}$$

where $m_{1'}$ and $m_{1''}$ are two different test masses when measured in terms of a standard unit of mass.

Now, using such an approximation I may now provide an estimate of the difference in acceleration of two objects with different masses due to gravity at the surface of the earth. I consider $m_{1'} = 5$ kg and $m_{1''} = 500$ kg, radius of the earth $r = 6.371 \times 10^6$ meter, mass of the earth $m_2 = 5.972 \times 10^{24}$ kg, $G = 6.674 \times 10^{-11}$ meter^3 kg^{-1} sec^{-2}, $c = 3 \times 10^8$ meter/sec. It is straightforward to check that $r \ggg s_1$ in case of both the masses because s_1 takes values 3.708×10^{-27} meter and 3.708×10^{-25} meter for 5 kg and 500 kg respectively. Using these data, one can calculate from eq.(34) that

$$g_{2\to 1'} - g_{2\to 1''} \simeq 113.148 \times 10^{-32}\text{meter/sec}^2. \tag{35}$$

It is now understandable that in the conventional units of measurement such a difference in acceleration is undetectable. Nevertheless, the interesting fact is that the more massive an object is, the slower it falls i.e. the acceleration is less. It will be interesting to see whether such an analysis can shed some light on the observations of the galaxies where the currently known and accepted theories of gravity (like that of Newton and Einstein) fail to provide satisfactory explanations leading to problems like missing mass or dark matter[21].

4.2.1 A digression: comments on experimental determinations of Eotvos parameter

I may add a few words to clarify the significance of the above analysis as far as experiments are concerned. Here, I have not distinguished between inertial mass and gravitational mass. These two concepts are already assumed to be equivalent while writing $g_{2\to 1}$ in eq.(33). Therefore, the results discussed here should not be associated with any experiment that determines Eotvos parameter (η), using test objects that are considered as a whole (classical). This is because η is a measure of the difference between the ratio of inertial mass and gravitational mass of two different *types* of objects i.e. objects with different compositions (say, iron and copper)[52]. Rather, the present analysis through eq.(33) and eq.(34), manifests that even with the assumption of the equivalence between inertial mass and gravitational mass, two objects of the same *type*, but having different magnitudes of mass, e.g. 5 kg iron and 500 kg iron, have different accelerations. This may be viewed as the analysis that showcases the approximate validity of "gravitational weak equivalence principle" that accounts for the back reaction of the test object, that has recently been discussed in refs.[53, 54](also see references therein). However, such an analogy should be drawn only at a superficial level because, unlike refs.[53, 54] and the concerned relevant literature that relies on the assumption of an absolute continuum (hence, "absolute differential calculus" in the words of Einstein on p.30 of ref.[16]), the present discussion does not take such mathematical structure for granted and aims to build on a demonstrably undecidable continuum.

There are modern experiments that measure η with atom interferometers where the test objects are not considered as a whole (quantum/not point-mass), but the test objects are of same type[16] with different magnitudes of mass viz. ^{85}Rb and ^{87}Rb e.g. see refs.[56, 57, 58, 59, 60]. The present analysis, being only concerned with objects considered as a whole (classical/point- mass), should not be associated with such experiments. However, a detailed analysis concerning a violation of the proposition (4) for the test object may lead to results in this framework that can be useful in such atom interferometer experiments, which is a possible work for the future.[To understand why I have treated the words "classical" and "quantum" as equivalent to "point-mass" and "not point-mass" respectively, from an axiomatic point of view, one may consult the discussion concerning "quantum gravity" in ref.[4].]

[16] The words "same composition" become meaningless because we are now dealing with atoms.

4.3 Non-singular small distance approximation

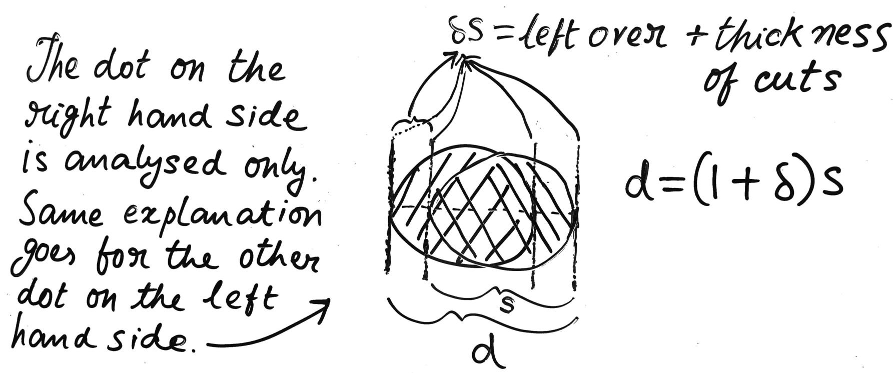
Now, I focus on the situation where the two dots overlap, which is written in the standard literature as "$r = 0$" in the Newton's law of gravity leading to the divergence of the force i.e. the singularity problem. Starting from the eq.(20), I assume $s_2 = n_0 s_1 : n_0 > 0$ and consider $s_1 < d < 2s_1$, $s_2 < d < 2s_2$ i.e. the two dots overlap. Alongside $0 < \delta_d^{s_1}, \delta_d^{s_2} < 1$, the relation $s_2 = n_0 s_1$ results in the following inequalities: $(n_0 - 1) < \delta^{s_1} < (2n_0 - 1)$, $(n_0^{-1} - 1) < \delta_d^{s_1} < (2n_0^{-1} - 1)$. If $n_0 > 1$, then $\delta_d^{s_2} < 0$ and if $n_0 < 1$, then $\delta_d^{s_1} < 0$, both of which violate the condition $0 < \delta_d^{s_1}, \delta_d^{s_2} < 1$ that is necessary for demonstrative purpose i.e. thickness of the cuts. Therefore, the only possibility is $n_0 = 1$. So, I consider $s_1 = s_2 = s$ (say), consequently $\delta_d^{s_1} = \delta_d^{s_2} = \delta_d^s$(say) and write the series expansion involving two SDEs with s, s, d from the observer's perspective, as follows:

$$\frac{F}{F_0} = 1 - \frac{2r}{s} + \frac{3r^2}{s^2} \cdots \quad : s = \epsilon_s^{\lambda_0} \lambda_0,\ r = \delta_d^s \epsilon_s^{\lambda_0} \lambda_0,\ 0 < \epsilon_s^{\lambda_0} \lll 1/2,\ 0 < \delta_d^s < 1. \tag{36}$$

Writing $s = Gm/c^2$ and $F_0 = c^4/G$, I recast eq.(36) as follows:

$$F = \frac{c^4}{G} \left[1 - \frac{c^2}{G} \frac{2r}{m} + \frac{c^4}{G^2} \frac{3r^2}{m^2} - \cdots \right]. \tag{37}$$

I have tried to give below a visual demonstration of two overlapping dots, but only the zoomed version. From the observer's perspective it is impossible to demonstrate visually because it is undecidable whether there is only one dot or two overlapping dots as $\epsilon_s^{\lambda_0} \lll 1/2$. From (36) and (37) it is evident



that as r becomes smaller and smaller than s, the subleading terms become more and more negligible and F approaches $F_0 = c^4/G$. This justifies the significance of F_0 as an upper bound on F and based on such justification this small distance analysis corresponds to "strong gravity" scenario.

Now, one may construct an acceleration by writing F/m, from eq.(37), as follows:

$$\frac{F}{m} = \frac{c^4}{mG} \left[1 - \frac{c^2}{G} \frac{2r}{m} + \frac{c^4}{G^2} \frac{3r^2}{m^2} - \cdots \right]. \tag{38}$$

However, unlike the case of large distance analysis where the leading term of $g_{2\to1}$ in eq.(33) does not depend on m_1, in eq.(38) all the terms are m-dependent. Thus, mass independent acceleration is impossible to achieve even in any approximate sense. Importantly, this acceleration "F/m" can not be interpreted as "gravitational field" if we take "field" as a concept meant for explaining "action at a distance". This is because the distinction between source object and test object is erased in this small distance analysis owing to the absence of a clear notion of distance (overlapping dots). Consequently, the concept of "field" can not be demonstrated by studying the motion of the test object as an effect of that "field", as if the source object and the test object are in a state of confinement.

The scenario is suggestive of the fact that the standard theories of gravity with geometric foundation, where the notion of gravitational field is inherent, are only large distance physics. However, it becomes a matter of concern regarding how to experimentally realize the small distance scenario – or may be it suggests that the scenario presents a situation beyond observations which are explicable with the concept of "gravitational field". Such a known scenario where the known theories of gravity fail is that of a black hole[26]. Nevertheless, in this current methodology, the infinite series in eq.(37) only suggests that there is nothing such as singularity, rather it is the foundation of physics on the first axiom of geometry (regarding "point") that has led to the theoretical conclusion of singularity. Ignorance of the perception of a dot by saying "a point has zero extension", to state the axiom, has created the problem of singularity. Therefore, the problem of singularity is a logico-linguistic catastrophe[40][17].

4.4 The approximate truth of a founding assumption of the singularity theorem by Hawking and Penrose

Now, in light of the present discussion, one may wonder about the status of the Hawking-Penrose singularity theorem, which is founded on a set theoretic basis[1, 2] and has also been credited with The Nobel Prize (to Penrose)[43] on the basis of experimental verification[26]. In view of this I venture to offer some critical comments in light of what I have discussed in subsection(4.2) and subsection(4.3).

Hawking and Penrose certainly legitimized and epitomized the singularity problem, based on analysis founded on set theoretic abstractions[1, 2]. Whereas, this present discussion paves a way to solve the problem by demonstrating that the singularity theorem is not an exact result. Rather, it holds only in a large distance approximation. I admit that a lot of work needs to be done along the lines of investigation that I have discussed here, but I believe there is a convincing enough argument that I can give in favour of my claim. To explicate this, I may note that one of the assumptions, for which the singularity theorem remains valid is that "*Einstein's equations hold*", as the authors declared in the *Abstract* of ref.[2]. Now, Einstein's formulation of general relativity, and hence his equations, are based on the hypothesis that all masses fall with same acceleration under the gravity of a source object and it is due to such hypothesis that the concept of "gravitational field due to a source object" is independent of the mass of the test object[20]. But, I have demonstrated through my analysis that this hypothesis is valid only in the large distance approximation in Section(4.1). Thus, Einstein's equations are only valid in the large distance approximation. Consequently, the singularity theorem is also valid only in the large distance approximation (even if I disregard any possible consequences of the present discussion concerning the other assumptions of the theorem).

[17]In this context I may note that, from the logico-linguistic or semantic perspective any attempt to write down a theory of "quantum gravity", the general motto being to get rid of the singularity problem, is founded on a contradiction[4].

5 Concluding Remarks

I believe that I have demonstrated, at least with partial satisfaction of the reader, how a singularity-free theory of gravity can be written. I conclude by bringing to the attention of the reader a few things which can be immediately noticed from this simple analysis. The notion of distance is only relative and not absolute. This is relativity of quantities i.e. largeness or smallness of some quantity only depends on the quantity with which it is being compared.

Further, I may note that the independence of the gravitational field (due to some source mass m_2) from the mass of the test object (m_1), is only restored when the conditions $d > \lambda_0 \ggg 2s_1, 2s_2$ are satisfied, where $s_1 = Gm_1/c^2, s_2 = Gm_2/c^2$. This manifests how the geometric foundation of physics goes hand in hand with the fact that all masses fall with same acceleration. This is because $d \gg 2s_1$ means any notion of distance while expressed as a line on paper to denote the trajectory of the test object, contains many more than just two points where the extension of the point is negligible compared to the chosen length unit which itself is a line with very many points. Since both, Newton's theory of gravity[7] and Einstein's general relativity[20], are founded upon such a thesis (i.e. all masses fall with same acceleration), the mass of the test object does not play a role in such theories. Therefore, the methodology that I have demonstrated may be of help in certain situations which are not satisfactorily explained by any of the above two theories e.g. the missing mass problem of galaxies[21]. I hope to develop along such directions in near future. Further, there is another issue that I have not focused on in this work and that is the condition $d > \lambda_0 \ggg 2s_2$ where s_2 is the dot that represents the source object as a whole. This condition does not have a direct implication in the calculation which I have shown, except the fact that it ensures the possibility of representation of the source object as a dot whose extension is negligible compared to the line that represents the chosen unit of length. An immediate question arises that what happens to the representation if $d > \lambda_0 > 2s_2$. I hope to elaborate on this issue in near future.

Last but not the least, I may conclude by mentioning that the case of two overlapping dots is a feature of the present analysis which is not possible to be demonstrated by any currently existing theories of gravity or "quantum gravity"[3]. This particular scenario, which manifests the singularity problem in standard literature of physics, is now explicated without any trouble at all, as I have explained in this non-standard scenario. Due to this single most reason, I believe, the present work is novel in its own right. What remains to be seen is whether it is possible to write theoretical physics in such singularity free scenario. Although I have provided a glimpse of, what I may call, Non-Standard Physics in Appendix(E), I plan to provide the details of such theoretical construction in near future.

Acknowledgment: This work has been supported by the Department of Science and Technology of India through the INSPIRE Faculty Fellowship, Grant no.- IFA18-PH208.

Declaration of Interests: The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. There is no conflict of interest.

A A critical commentary on Einstein's views regarding "Physical Meaning of Geometrical Propositions"

In what follows, I shall offer some critical comments on a few relevant statements by Einstein from the first chapter of ref.[20] in order to show how Einstein was not operational and rather relied on logic, which by nature is abstract and detached from immediate truths of experience as Einstein himself pointed out. So, what Bridgman called as "operational" was only a contextual characteristic of Einstein's reasoning that founded his relativity theories i.e. while Einstein explained the concepts verbally he was operational because his explanations were directly attached to experienced facts, but while he did mathematical analysis he relied on the axioms of geometry. I analyze Einstein's statements part by part as follows.

"*Geometry sets out from certain conceptions such as "plane," "point," and "straight line," with which we are able to associate more or less definite ideas, and from certain simple propositions (axioms) which, in virtue of these ideas, we are inclined to accept as "true."* "

It is manifest from Einstein's words that geometric conceptions are associated with "more or less definite" ideas i.e. geometric conceptions are approximate and not exact or definite. Furthermore, such ideas are assumed to be universal truths and that is why we need to be "inclined to accept" such truths in order to proceed with the axiomatic framework of geometry. Such inclinations or biases are the necessities for working with such axiomatic framework irrespective of whether such assumed truths defy experience or not. Indeed Einstein pointed this out shortly.

"*The question of the "truth" of the individual geometrical propositions is thus reduced to one of the "truth" of the axioms. Now it has long been known that the last question is not only unanswerable by the methods of geometry, but that it is in itself entirely without meaning.* "

Here, Einstein pointed out that the "truth" of any axiom of geometry can not be verified by remaining within the axiomatic framework of geometry. This certainly means that if any one axiom of geometry is considered as "false", then the basis of such consideration can not be geometry itself. The experience of seeing a dot, which is an act of being operational, is thus necessary to be taken into consideration if the limited validity of the axioms of geometry is to be explored. And this is reflected from Einstein's own admission that I point out next.

"*The concept "true" does not tally with the assertions of pure geometry, because by the word "true" we are eventually in the habit of designating always the correspondence with a "real" object; geometry, however, is not concerned with the relation of the ideas involved in it to objects of experience, but only with the logical connection of these ideas among themselves.*"

This statement by Einstein clearly justifies in what sense he was not operational and it was because of his reliance on the axioms of geometry to write down theories. As Einstein emphasized, the word "true" is only meaningful while we relate our ideas to some "real" objects of experience. So, considering Bridgman's notion of being "operational", the axioms of geometry are not operational (or anti-operational) as those axioms are statements which do not explain our experience. Further, considering Einstein's explication of the word "true", a statement that expresses the visible dot of the pencil can be only considered to be "true" by virtue of our experience of vision of the dot. Therefore, I may conclude that although Einstein was operational while explaining concepts based on experience of observed phenomena, but he was not operational due to his use of the axioms of geometry to write down his theories.

B More clarifications on the statement regarding "a point" and "a line"

Here I clarify certain issues with some expected objections in mind those may be raised by the logic minded reader. I enlist such expected objections and the possible refutations as follows:

- Is my statement a definition of "a point" or "a line"? I should say "it is neither" and "it is both" – the answer depends on the way the situation is analyzed. It is *neither* because I have not considered the term "definition" at all (except on a different occasion in the sense of symbolic abbreviation – see footnote(9)). Rather, I have adopted a practical method of demonstrating facts in simple language which depends of the mindset of the reader to be deemed as acceptable. If I consider the term "definition" as "explication", then I have explicated *both* "a point" and "a line" in relation to each other. This is because I can not realize any of those concepts in isolation, but only in relation to each other. When I put a dot on the paper, it seems of negligible extension only if I have the experience of drawing a line with respect to which I write the dot's extension to be "negligible". Otherwise, the word "negligible" by itself is incomplete in the sense that one can always question "negligible with respect to what?"

- I can draw a line only after putting a dot. So, is my statement circular in reasoning? The answer is "yes" and "no" depending on the way one chooses to analyze the situation. The statement *is* circular if "a dot" ("a point") and "a line" are considered as isolated and logical truths, as it is done in accord with the axioms of geometry. However, in that process, the truth of the experience is denied – an anti-operational process of reasoning. The statement *is not* circular if the experience of putting a dot and drawing a line is truthfully analyzed. Certainly the dot needs to be put first and then the line can be drawn while I draw with a pencil. However, the negligible extension of a dot can be realized only after drawing, and in relation to, a line. So, neither a dot nor a line has in itself the truth of experience in isolation. Rather, both can be realized in relation to the other. Therefore, my statement is just an expression of demonstration based on experience. I do not want to categorize only parts of the statement into logical and hence, isolated absolute truths that only leads to logical paradoxes like this objection of circular reasoning.

So, instead of putting logical analysis into the context, I consider my investigation as a series of reasonable expressions that explicate my experience. Nevertheless, I must admit that such operational process of reasoning brings in the danger of consideration of an infinite process of reasoning because one can raise a legitimate doubt regarding whether I can ever stop expressing my experience e.g. colour of the paper, colour of the pencil, the pressure I apply to put the dot and so on. So, I declare the incompleteness of my expression and my choice to cut off the reasoning where I find it suitable and sufficient for the present context.

Even after all these explanations, I expect the modern scientist, who puts his complete trust in logic and hence, his belief in complete definitions, to be unwilling to believe in the process of reasoning that I have adopted here. To germinate the seed of doubt in such a mind, I have analyzed Riemann's definition of the concept of "line" in Appendix(D).

C Born's wish of uncertain numbers, Klein's vision of "a small spot" and "fusion of arithmetic and geometry": an unfulfilled desire

The visual experience of the dot and the indispensability of the cuts, render the explication of numbers to be inexact. Such a demonstration process is always accompanied with errors due to the mode of demonstration itself and without such demonstration the ideas remain abstract and can not be applied for practical purpose, unless it is lied about. Apart from Descartes' doubt regarding the exactness of numbers being explicated in such a way (see footnote (10)), I find it interesting and worthy to note Born's and Klein's views regarding such issues so that the importance of the present work is manifested in a more convincing manner.

Born wrote the following on page no. 81 of ref.[34]:

"*Of course, I do not intend to banish from physics the idea of a real number. It is indispensable for the application of analysis. What I mean is that <u>a physical situation must be described by means of real numbers in such a way that the natural uncertainty in all observations is taken into account</u>.*"

I believe, Born would have agreed that demonstration of the concept of number by drawing a line and making cuts, which nevertheless begins with putting a visible dot on the paper, is both a physical act and a physical observation as well. Thus, the uncertainty (ϵ, δ etc.), that is incorporated in my demonstration, is what Born tried to indicate (my guess). Assuming Born's agreement, I may consider that the demonstration procedure that I have adopted and the way I have analyzed the situation, can be considered as quite believable by the modern authorities of science. However, I may go further to bring to light some more convincing historical visions along such lines of thought, which Born himself provided in the statements which followed the above quoted ones. Born's comment concerning Klein's vision, that followed the above quoted statement only justifies the essence of the present discussion in an even more explicit and convincing way:

"*Felix Klein called for a similar step to be taken in geometry. Besides abstract, exact geometry, he desired to have a practical geometry, in which <u>a point is replaced by a small spot, straight lines by narrow strips</u>, etc. However, nothing much resulted from this.*"

Although Born did not analyze further why Klein could not accomplish what he wanted to, it appears to me that once "a point" is replaced by "a small spot", then the question arises that "small with respect to what". *An* obvious, and the simplest (according to me), answer is that "with respect to any other extension that one may call a line" i.e. the relational aspect of the scenario (i.e. the visual cognition) needs to be necessarily taken into account and the involvement of units become a requirement because neither the dot nor the line, by itself, explicates the concept of number that is represented by numeral. Then the scenario can not be categorized as either "arithmetic" or "geometry", but somewhere in the middle – a practical method of demonstration and the associated analysis, which Klein himself might have called "*fusion of arithmetic and geometry*" and not any one in isolation – see page no. 2 of ref.[35]. However, any act of demonstration of this "fusion", which being a "practical method", must rely on a mode of demonstration. This mode of demonstration is certainly a physical act that leads to the "physical situation" (in Born's words) involving the extension of the dot that is negligible with respect to the extension of a line. Certainly Klein highlighted the issue of inexactness associated with such an explication of the scenario. However, what was missing in Klein's analysis was this understanding of the relational nature of the underlying situation that Born also could not get the grasp of.

Nevertheless, it is worth clarifying certain aspects of this work in tandem with some of the statements of Klein from pp. 42-43 of ref.[36]:

"*If we now ask how we can account for this distinction between the naive and refined intuition, I must say that, in my opinion, the root of the matter lies in the fact that the naive intuition is not exact, while the refined intuition is not properly intuition at all, but arises through the logical development from axioms considered as perfectly exact. To explain the meaning of the first half of this statement it is my opinion that, in our naive intuition, when thinking of a point we do not picture to our mind an abstract mathematical point, but substitute something concrete for it. In imagining a line, we do not picture to ourselves "length without breadth", but a strip of a certain width.*"

These are the statements, from which I have quoted the appropriate words while demonstrating the impossibility of a complete refinement of the act of counting dots in Section(3) and therefore, the importance of the condition $(N_d^{s_i} + 2)\delta_I^{s_i} s_i < s_i$ which I imposed. Added to these statements, it is important to note what Klein wrote next regarding the term "definition" while it comes to the words "point", "line", etc.

"*Now such a strip has of course always a tangent....; i.e. we can always imagine a straight strip having a small portion (element) in common with the curved strip; similarly with respect to the osculating circle. The definitions in this case are regarded as holding only approximately, or as far as may be necessary. The "exact" mathematicians will of course say that such definitions are not definitions at all. But I maintain that in ordinary life we actually operate with such inexact definitions. Thus we speak without hesitancy of the direction and curvature of a river or a road, although the "line" in this case has certainly considerable width.*"

Certainly the term "definition" is logical and has an inherent appeal of being complete or exact or perfect. Then, to be logical a complete refinement of intuition is necessary and then, in the process, directly experienced truth needs to be ignored in the case of the dot or the line. This is why, considering such a view in accord with Klein, I have used terms like "explication", "demonstration", etc. instead of the term "definition" in the context "point", "line", etc. In agreement with Klein's view, Kant would prefer the word "*exposition*" instead of the word "definition":

"*Instead of the term, definition, I prefer to use the term, exposition, as being a more guarded term, which the critic can accept as being up to a certain point valid, though still entertaining doubts as to the completeness of the analysis.*"

– see page no. 144, Vol. 1 of ref.[42]. For a very simple demonstration of such a logical dilemma while considering the term "definition", one may consult Section (2.1.2) of ref.[40] and the relevant discussion regarding Frege's demonstration of definition of a concept analyzed in Section (2.1) of ref.[41].

D Analyzing Riemann's expressions regarding "The hypotheses on which geometry is based" from the logician's perspective

In what follows I analyze some of the statements of Riemann in ref.[11] so as to bring forth the subtleties of reasoning that gets associated if each of his verbal statements are carefully analyzed. The two issues that I intend to raise concern about are the following: (i) Riemann's definition of "a line" is circular in reasoning (ii) Riemann's statements regarding "infinitesimal" are incomplete.

D.1 Circular reasoning in Riemann's definition of the concept of "line"

Riemann wrote on pp 261-262 of ref.[11]:

"*Measurement requires that the measure of the entities being measured must be independent of their location, and this can be the case in more than one way.*"

If I suppose, a length L being measured in terms of some length unit λ_0, then the number $n_L^{\lambda_0}$ that is yielded must be independent of the location where the measurement process takes place. I believe this is what Riemann meant in the above statement. Einstein assumed that such measurement process is exact and does not contain any error as he wrote on p 4 of ref.[20] "*that there is nothing left over, i.e. that the measurement gives a whole number.*". Therefore, $n_L^{\lambda_0}$, which represents the number of times λ_0 can be superposed on L, can only take values $1, 2, 3, \cdots$. Now, Riemann went on to write,

"*The assumption which first suggests itself, and which I intend to pursue here, is that the length of lines is independent of their position, so that every line can be measured by comparing it with any other line.*"

The word "location" is now replaced by "position". Then Riemann wrote the following.

"*If the determination of the position of a point in a given n-dimensional manifold is reduced to the determination of n variables $x_1, x_2, x_3, \cdots, x_n$, then a line may be defined by the statement that the quantities x are given functions of a single variable.*"

Here, I raise the following question: *How is the position of a point determined?* Since Riemann did not explain further the word "determination", I consider the following explanation.

- I choose some origin O and draw a line by joining O and the point P (say). The line OP is the distance d of the point P from O.

- I choose another line λ_0 such that it can be superposed multiple times on d to generate a number $n_d^{\lambda_0}$ (say). Then, I write $d = n_d^{\lambda_0}\lambda_0$.

The above steps explicate the meaning of "determination" of the position of a point. Then the quantities x (which, I believe, should be written as $x_i : i = 1, 2, 3, \cdots, n$) should be written in terms of λ_0. Therefore, this act of "determination" is itself a measurement process. Then, according to Riemann's own statement, the yielded number $n_d^{\lambda_0}$ should be independent of the "location" of the lines that represent the distance d and the length unit λ_0. The problem is now to give meaning to the word "location/position". This is because the distance d itself is now a line that is getting measured for the determination process to be carried out. Then, one needs to choose another origin, say O', with respect to which the "location/position" of d and λ_0 needs to be "determined" so that the previous determination holds any meaning according to Riemann. However, the new "determination" is again a measurement process and to give meaning to it, another origin, say O'', needs to be chosen. The process goes on. Hence, the term "determination" can not be completely *defined* and can only be *explicated* with partial satisfaction as there is always a doubt retained in the process of reasoning owing to such self-inquiry[40].

Further, it must be clear from the above explanation that in order to carry out the "determination" process, the lines need to be drawn i.e. the concept of "line" needs to be used. However, instead of analyzing his own words in such a way, Riemann "defined" what a "line" is, without explication of the term "determination" for which he needed to use the concept of a "line" in the first place. Since geometry is a system of logical truths or axioms then, viewing from the logician's perspective, such definition of "line" appears to be based on circular reasoning i.e. Riemann's definition of "line" is not a logical definition. However, if one ignores the logical rigor of such a logical system of thoughts then Riemann's analysis is definitely useful in an operational way because we do general relativity based on such concepts. Therefore, the foundations of Riemannian geometry, which forms the basis of general relativity, are both logical and not logical in the same process of reasoning – logical because such a definition of "line" is accepted to be true and considered as axiom; illogical because such a definition is circular in reasoning and hence, can not be considered as logical by the logician.

D.2 Riemann's incomplete statement: "infinitesimal" with respect to what?

Riemann continued to write the following:

"*The problem then is to find a mathematical expression for the length of a line, and for this purpose we need to consider the quantities x as expressible in terms of units.*"

As I have explained earlier while explicating the term "determination", Riemann did acknowledge that the quantities x (x_i) should be expressed in terms of units. However, Riemann's verbal statements are not truthfully translated into his equations because the units are not written explicitly. This is important to note because of what Riemann wrote next.

"*I shall handle this problem only under certain restrictions, and confine myself in the first place to lines in which the relations between the quantities dx - the associated variations of the variables x - vary in continuous fashion. We can then visualize the line as being divided up into elements, within which the ratios of the increments dx can be regarded as constant, and the problem reduces to finding a general expression for line element starting from a given point, which will involve the variables x as well as the variables dx.*"

Here, there is no clarification regarding whether the elements of the line are bigger than or smaller than or equal to the length unit. This is important because of what Riemann wrote next.

"*Secondly, I shall assume that the length of the line element, disregarding quantities of the second order of magnitude, remains unchanged if all its points undergo the same infinitesimal displacement.*"

I believe that by the word "infinitesimal" Riemann meant "infinitesimally small". If not, then the word "infinitesimal" needs to be clarified in a more elementary fashion. If yes, then an obvious question arises about the displacement and that is, infinitesimally small with respect to what? Is it with respect to the length unit or the quantity x? If it is with respect to the length unit, then how does the theory look like when smaller length units are chosen because in that case the displacement does not remain "infinitesimal" anymore? If it is with respect to the quantity x, then the theory should be written in such a way that the quantity x must have, as Born would write (see Appendix(C)), a "natural uncertainty" much greater than dx irrespective of the role of the length unit. Neither do I find any answer to such basic questions nor do I find clarifications regarding such basic doubts anywhere in ref.[11]. Therefore, I find Riemann's use of the word "infinitesimal" to have only an incomplete sense.

E A glimpse of non-standard physics

In this work, I have used the concept of "mass" as an independent physical dimension so as to draw the connection with standard physics literature by writing $s_i = Gm_i/c^2$. However, the relevant equations can be written in terms of s_i, in relation to λ_0, only and this will result in the appearance extremely small numbers in the associated analysis which is quite akin to what one encounters in Non-Standard Analysis (NSA) [22][18]. Goedel asserted that "*there are good reasons to believe that non-standard analysis, in some version or other, will be the analysis of the future.*" (see the *Preface* of ref.[22]). Certainly, what I have discussed in this work has a priori nothing to do with NSA that is founded on mathematical logic and does not involve physical dimensions. However, it would not be a criminal offense to consider it as an "other version" of NSA. I may call this *Non-Standard Physics* (NSP) so as to distinguish it from standard physics literature and from NSA due to the distinctions that I mentioned before. While a detailed discussion regarding NSP is beyond the scope of this article, however, some immediate results from NSP can be showcased so as to convince the reader that it is a clear possibility. In NSP, instead of being considered as an independent physical dimension, the concept of "point mass" founded on which is classical mechanics of standard physics, is expressed as an intrinsic length of an object, considered as a whole, that is extremely small compared to the chosen conventional length unit. So, the unit of intrinsic length, if called "kilogram" and abbreviated as "kg", then unlike standard physics, now we have "kg $\lll$ meter", where "meter" is the chosen unit of length. Any other such intrinsic lengths are some multiples of "kg". I provide below a comparison between the standard physics and NSP, in light of the choice of units that we make in the beginning of classical mechanics while studying the laws of motion, followed by a simple problem to explicate the situation.

PLEASE TURN OVER.

[18]For modern texts see, for example, refs.[23, 24]

Choice of units

Standard Physics		Non-standard Physics
Force on an object, considered as a whole and represented as a "point mass", is written as $F \propto ma$. Therefore, $$F = kma,$$ where k is a proportionality constant with appropriate physical dimension. For $m =$ 1kg, $a =$ 1meter/sec^2, we have $$F = k \text{ kg.meter/sec}^2.$$ We choose k such that 1 unit of mass, having 1 unit of acceleration is equivalent to 1 unit of force. So, we choose $k = 1$ to write $$1\text{N} = 1\text{kg.meter/sec}^2,$$ where N stands for 'Newton', the unit of force. This is a convention to write the unit of force that is derived from the units of mass and acceleration.		Force on an object, considered as a whole and represented as "an intrinsic length that is extremely small compared to the chosen conventional unit of length", is written as $F \propto sa$. Therefore, $$F = ksa,$$ where k is a proportionality constant with appropriate physical dimension. Chosen conventional unit of length is called "meter". So, we write $s \lll$meter. Unit of s (not m) is called "kilogram", abbreviated as "kg" such that kg $\lll$ meter. Then, (instead of "$m =$ 1kg") we write "$s =$ 1kg$= 1.\epsilon_{kg}^{meter}$ meter $= \epsilon_{kg}^{meter}$ meter : $0 < \epsilon_{kg}^{meter} \lll 1$". So, for $s =$ 1kg$= \epsilon_{kg}^{meter}$ meter, $a = 1$ meter/sec^2, we have $$\begin{aligned} F &= k(1\text{kg}).(1\text{meter/sec}^2) : 0 < \epsilon_{kg}^{meter} \lll 1 \\ &= k\epsilon_{kg}^{meter} \text{ meter}^2/\text{sec}^2. \end{aligned}$$ We choose k such that 1 unit of intrinsic length (extremely small compared to conventional length unit), having 1 unit of acceleration is equivalent to 1 unit of force. So, we choose $k = 1$ to write $$1 \text{ N} = \epsilon_{kg}^{meter} \text{ meter}^2/\text{sec}^2,$$ where N stands for 'Newton', the unit of force.

PLEASE TURN OVER.

A simple problem

A force of 5 N gives a mass m_1, an acceleration of 10 meter/sec^2 and a mass m_2 an acceleration of 20 meter/sec^2. What acceleration would it give if both the masses were tied together?

Solution in Standard Physics		**Solution in Non-standard Physics**
We note that N = kg.meter/sec^2. Now, we solve the problem as follows. $5\text{ N} = m_1.10\text{ meter/sec}^2 \quad \Leftrightarrow \quad m_1 = \frac{1}{2}\text{ kg}$ $5\text{ N} = m_2.20\text{ meter/sec}^2 \quad \Leftrightarrow \quad m_2 = \frac{1}{4}\text{ kg}$ If a is the acceleration of the joint mass, then we can write $5\text{N} = (m_1 + m_2)a = \left(\frac{1}{2} + \frac{1}{4}\right)\text{kg}.a$ $\Leftrightarrow \quad a = \frac{20}{3}\text{meter/sec}^2.$		First we replace m_1, m_2 by s_1, s_2 and also we note that N= ϵ_{kg}^{meter} meter^2/sec^2 $: 0 < \epsilon \lll 1$. Now, we solve the problem as follows. $5\text{N} \;=\; s_1.10\text{ meter/sec}^2 \Leftrightarrow s_1 = \frac{1}{2}\epsilon_{kg}^{meter}\text{ meter},$ $5\text{N} \;=\; s_2.20\text{ meter/sec}^2 \Leftrightarrow s_1 = \frac{1}{4}\epsilon_{kg}^{meter}\text{ meter}.$ If a is the acceleration of the tied collection of bodies, then we can write $5\text{N} = (s_1 + s_2)a = \left(\frac{1}{2} + \frac{1}{4}\right)\text{kg}.a$ $\Leftrightarrow \quad a = \frac{20}{3}\text{meter/sec}^2.$

In view of this, I may write that it now becomes just a matter of further effort to understand how I can write down the known "laws" of standard physics as only approximate truths from NSP.

F Further motivations to write $s_i = Gm_i/c^2$

Here are some reasons that I find compelling to motivate the relation $s_i = Gm_i/c^2$.

F.1 Gravitational field, potential, back reaction of test mass

Let me focus on how we define gravitational field g, due to a source mass m_2, by considering Newton's law of gravitation, an experimentally verified hypothesis, as the premise. The reasoning broadly consists of the following steps.

1. **Step 1:** According to Newton's law of gravitation, the gravitational force of interaction between a source mass m_2 and a test mass m_1 is given by

$$F = \frac{Gm_1 m_2}{r^2}. \tag{39}$$

   We write the acceleration of the test mass m_1 as

$$\frac{F}{m_1} = \frac{Gm_2}{r^2}. \tag{40}$$

Here, I have disregarded the distinction between inertial mass and gravitational mass.

2. **Step 2:** We argue that the gravitational back reaction of the test mass m_1 must be negligible so that the acceleration F/m_1 can be treated as the gravitational field due to the source mass m_2, symbolized as "g". Thus, g is defined as follows:

$$g := \lim_{m_1 \to 0} \frac{F}{m_1} = \frac{GM}{r^2}. \tag{41}$$

That is, the condition "$\lim_{m_1 \to 0}$" is considered as the computational step corresponding to the argument concerning the gravitational back reaction of the test mass m.

3. **Step 3:** Then, we generally introduce the gravitational potential $\Phi = Gm_2/r$ by writing

$$g = -\frac{d\Phi}{dr} \quad : \quad \frac{d\Phi(r)}{dr} := \lim_{h \to 0} \frac{\Phi(r+h) - \Phi(r)}{h}, \tag{42}$$

where h is some length that represents an infinitely small change of r. Validity of this definition is verified by the following steps of calculation.

$$\begin{aligned}
g &= -\frac{d\Phi}{dr} = -\lim_{h \to 0} \frac{\Phi(r+h) - \Phi(r)}{h} \\
&= -GM \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{(r+h)} - \frac{1}{r} \right] \qquad [\text{using } \Phi = GM/r] \\
&= GM \lim_{h \to 0} \frac{1}{h} \left[\frac{h}{r^2 + rh} \right] \\
&= GM \lim_{h \to 0} \left[\frac{1}{r^2 + rh} \right] \qquad [h \text{ cancels out as } h \neq 0] \\
&= GM \left[\frac{1}{r^2 + r.0} \right] \qquad [h = 0 \text{ is used to calculate the limit}] \\
&= \frac{GM}{r^2}.
\end{aligned}$$

Ignoring the objections regarding statements like "$m_1 \to 0$", "$h \to 0$" that I have discussed in the very beginning of this article and in refs.[4, 41], I make the following observations.

In Step 1, F/m_1 is equal to Gm_2/r^2. Therefore, "$\lim_{m_1 \to 0}$" should apply to both F/m_1 and Gm_2/r^2 in Step 2. However, it does not make sense to write "$\lim_{m_1 \to 0} Gm_2/r^2$" because Gm_2/r^2 contains no m_1-dependent term. In case we would have had an expression like

$$\frac{F}{m_1} = \frac{Gm_2}{r^2} + m_1\text{-dependent terms that vanish as } m_1 \to 0,$$

then only we could have written

$$\lim_{m_1 \to 0} \frac{F}{m_1} = \lim_{m_1 \to 0} \left[\frac{Gm_2}{r^2} + m_1\text{-dependent terms that vanish as } m_1 \to 0 \right] = \frac{Gm_2}{r^2}. \tag{43}$$

Considering this, together with the arguments that has been discussed in section(2.2), it is suggestive of the fact that the hypothesis should be of such a form that the following is true:

$$\frac{F}{m_1} = \frac{Gm_2}{r^2} + \text{sub-leading terms, depending on } m_1 \text{ and } c\text{, such that they vanish as } m_1 \to 0.$$

So, there are clear indications of missing terms in the hypothesis concerning gravitational two body interaction. The question remains how to look for those terms and here goes the clue.

Using expressions (41) and (42), I may write

$$\lim_{m_1\to 0} \frac{F}{m_1} = -\lim_{h\to 0} \frac{\Phi(r+h) - \Phi(r)}{h} \quad : F = Gm_2m_1/r^2, \Phi = Gm_2/r. \tag{44}$$

It does not make sense that the limiting condition on the left hand side concerns mass, and on the right hand side, concerns length. However, it can be given sense if mass is related to length by some means. A possibility is to write $s_i = Gm_i/c^2 \ \forall i \in [1, 2]$. So, $s_1 = Gm_1/c^2$. It is evident that $s_1 \to 0$ as $m_1 \to 0$ and $c \to \infty$. This provides the ground for speculation that s_1 may be playing the role of h and is involved in the sub-leading terms of (43) in positive powers.

F.2 Some specific questions regarding Schwarzschild metric

In ref.[11] Riemann did not specify that the line element, the displacement, etc. are infinitesimal or infinitely small with respect to which length (see section (D.2)). Such line of inquiry indeed justifies the consideration of the relation $s_i = Gm_i/c^2$ in the following way. One of the most widely used "infinitesimal line element" in the literature of general relativity, as also pointed out in the very beginning of ref.[26], is the Schwarzschild metric[27]:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

So, I inquire, ds is infinitesimal or infinitesimally small with respect to which of the following: (i) $s_M = GM/c^2$ where M is the source mass (ii) r, the coordinate distance (iii) the unmentioned unit of length (iv) $s_m = Gm/c^2$ where m is the test mass that plays no role otherwise.

References

[1] R. Penrose, *Gravitational Collapse and Space-Time Singularities*, Phys. Rev. Lett. 14, 57 (1965), https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.14.57. 3, 16, 23

[2] S. W. Hawking, R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. Lond. A. 314, 529-548 (1970), https://royalsocietypublishing.org/doi/10.1098/rspa.1970.0021. 3, 16, 23

[3] Quantum Gravity, *Stanford Encyclopedia of Philosophy*, https://plato.stanford.edu/entries/quantum-gravity/. 3, 24

[4] A. Majhi, *Cauchy's logico-linguistic slip, the Heisenberg uncertainty principle and a semantic dilemma concerning "quantum gravity"*, Int. J. Theor. Phys.61, 55 (2022); https://doi.org/10.1007/s10773-022-05051-8. 3, 5, 21, 23, 34

[5] S. Weinberg, *Chapter 16: Ultraviolet divergences in quantum theories of gravitation* in *General Relativity - An Einstein centenary survey* edited by S. W. Hawking and W. Israel, Cambridge University Press (1979). 3

[6] I. Newton, *Principia, Volume I - The Motion of Bodies*, University of California Press (1966); English translation by A. Motte. 3, 4, 6, 16

[7] I. Newton, *Principia, Volume II - The System of the World*, University of California Press (1966); English translation by A. Motte. 3, 4, 6, 24

[8] Euclid, J. L. Heiberg, R. Fitzpatrick - Euclid's Elements of Geometry (2008). 3

[9] D. Hilbert, *The Foundations of Geometry*, The Open Court Publishing Company - Reprinted Edition (1950); Project Gutenberg eBook: https://www.gutenberg.org/files/17384/17384-pdf.pdf. 3

[10] N. Lobachevski, *Geometrical researches on the theory of parallels*, Open Court Publishing Co. (1914). 3

[11] B. Riemann, *The hypotheses on which geometry is based*, pp 257-272 in *Bernhard Riemann Collected Papers*, Kendrick Press (2004). 3, 28, 29, 30, 35

[12] A. Einstein, *Annalen der Physik*, 17, 891-921 (1905). 3, 4, 6

[13] A. Einstein, B. Podolsky, N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. 47, 777 (1935).

[14] P. W. Bridgman, *The Logic of Modern Physics*, The Macmillan Company (1960). 3, 4, 5

[15] E. Mach, *The Science Of Mechanics - A Critical and Historical Account of its Development*, 4th Edition (2007). 3, 4

[16] A. Einstein, *The Collected Papers of Albert Einstein, Volume 6, The Berlin Years: Writings, 1914–1917, English Translation of Selected Texts*, Princeton University Press (1997). 3, 7, 21

[17] P. W. Bridgman, *Dimensional analysis*, Yale University Press (1963). 4, 8

[18] H. Goldstein, C. P. Poole, J. L. Safko, *Classical Mechanics*, Addison Wesley (2001). 6

[19] R. Descartes, D. E. Smith, M. L. Latham, *The Geometry of Rene Descartes*, Dover Publications, Inc. (1954). 10, 14

[20] A. Einstein, *Relativity: The Special and the General Theory*, Forgotten Books (2010). 5, 6, 8, 13, 15, 23, 24, 29

[21] R. H. Sanders, *The Dark Matter Problem: A Historical Perspective* (2010). 21, 24

[22] A. Robinson, *Non-standard analysis*, North Holland Publishing Company, Amsterdam (1966). 31

[23] H. J. Keisler, *Elementary Calculus: An Infinitesimal Approach, Third Edition*, Dover Publications (2012). 31

[24] H. J. Keisler, *Foundations of Infinitesimal Calculus* (2011), online link. 31

[25] A. A. Fraenkel, Y. Bar Hillel, A. Levy, *Foundations of Set Theory - Studies in Logic and The Foundations of Mathematics, Volume 67*, Elsevier (1973). 17

[26] The Event Horizon Telescope Collaboration, First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole, *The Astrophysical Journal Letters, 875:L1 (17pp)* (2019), https://arxiv.org/abs/1906.11238. 23, 35

[27] K. Schwarzschild, *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, 7: 189-196 (1916). 35

[28] C. M. Will, Living Rev. Relativity 17 (2014), 4, https://arxiv.org/abs/1403.7377. 7, 17

[29] A. Einstein, *Physics and Reality*, Journal of the Franklin Institute Volume 221, Issue 3, March 1936, Pages 349-382, https://doi.org/10.1016/S0016-0032(36)91047-5. 16

[30] H. Poincare, *Science and Hypothesis*, The Walter Scott Publishing Co. Ltd., New York (1905). 4

[31] E. Regis, *Who got Einstein's office?*, Basic Books (1988). 16

[32] E. V. Huntington, *Georg Cantor - The continuum, and other types of serial order - with an introduction to Cantor's transfinite numbers*, Dover Publications (2003). 16

[33] K. Goedel, *The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis With the Axioms of Set Theory*, Princeton University Press (1940). 16

[34] M. Born, *Physics in My Generation*, Springer New York (1968). 14, 27

[35] F. Klein, *Elementary Mathematics from an Advanced Standpoint, Geometry*, Dover Publications (2004). 27

[36] F. Klein, *The Evanston colloquium: Lectures on mathematics delivered from Aug. 28 to Sept. 9, 1893 before members of the Congress of Mathematics held in connection with the world's fair in Chicago at Northwestern University, Evanston, Ill.* 12, 28

[37] N. Gisin, *Mathematical languages shape our understanding of time in physics*, Nat. Phys. 16, 114–116 (2020), http://doi.org/10.1038/s41567-019-0748-5. 5, 14

[38] N. Gisin, Synthese 199, 13345-13371 (2021); https://philpapers.org/archive/GISIIP-2.pdf. 5, 14

[39] L. E. J. Brouwer, *Consciousness, philosophy, and mathematics*, Proceedings of the Tenth International Congress of Philosophy (Amsterdam, August 11-18, 1948), North-Holland Publishing Company, Amsterdam1949, pp. 1235–1249. 5, 16

[40] A. Majhi, *A Logico-Linguistic Inquiry into the Foundations of Physics: Part 1*, Axiomathes (2021), https://doi.org/10.1007/s10516-021-09593-0. 3, 4, 5, 12, 13, 16, 17, 23, 28, 29

[41] A. Majhi, *Logic, Philosophy and Physics: A Critical Commentary on the Dilemma of Categories*, Axiomathes (2021), https://doi.org/10.1007/s10516-021-09588-x. 4, 5, 13, 28, 34

[42] W. B. Ewald, *From Kant to Hilbert: A source book in the foundations of mathematics, Vol. 1 & 2*, Oxford University Press, USA (2007). 28

[43] Press release: The Nobel Prize in Physics 2020, https://www.nobelprize.org/prizes/physics/2020/press-release/. 23

[44] A. Kanamori, *Cantor and Continuity* (2019), https://www.semanticscholar.org/paper/Cantor-and-Continuity-Kanamori/a6adc590d37f6772fac2db9bdab73e80bbedd70b. 13, 16

[45] H. Cavendish, *Philosophical Transactions of the Royal Society of London*, Vol. 88 (1798), pp. 469-526, https://royalsocietypublishing.org/doi/10.1098/rstl.1798.0022. 6

[46] C. V. Boys, *The Newtonian Constant of Gravitation*, Nature, vol. 50, 330 - 334 (1894); https://doi.org/10.1038/050571a0. 6

[47] T. Quinn, C. Speake, H. Parks and R. Davis, *The BIPM measurements of the Newtonian constant of gravitation, G*, Phil. Trans. R. Soc. A372: 20140032, https://doi.org/10.1098/rsta.2014.0032. 6

[48] Qing Li et al., *Nature*, vol. 560, 582-588 (2018); https://www.nature.com/articles/s41586-018-0431-5. 6

[49] T. J. Quinn, C. C. Speake, S. J. Richman, R. S. Davis, and A. Picard, *A New Determination of G Using Two Methods*, Phys. Rev. Lett. 87, 111101 (2001); https://doi.org/10.1103/PhysRevLett.87.111101. 6

[50] P. J. Mohr, D. B. Newell and B. N. Taylor, *CODATA recommended values of the fundamental physical constants: 2014*, Rev. Mod. Phys. 88, 035009 (2016), https://arxiv.org/pdf/1507.07956v1.pdf. 7

[51] G. Frege, *The Foundations of Arithmetic*, Northwestern University Press (1980), translated into English by J. L. Austin. 18

[52] R. v. Eötvös, D. Pekár, and E. Fekete, Ann. Phys. (Leipzig) 68 (1922) 11. 21

[53] E. D. Casola, S. Liberati, S. Sonego, *Nonequivalence of equivalence principles*, Am. J. Phys. 83, 39 (2015), https://arxiv.org/pdf/1310.7426.pdf 7, 21

[54] E. D. Casola, S. Liberati, S. Sonego, Physical Review D 89, 084053 (2014), https://arxiv.org/pdf/1401.0030.pdf. 7, 21

[55] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman and Company (1973). 7

[56] S. Fray et al, *Atomic Interferometer with Amplitude Gratings of Light and Its Applications to Atom Based Tests of the Equivalence Principle*, Phys. Rev. Lett. 93, 240404 (2004), https://doi.org/10.1103/PhysRevLett.93.240404 21

[57] A. Bonnin et al, *Simultaneous Dual-Species Matter-Wave Accelerometer*, Phys. Rev. A 88, 043615 (2013), https://doi.org/10.1103/PhysRevA.88.043615. 21

[58] D. Schlippert et al,*Quantum Test of the Universality of Free Fall*, Phys. Rev. Lett. 112, 203002 (2014), https://doi.org/10.1103/PhysRevLett.112.203002. 21

[59] P. Asenbaum, C. Overstreet, M. Kim, J. Curti, M. A. Kasevich, Phys. Rev. Lett. 125, 191101 (2020), https://arxiv.org/pdf/2005.11624.pdf. 21

[60] L. Zhou et al., Phys. Rev. Lett. 115, 013004 (2015), https://arxiv.org/pdf/1503.00401.pdf. 21