On The Simple Identity $(1/(x-1)) + (1/(x-2)) = (2x-3)/((x-1)(x-2))$ and The Expression That $g(z,a) + \log|z-a|$ Is Harmonic Around $z = a$ From The Viewpoint of The Division By Zero Calculus

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Abstract: In this note, we will refer to the simple identity $(1/(x-1)) + (1/(x-2)) = (2x-3)/((x-1)(x-2))$ and the expression that $g(z,a) + \log|z-a|$ is harmonic around $z = a$ from the viewpoint of the division by zero calculus that are very popular expressions in elementary mathematics.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

Key Words: Division by zero, division by zero calculus, identity, Green function, log function, singular point.

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1 Introduction

In this note, we will refer to the simple identity $(1/(x-1)) + (1/(x-2)) = (2x-3)/((x-1)(x-2))$ and the expression that $g(z,a) + \log|z-a|$ is harmonic around $z = a$ from the viewpoint of the division by zero calculus that are very popular expressions in elementary mathematics. With these simple and very popular expressions, we would like to show clearly the importance of the division by zero calculus for some general people in a self-contained way.

2 Essences of division by zero and division by zero calculus

We will state very elementary facts and so, in order to state the contents in a self contained way, we state first the essences of division by zero and division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{2.1}$$

we will **define**

$$f(a) = C_0. \tag{2.2}$$

For the correspondence (2.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (2.1), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

However, we can consider the general definition of the division by zero calculus.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{2.3}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense as in stated in [1, 2, 3, 4].

In particular, note that for $a > 0$

$$\left[\frac{a^n}{n}\right]_{n=0} = \log a.$$

This will mean that the concept of division by zero calculus is important.

Note that

$$(x^n)' = nx^{n-1}$$

and so

$$\left(\frac{x^n}{n}\right)' = x^{n-1}.$$

Here, we obtain the right result for $n = 0$

$$(\log x)' = \frac{1}{x}$$

by the division by zero calculus.

3 Statement of results

We recall the simple identity

$$\frac{1}{x-1} + \frac{1}{x-2} = \frac{2x-3}{(x-1)(x-2)}.$$

In our usual (popular) mathematics, we can not consider the identity for the singular points $x = 1$ and $x = 2$. By our new concept of the division by zero calculus, we can now consider the functions even at the singular points.

For example, for the point $x = 1$

$$\frac{1}{x-1} + \frac{1}{x-2} \longrightarrow \frac{1}{1-1} + \frac{1}{1-2} = \frac{1}{0} + \frac{1}{-1} = 0 + (-1) = -1.$$

Meanwhile, the left hand side is also -1 at the point $x = 1$ by the division by zero calculus. Therefore, the identity holds for every x; that is IDENTITY in the real sense. No exceptional.

Next, when we define the Green function, we use the expression that $g(z, a) + \log |z - a|$ is harmonic around $z = a$. However, the function $\log |z - a|$ has not any meaning at $z = a$. However, we are considering the sum at even the point $z = a$; this statement will be strange. We are referring the property at the singular point $z = a$.

We added something we shouldn't think about, and then it became meaningful where we shouldn't think about it.

Isn't there something wrong? Modern mathematics.

When we added something that didn't make sense, it became meaningful. However, how can mathematics add meaningless things in the first place?

In our viewpoint, the function $\log |z - a|$ takes zero at the singular point $z = a$ and so the usual representation is now no problem.

We can consider a similar problem:

We can consider the function

$$f(z, a) = z - a$$

for all complex numbers. However, we can not consider the function

$$F(z, a) := f(z, a)^{-1} := \frac{1}{z - a}$$

for $z = a$. Conversely, for the function $F(z, a)$ that is not defined at $z = a$, we have

$$f(z, a) := F(z, a)^{-1} := z - a$$

that has the meaning at $z = a$.

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