ON THE PROBLEM OF AXIOMATIZATION OF GEOMETRY

Temur Z. Kalanov
Home of Physical Problems, Yozuvchilar (Pisatelskaya) 6a,
100128 Tashkent, Uzbekistan
tzk_uz@yahoo.com, t.z.kalanov@mail.ru, t.z.kalanov@rambler.ru

Abstract. An analysis of the foundations of geometry within the framework of the correct methodological basis – the unity of formal logic and rational dialectics – is proposed. The analysis leads to the following result: (1) geometry is an engineering science, but not a field of mathematics; (2) the essence of geometry is the construction of material figures (systems) and study of their properties; (3) the starting point of geometry is the following system principle: the properties of material figures (systems) determine the properties of the elements of figures; the properties of elements characterize the properties of figures (systems); (4) the axiomatization of geometry is a way of construction of the science as a set (system) of practical principles. Sets (systems) of practice principles can be complete or incomplete; (5) the book, "The Foundations of Geometry" by David Hilbert, represents a methodologically incorrect work. It does not satisfy the dialectical principle of cognition, "practice $\rightarrow$ theory $\rightarrow$ practice," because practice is not the starting point and final point in Hilbert's theoretical approach (analysis). Hilbert did not understand that: (a) scientific intuition must be based on practical experience; intuition that is not based on practical experience is fantasy; (b) the correct science does not exist without definitions of concepts; the definitions of geometric concepts are the genetic (technological) definitions that shows how given material objects arise (i.e., how a person creates given material objects); (c) the theory must be constructed within the framework of the correct methodological basis: the unity of formal logic and rational dialectics. (d) the theory must satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. Therefore, Hilbert cannot prove the theorem of trisection of angle and the theorem of sum of interior angles (concluded angles) of triangle on the basis of his axioms. This fact signifies that Hilbert's system of axioms is incomplete. In essence, Hilbert's work is a superficial, tautological and logically incorrect verbal description of Figures 1-52 in his work.

Keywords: geometry, engineering, applied mathematics, general mathematics, methodology of mathematics, philosophy of mathematics, history of mathematics, higher education, formal logic, dialectics, epistemology.

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Introduction

As is known, science and technology are developed in an inductive way. This means that new scientific knowledge is not a consequence of old scientific knowledge. New knowledge is a guess (discovery). There is an epistemological principle (for example, in mathematics and physics), which states that the relation between old scientific knowledge and new scientific knowledge must satisfy the following condition: old knowledge must be a consequence of new knowledge. This condition would be an expression of the dialectical law of "negation of negation" if this condition had the following logical formulation: old correct knowledge should be a consequence of new correct knowledge. Correct knowledge is achieved within the framework of correct methodological basis and, therefore, satisfies the correct criterion of truth.

As is shown in my works (for example, [1-53]), the foundations of theoretical physics and mathematics do not satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. The unity of formal logic and rational dialectics is the correct methodological basis of science. (In other words, the concepts “correct methodological basis” and “correct criterion of truth” are identical concepts). But scientists, as the analysis of the literature shows, ignore the correct methodological basis. The purpose of this work is to propose the arguments that show scientifically impossibility of complete axiomatization of geometry within the framework of correct methodological basis: the unity of formal logic and rational dialectics.

1. The essence of geometry

1) As is known, all material objects are bounded objects [54]. Bounded material objects have set of properties: for example, physical, chemical, geometric properties. The theoretical study of the properties of bounded material objects is carried out using applied mathematics. Applied mathematics operates with quantities that represent the measure: the unity of the qualitative and quantitative determinacy of a material object. Therefore, the quantities in applied mathematics have dimensions. Unlike applied mathematics, pure mathematics operates with mathematical quantities that have only quantitative determinacy. The quantities in pure mathematics do not have dimensions: the values of mathematical quantity in pure mathematics are unnamed (abstract) numbers. Applied mathematics and pure mathematics are the science of operations with numbers.

2) Geometry studies the geometric properties of rigid (solid) and flexible bounded material bodies. As is known, unbounded material bodies do not exist in reality. The geometric property of rigid (solid) and flexible bounded material bodies is manifested in the extension of material objects. The measurable (measurand) length is characterized by the dimension “meter”. The dimension “meter” denotes the unity of qualitative and quantitative determinacy. As is known, a bounded material body (or part of a body) has only three extensions (three dimensions): length, width, and height. The relationship between the length, width and height of the body is called the form of the body (or the form of the part of the body). In the point of view of geometry, a form is the only essential property of a material body (or part of body). In this case, the bounded material body is called a geometric body, i.e., a geometric figure. A given geometric body (figure) can be decomposed into its component parts (fragments). Then the component parts (fragments) can be connected, combined, forming the original body. This means that the study of the form is carried out by means of geometric (material) constructions and decompositions (dissections) using applied mathematics (i.e., using real numbers) [54]. But the construction and decomposition of a form are not mathematical operations.

3) The description of the form of a bounded material body is carried out within the framework of geometric models of rigid (solid) and flexible bodies. A geometric model of a bounded material body is a system (figure) consisting of the following material elements: points, bounded lines and bounded surfaces, which are elementary models of bounded material bodies (or parts of bodies). The concepts of a point, a bounded line and a bounded surface are as follows. A point is an elementary geometric model of a bounded body (or part of body), the three dimensions of which can be neglected (for example: material point in physics, material point drawn with a pencil or paint, the mark, rivet, the notch, the weld, welded joint, hinge, bolt, universal joint). A bounded line is an elementary geometric model of a bounded body (or part of body), two dimensions of which can be neglected (for example: material line drawn with a pencil, chalk or compasses, bar, reinforcement bar, cord, cable, string). A bounded surface is an elementary geometric model of a bounded body (or part of body), one size of which can be neglected (for example: paper, carpet, sheet, board, metal plate). Each elementary geometric model is the unity of qualitative and quantitative determinacy. Thus, the difference between the geometric models of bodies is established by comparing the qualitative and quantitative determinacy of these models. Measurement results are expressed in numbers. Therefore,

comparison of sizes (i.e., numbers) is a mathematical (quantitative) operation, the result of which is indicated by the following symbols: “$=, \approx, \neq$”, “$>, \geq, >>$”, “$<, \leq, <<$”.

4) A point, a bounded line and a bounded surface are mutually independent (free) material elements [54]. The independent existences of elements – a material point p, a bounded straight line a, and a bounded plane α – are defined graphically as follows (Figure 1).

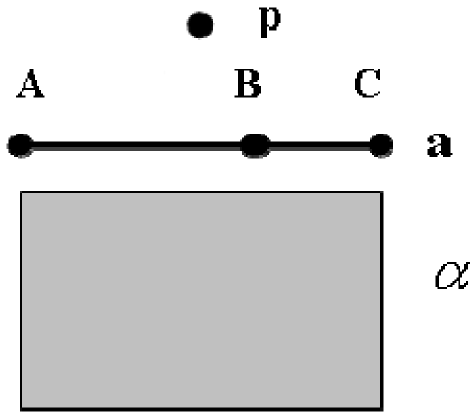

Figure 1. Graphic (material) definition of the following independent geometric elements: a point p, a bounded straight line a and a bounded plane α.

This means that the existence, position and properties (as the unity of qualitative and quantitative determinacy) of any element do not depend on the existence, position and properties of other elements. In particular, mutual movements (changes) of free elements are independent.

The independence of the existence of elements – a material point p, a bounded straight line a and a bounded plane α – means that each element is not genetically determined by other elements. For example, a bounded straight line a can be made (drawn on paper or board) with a pencil or a piece of chalk. But points (on paper or blackboard) or pieces of chalk do not genetically determine (define) a line. Similarly, a bounded straight line a and a point p not lying on a do not determine (define) genetically bounded plane α.

5) Connection between the elements puts restraints (limitations) on mutual positions and movements of the elements (Figure 2).

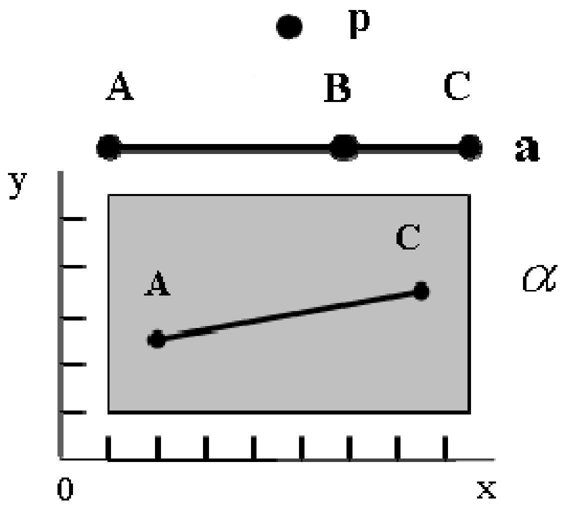

Figure 2. Graphical (material) definition of the following geometric elements: a point p, a bounded straight line a, a bounded plane α. The XOY coordinate system is connected with the plane α. The XOY coordinate system defines the positions of the plane α and the segment of the straight line a lying on the plane.

For example, if any material point is fixed (i.e., if it exists) on the material plane, then a bounded material straight line can be fixed on the plane by this point [54]. If this point does not coincide with the end point of the bounded straight line, then the bounded straight line exists on the plane. “Plane + bounded straight line” is a system. The bounded straight line in the system “plane + bounded straight line” can have one degree of freedom of movement: rotation around the point of fixation. But if the bounded straight line is rigidly connected with the plane by two points (for example, nails), then the bounded straight line have no degree of freedom of movement (Figure 2).

6) All free material points are identical. If a material point p is not fixed on a bounded straight line a or on a bounded plane α, then the point p does not exist (does not belong) on the bounded straight line a or on the bounded plane α. In this case, the material point p have no a name (designation), because a free point does not designate (does not characterize) a place. Place exists only on a bounded straight line and on a bounded surface. If a material point p is fixed (belongs) on a bounded straight line a or on a bounded plane α, then the point p has a name (designation), because this point names (designates, characterizes) a certain place on the bounded straight line a or on the bounded plane α. Letters A, B, C denote (name, mark) identical points p situated (fixed) in different (distinct) places on the bounded straight line a or on the bounded plane α. Therefore, these points p have different names (designations), which characterize the different positions (places) of the points p. But, for example, the relationship $A = B$, expressing the identity of different (distinct) places, is a formal-logical error: the relationship $A = B$ contradicts to the formal-logical law of the lack of contradiction, $A \neq B$.

Points A and C belong to the bounded straight line a and are called endpoints of the bounded straight line a (Figures 1 and 2). The positions (places) of points on a bounded straight line are determined using a ruler or compass. A ruler or compass determines (measures) the distance between endpoints A and C as well as the distance of point B from the endpoints of the line a.

7) Point B is between points A and C (Figures 1 and 2) if the following mathematical (quantitative) relationships are satisfied:

$$d_{AB}^{(a)} < d_{AC}^{(a)}, \quad d_{BC}^{(a)} < d_{AC}^{(a)}, \quad d_{AB}^{(a)} + d_{BC}^{(a)} = d_{AC}^{(a)}$$

where d is the length (distance); $d_{AB}^{(a)}$, $d_{BC}^{(a)}$ and $d_{AC}^{(a)}$ are the lengths of line segments $\overline{AB}$, $\overline{BC}$ and line $\overline{AC}$, respectively.

If the segments $\overline{AB}$ and $\overline{BC}$ form the angle $\varphi = \angle\left(\overline{BA}, \overline{BC}\right)$, $0^\circ < \varphi < 180^\circ$, then the bounded straight line a is called a broken line.

If there exist (fasten, fix), for example, points K, L, M on a bounded straight line a, then the order of arrangement (situation, disposition) of these points is determined by the following mathematical (quantitative) relationships:

$$d_{AK}^{(a)} < d_{AL}^{(a)} < d_{AM}^{(a)}$$

where $d_{AK}^{(a)}$, $d_{AL}^{(a)}$ and $d_{AM}^{(a)}$ are the lengths of lines $\overline{AK}$, $\overline{AL}$, and $\overline{AM}$, respectively.

If there exist two independent (free) bounded straight lines a and b, then these lines can be connected using a material point p. In this case, the material point denotes the place of connection (intersection) of the lines and has the name P. Point P is the vertex of the angles formed by the lines a and b.

If bounded straight lines a and b are fixed by points on the bounded material plane α, then there are two following mutual positions of the lines a and b: the position of parallelism of the lines, and the position of non-parallelism of the lines [54]. Lines a and b are called parallel if these lines are equidistant lines, i.e., if the distance $d_{a,b}$ between the lines a and b is constant at any points of these lines: $d_{a,b} = const$. If the distance $d_{a,b}$ between the lines a and b is not constant at any point of these lines, then the lines are not parallel. This definition of parallel lines is a precise (exact, accurate, rigorous, strict) and correct definition based on the use of surface gage. (In other words, the genetic definition of the parallelism of lines is the following: a line b is called a parallel to line a if the line b at any point is the equidistant line generated by surface gage).

Also, an example of equidistant curved lines is concentric circles.

If the bounded straight line a is fixed on the bounded plane α by points A and C, then the position of the line a on the bounded plane α is determined by the positions of points A and C (Figure 2). And the positions of points A and C are determined by the system of rulers (i.e., the XOY coordinate system) connected to the bounded plane α. But the bounded plane α and the position of the bounded plane α are not defined by a line a and a point not lying on the line a.

8) The connection (relation) between elements can characterize the essential properties (features) of elements.

(a) The property of a bounded straight line can be expressed as follows: any point of a bounded straight line is equidistant from the two fixed poles (explanation: fixed pole points do not belong to this bounded straight line). This property is an essential feature of a bounded straight line. Therefore, the correct definition of a bounded straight line is the following: a bounded line is called a bounded straight line if any point of the bounded line is equidistant from the two fixed poles that do not belong to this bounded line.

(b) The correct definition of a bounded plane is the following: a bounded surface is called a bounded plane if this bounded surface has the following essential property (essential feature): any bounded straight line lies (are) on this bounded surface (i.e., any point of any bounded straight line is on this bounded surface) (Figure 2).

(c) The correct definition of the three-dimensional coordinate system $XOYZ$ is the following: a system of three bounded planes is called a three-dimensional coordinate system $XOYZ$ if this system has the following constructive property (essential feature): the system consists of three intersecting material right-angled planes XOY, XOZ, and YOZ with metric rulers; O is the point of intersection of the planes XOY, XOZ, and YOZ; lines of intersections of the planes are intersecting bounded straight lines; the angles between intersecting straight lines are 90°. The essential (measuring, informational) property of the coordinate system $XOYZ$ is the following: if a material object is in the coordinate system $XOYZ$, then the set of positions (coordinates) of the material object in the coordinate system $XOYZ$ determines (and is called) the geometric space of this material object in the material rectangular (Cartesian) coordinate system $XOYZ$.

9) The concepts of a point, a bounded straight line, a segment of a bounded straight line and a bounded surface are geometric concepts. The definitions of geometric concepts are a genetic (technological) definitions that show how a given material objects arise (i.e., how a person creates given material objects). In particular, a point, a bounded straight line, a segment of a bounded straight line and a bounded surface drawn on paper are a material manifestation (expression) of the idea of a point, a bounded straight line, a segment of a bounded straight line, and a bounded surface. In other words, the drawn geometric objects are material objects that exist as a materialization (material manifestation) of the idea.

For example, a drawing (ornament) on a carpet is the materialization of an idea that exists in the designer's head. Complicated composition can only be created with the help of templates, because the compasses and the ruler are inadequate tools. Consequently, definitions of geometric concepts are not mathematical (quantitative, numerical) definitions. (It must be emphasized that the process of scientific thinking (abstract thinking, logical thinking) always relies on visual (material) images (i.e., on sensibly perceived material) and leads to the formulation (construction) of concepts, propositions, systems of propositions, and theories. Thinking (thought) that relies on perceivable material is called intuition).

10) Geometric elements can be concatenated (combined, connected) to each other forming a geometric system (points can be connecting elements). A geometric system can or cannot have degrees of freedom of structural movement (subject to the properties of the connecting elements [54]). (The structural movement of a given system is a change in the form of a given system without destroying this system [54]. This change carried out by man). The connection of geometric elements is manifested (reflected) in human thinking as a connection (system) of concepts.

The geometric model of a material body as a system composed (constructed) of material elements obeys to the system principle: the properties of the system determine the properties of the elements of the system; properties of elements of system characterize the properties of the system [54]. The system principle is a concretization of the laws of dialectics.

11) Definitions of geometric spaces.

(a) The set of positions (places) of a point p on a bounded straight line a is called the geometric space (geometric states) of a point p on a bounded line a.

(b) The set of positions (places) of a point p on a bounded plane α is called the geometric space (geometric states) of a point p on a bounded plane α.

(c) The set of positions (places) of a material point p in the bounded coordinate system $XOYZ$ is called the geometric space (geometric states) of a material point p in the bounded coordinate system $XOYZ$.

(d) The set of positions (places) of a bounded straight line a on a bounded plane α is called the geometric space (geometric states) of a bounded straight line a on a bounded plane α.

(e) The set of positions (places) of a bounded line a in a bounded coordinate system $XOYZ$ is called the geometric space (geometric states) of a bounded line a in a bounded coordinate system $XOYZ$.

(f) The set of positions (places) of a bounded plane α in a bounded coordinate system $XOYZ$ is called a geometric space (geometric states) of a bounded plane α in a bounded coordinate system $XOYZ$.

(g) Unbounded planes cannot intersect one another. Bounded planes can intersect one another and have either one common point (Figure 3) or set of common points (Figure 4):

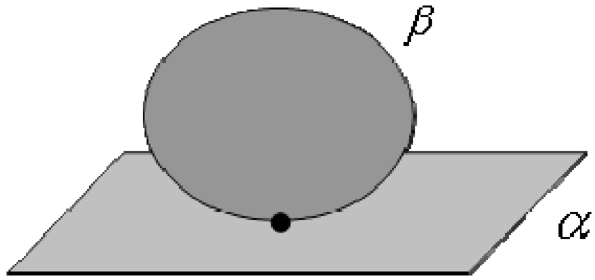

Figure 3. Intersection of bounded planes α and β at one point.

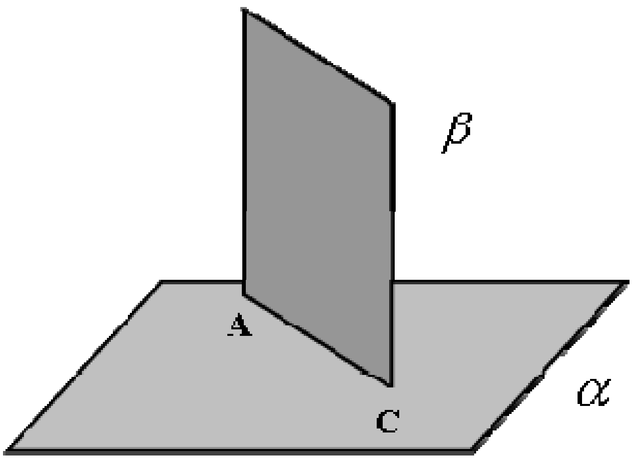

Figure 4. Intersection of bounded planes α and β in line AC containing set of points.

In this case, the set of positions of intersecting planes α and β in the bounded $XOYZ$ coordinate system is called the geometric space (geometric states) of intersecting planes α and β in the bounded $XOYZ$ coordinate system.

2. The essence of the axioms of geometry

1) The definition of the axiom is as follows. “An axiom is: (a) a scientific assertion (statement) that is accepted without logical proof; (b) an obvious, convincing and true starting point of the theory” (Russian Wikipedia). In other words, an axiom is a verbal expression of an empirical fact. In the point of view of formal logic, an axiom is a proposition (a system of concepts) based on practical experience. Therefore, an axiom is a true proposition. The explanation of the axiom represents a theorem and a theory. Examples of axioms are the following: “All men are mortal. Socrates is a man. Hence, Socrates is mortal”, “Day is replaced by night (i.e., there is a cyclical change in day and night)”, “Summer is replaced by winter (i.e., there is a cyclical change in seasons)”.

A theorem is a statement that is based on logical proof. Usually, a theorem contains a condition and a conclusion.

2) In the point of view of formal logic, the axioms and theorems of geometry are systems of concepts. Definitions of geometric concepts are genetic (technological) definitions of concepts

and systems of concepts (i.e., genetic definitions of elements and of properties of a geometric figure). But the properties of the elements do not determine the properties of the figure. The properties of a given figure can be determined if and only if one constructs the given figure as a system of certain material elements. Determination of the list of these material elements consists, first of all, in the decomposition of a given figure into a set of concrete (specific) elements. (Set of concrete (specific) elements can be created using templates (gauges) and other complex devices (gadgets), because a compass and a ruler are inadequate tools in a general case). This set of concrete (specific) elements characterizes the given figure, but not an arbitrary figure. That is why a set of axioms that does not contain a genetic definition of a given figure is always an incomplete set.

In this point of view, the starting point of geometry should be chosen as follows: (a) the axiom of the existence of geometric elements and figures: geometric elements and figures exist if they can be genetically determined as material objects; (b) the axiom of the identity of geometric figures: two geometric figures are called identical (congruent) to each other if they are copies of each other.

3. Formal logic as the methodological basis for construction of axioms

1) By definition, formal logic is the science of the laws of correct thinking. The starting point and fundamental element of formal logic is a concept. A concept is a form of thought that expresses the essential features of objects and phenomena. A concept is expressed in a word or in several words (grammatical sentences). Concepts (thoughts) cannot be expressed without words and grammatical sentences. Concepts have material nature.

2) The basis of formal logic is a system of concepts. The connection of concepts forms the structure of the system. The connection of concepts is expressed by the following words: “is”, “is not”, “if… is…, then…”, “if… is not…, then…”, “consequently”.

3) Proposition as a logical form of verbal expression (utterance) of thought is the essence of formal logic. The definition of proposition is the following: proposition is a statement (i.e., the act of thinking and verbal expression of thought) about the existence or non-existence of an object or phenomenon; proposition is a statement about the properties of an object or phenomenon of reality; proposition is expressed in the statement of the existence or absence of certain features of objects and phenomena. A proposition connects concepts that logically express objects. There exist no true propositions that connect concepts without objects. Also, there are no true propositions that connect objects without concepts of objects (in this case, the connection between objects is not a logical connection!). Therefore, a proposition has the following two properties: (a) the property of assertion or negation; (b) the property of truth or false. This property is expressed in the following words: “truth” or “false”.

4) The connection (combination) of propositions which represents deriving (extracting) a new proposition from one or more propositions is called inference. The new proposition is called a conclusion (in Latin: conclusio). Those propositions from which a new proposition is derived (extracted, follows) are called premises (in Latin: praemissae). The relation between premises and conclusion is the relation between cause and effect. Inference is based on the law of sufficient reason.

5) Inferences are divided into the following two groups: direct inferences and mediated inferences. If a conclusion (proposition) is made from only one premise (proposition), then the inference is called direct inference. If a conclusion (proposition) is made from several premises (propositions), then the inference is called mediated inference.

4. The essence of David Hilbert’s axioms

In the conventional point of view, “the axioms are the foundation for a modern treatment of Euclidean geometry. Well-known modern axiomatizations of Euclidean geometry are those of

David Hilbert, of Alfred Tarski and of George Birkhoff ”(Wikipedia). The axioms of David Hilbert, of Alfred Tarski and of George Birkhoff do not contradict to one another and, therefore, are in effect identical [56-68]. The axioms are the followings: Incidence; Order; Congruence; Parallels; Continuity.

The essence of the axioms can be understood by the example of the work of David Hilbert [56]. “Hilbert’s axioms are a set of 20 assumptions proposed by David Hilbert in 1899 in his book “Grundlagen der Geometrie” (“The Foundations of Geometry”). Hilbert’s set of axioms is constructed with six primitive notions: three primitive terms: point; line; plane; - and three primitive relations:

(a) betweenness (a ternary relation linking points);

(b) lies on (three binary relations, one linking points and straight lines, one linking points and planes, and one linking straight lines and planes);

(c) congruence (two binary relations, one linking line segments and one linking angles).
Line segments, angles, and triangles may each be defined in terms of points and straight lines, using the relations of betweenness and containment. All points, straight lines, and planes in the axioms are distinct” (Wikipedia).

The essence of the work “The Foundations of Geometry” by David Hilbert is the following.
“Contents.

INTRODUCTION.

Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental principles are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space. The following investigation is a new attempt to choose for geometry a simple and complete set of independent axioms and to deduce from these the most important geometrical theorems in such a manner as to bring out as clearly as possible the significance of the different groups of axioms and the scope of the conclusions to be derived from the individual axioms” [56].

Objections to the introduction:

1) Intuition is not an object of logical analysis.

2) The term “fundamental principles” is not identical with the term “axioms”. In the point of view of formal logic, the concept “principle” (as the basic starting point of the theory) is broader than the concept “axiom”.

“THE FIVE GROUPS OF AXIOMS.

§1. THE ELEMENTS OF GEOMETRY AND THE FIVE GROUPS OF AXIOMS.

Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters A, B, C, …; those of the second, we will call straight lines and designate them by the letters a, b, c, …; and those of the third system, we will call planes and designate them by the Greek letters α, β, γ, … . The points are called the elements of linear geometry; the points and straight lines, the elements of plane geometry; and the points, lines, and planes, the elements of the geometry of space or the elements of space. We think of these points, straight lines, and planes, which we indicate by means of such words as "are situated", "between", "parallel", "congruent", "continuous", etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry. These axioms may be arranged in five groups. Each of these groups expresses, by itself, certain related fundamental facts of our intuition. We will name these groups as follows:
I, 1–7. Axioms of connection.
II, 1–5. Axioms of order.
III. Axiom of parallels (Euclid's axiom).
IV, 1–6. Axioms of congruence.
V. Axiom of continuity (Archimedes's axiom)" [56].

Objections to §1:
1) The concepts of point, straight line, plane, and space are not defined. In the point of view of formal logic, this is a gross mistake.
2) In the point of view of system approach (system analysis), groups of things (objects) are not systems of things (objects). In the point of view of formal logic, names and letter designations (denotations) of things (objects) are not definitions of things (objects) and concepts.
3) The set of free points is not a system. All free points are identical geometric objects and cannot be denoted by the letters $A, B, C, ...$. The sets of free straight lines and planes are not systems.
4) Points, straight lines, and planes cannot have any mutual relations which are indicated by means of such words as "are situated", "between", "parallel", "congruent", "continuous", etc. because the concepts of points, straight lines, and planes are undefined. Relations are facts of practice, but not intuition.

"§2. GROUP I: AXIOMS OF CONNECTION.

The axioms of this group establish a connection between the concepts indicated above; namely, points, straight lines, and planes. These axioms are as follows:
I, 1. Two distinct points A and B always completely determine a straight line a. We write $AB = a$ or $BA = a$. Instead of "determine," we may also employ other forms of expression; for example, we may say A "lies upon" a, A "is a point of" a, a, "goes through" A "and through" B, a "joins" A "and" or "with" B, etc. If A lies upon a and at the same time upon another straight line b, we make use also of the expression: "The straight lines" a "and" b "have the point A in common," etc.
I, 2. Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$.
I, 3. Three points A, B, C not situated in the same straight line always completely determine a plane α. We write $ABC = a$. We employ also the expressions: A, B, C "lie in" α; A, B, C "are points of" α, etc.
I, 4. Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane.
I, 5. If two points A, B of a straight line a lie in a plane α, then every point of a lies in α. In this case we say: "The straight line a lies in the plane α," etc.

I, 6. If two planes α, β have a point A in common, then they have at least a second point B in common.
I, 7. Upon every straight line there exist at least two points, in every plane at least three points not lying in the same straight line, and in space there exist at least four points not lying in a plane.
Axioms I, 1–2 contain statements concerning points and straight lines only; that is, concerning the elements of plane geometry. We will call them, therefore, the plane axioms of group I, in order to distinguish them from the axioms I, 3–7, which we will designate briefly as the space axioms of this group. Of the theorems which follow from the axioms I, 3–7, we shall mention only the following:
Theorem 1. Two straight lines of a plane have either one point or no point in common; two planes have no point in common or a straight line in common; a plane and a straight line not lying in it have no point or one point in common.
Theorem 2. Through a straight line and a point not lying in it, or through two distinct straight lines having a common point, one and only one plane may be made to pass" [56].

Objections to §2:
1) The concepts of point, straight line, plane, and space are not defined.
2) The term "determine" is not identical with the terms "lies upon", "goes through", etc. Therefore, the term "determine" cannot be used instead of the terms "lies upon", "goes through", etc.
3) The expressions $AB = a$ and $BA = a$ are not definitions of a straight line. These expressions are erroneous conventional notations: the left side denotes points; the right side denotes the line. In the point of view of formal logic, it is a blunder because these expressions contradict to the law of lack of contradiction: $AB \neq a$ and $BA \neq a$.
4) Points, straight lines, and planes are independent elements. A straight line exists independently of the existence of points. Therefore, points A and B do not define (do not determine) a straight line. Points A and B exist on the line and represent (designate) distinct places on the line.
5) Statements "Two distinct points A and B always completely determine a straight line a. We write "$AB = a$ or $BA = a$" and "Any two distinct points of a straight line completely determine that line; that is, if $AB = a$ and $AC = a$, where $B \neq C$, then is also $BC = a$" are tautology. Really, these statements have the following sense (meaning): if distinct points A, B, C lie upon (situated in) a straight line a, then the points A, B, C lie upon (situated in) the straight line a.
6) The statement "Any three points A, B, C of a plane α, which do not lie in the same straight line, completely determine that plane" is erroneous because: (a) a plane exists independently of the existence of points; (b) points lying in the plane do not define (do not determine) the position of the plane. In addition, this statement is the following tautology: Any three points A, B, C of a plane α, which do not lie in the same straight line, completely lie in (situated in) that plane.
7) As is shown in Figures 3 and 4, the statement "If two planes α, β have a point A in common, then they have at least a second point B in common" is erroneous. In addition, unbounded planes cannot intersect one another (each other).
8) The concept of space is not defined.

"§3. GROUP II: AXIOMS OF ORDER.

The axioms of this group define the idea expressed by the word "between," and make possible, upon the basis of this idea, an order of sequence of the points upon a straight line, in a plane, and in space. The points of a straight line have a certain relation to one another which the word "between" serves to describe. The axioms of this group are as follows:
II, 1. If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A.

Fig. 1.

II, 2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D. Fig. 2.
II, 3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.
II, 4. Any four points A, B, C, D of a straight line can always be so arranged that B shall lie between A and C and also between A and D, and, furthermore, that C shall lie between A and D and also between B and D.
Definition. We will call the system of two points A and B, lying upon a straight line, a segment and denote it by AB or BA. The points lying between A and B are called the points of the segment AB or the points lying within the segment AB. All other points of the straight line are referred to as the points lying outside the segment AB. The points A and B are called the extremities of the segment AB.
II, 5. Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC. Axioms II, 1–4 contain statements concerning the points of a straight line only, and, hence, we will call them the linear axioms of group II. Axiom II, 5 relates to the elements of plane geometry and, consequently, shall be called the plane axiom of group II.**”** [56].

Objections to §3:
1) The statement “II, 1. If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A” is a tautology because “A and C” is identical with “C and A”: $\overline{AC} = \overline{CA}$. The relationship $\overline{AC} = \overline{CA}$ means that points A and C can be rearranged (permuted).
2) The statement “Definition. We will call the system of two points A and B, lying upon a straight line, a segment and denote it by AB or BA” is erroneous because the system of two points A and B is not a segment.
3) All statements represent the perfunctory, superficial and fallacious verbal description of Figures 1, 2, 3.

“§4. CONSEQUENCES OF THE AXIOMS OF CONNECTION AND ORDER.

By the aid of the four linear axioms II, 1–4, we can easily deduce the following theorems” [56].

Objections to §4:
All statements represent the perfunctory, superficial and fallacious verbal description of Figures 4, 5, 6, 7.

“§5. GROUP III: AXIOM OF PARALLELS. (EUCLID’S AXIOM.)

The introduction of this axiom simplifies greatly the fundamental principles of geometry and facilitates in no small degree its development. This axiom may be expressed as follows:

III. In a plane α there can be drawn through any point A, lying outside of a straight line a, one and only one straight line which does not intersect the line a. This straight line is called the parallel to a passing through the given point A.

This statement of the axiom of parallels contains two assertions. The first of these is that, in the plane a, there is always a straight line passing through A which does not intersect the given line a. The second states that only one such line is possible. The latter of these statements is the essential one, and it may also be expressed as follows:

Theorem 8. If two straight lines a, b of a plane do not meet a third straight line c of the same plane, then they do not meet each other.

For, if a, b had a point A in common, there would then exist in the same plane with c two straight lines a and b each passing through the point A and not meeting the straight line c. This condition of affairs is, however, contradictory to the second assertion contained in the axiom of parallels as originally stated. Conversely, the second part of the axiom of parallels, in its original form, follows as a consequence of theorem 8.

The axiom of parallels is a plane axiom" [56].

Objections to §5.

In a practical point of view, this axiom is not precise (exact, accurate) and correct. In the point of view of formal logic, the exact and correct formulation is based on the use of surface gage. The genetic definition of parallel lines is as follows: a line b is called the parallel to a given line a if the line b passes through a point A not lying upon the line a and is the equidistant line generated by surface gage.

Further citation of Hilbert's work has no sense (meaning content) because Hilbert's work [56] represents the perfunctory, superficial and fallacious verbal description of Figures 1-52 in his work.

5. Discussion

1. Geometry is the science of the properties of material geometric systems (figures) constructed of material geometric elements [2, 55]. Geometry is based on the following system principle: the properties of the system determine the properties of the elements of the system; properties of the elements of the system characterize the properties of the system. Therefore, geometry is an engineering science [2, 54] which uses applied mathematics. For example, a car designer first materializes an idea in the form of a drawing: the designer draws the form of a car, then he decomposes this form into its component parts and draws the component parts of the form. The material-processing robots make parts and production-line robots connect the parts into one whole (system). This means that the whole (i.e., system, form) determines the properties of the parts (i.e., elements of form). Therefore, a correct geometric theory should be based on the dialectical principle of cognition: "practice $\to$ theory $\to$ practice".

2. The system of Hilbert's axioms is incomplete because within the framework of this system it is impossible to prove the theorem of the trisection of an angle and the theorem of the sum of the interior angles (concluded angles) of a triangle [57-68].

The proof of the angle trisection theorem is based on the following material operations [1]: (1) construction of a circle; (2) construction of a given central angle; (3) extraction the arc on which the central corner rests; (4) straightening of the extracted arc (i.e., converting of the arc to a straight line segment); (5) division of the straight line segment into three identical parts using a proportional compass (whole-and-half compasses); (6) designation of two marks (points) on the straight line segment; (7) transformation (bending) of the straight line segment with marks (points) into the initial arc; (8) insertion of the arc with marks (points) into the circle; (9) drawing two straight lines from the center of the circle through the marks (points) on the arc.

The proof of the theorem of the sum of the interior angles (concluded angles) of a triangle is based on the following statements [2, 40-44]: (1) the sides of a triangle are material straight line segments; (2) the vertices of the triangle represent material universal joints; (3) universal joints allow structural (internal) movement of the triangle; (4) the structural movement of a triangle is a change in the angles and lengths of the sides of the triangle; (5) the sum of the interior angles (concluded angles) of a triangle is equal to 180° in general case if some concluded angle is equal to 180° in special case

3. In a practical and logical points of view, a correct system of axioms is a set of practical techniques (methods, principles) for constructing a given geometric figure. Set of practical techniques (methods, principles) for constructing given geometric figures can be complete in some cases, but can be incomplete in other cases. This is explained by the fact that practice is the supporting and developing points in the inductive process of cognition. An inductive process – an unlimited process – obeys to the dialectical principle of cognition: “practice $\rightarrow$ theory $\rightarrow$ practice”.

4. The book, “Grundlagen der Geometrie” (“The Foundations of Geometry”). by David Hilbert, is a methodologically wrong work. It does not satisfy the dialectical principle of cognition, “practice $\rightarrow$ theory $\rightarrow$ practice,” because practice is not the starting point and final point of Hilbert’s theoretical approach (analysis). Hilbert did not understand that: (a) scientific intuition must be based on practical experience; intuition that is not based on practical experience is fantasy; (b) the definitions of geometric concepts are the genetic (technological) definitions that show how given material objects arise (i.e., how a person creates given material objects); (c) the theory must be constructed within the framework of the correct methodological basis: the unity of formal logic and rational dialectics. (d) the theory must satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. Therefore, Hilbert cannot prove the theorem of trisection of angle and the theorem of sum of interior angles (concluded angles) of triangle on the basis of his axioms. This fact signifies that Hilbert’s system of axioms is incomplete. In essence, Hilbert’s work is a superficial, tautological and logically incorrect verbal description of Figures 1-52 in his work.

Conclusion

The analysis of the foundations of geometry within the framework of the correct methodological basis – the unity of formal logic and rational dialectics – leads to the following result:

1) Geometry is an engineering science, but not a field of mathematics.

2) The essence of geometry is the construction of material figures (systems) and study of their properties.

3) The starting point of geometry is the following system principle: the properties of material figures (systems) determine the properties of the elements of figures; properties of elements characterize the properties of figures (systems).

4) Axiomatization of geometry is a way of construction of the science as a set (system) of practical principles. Sets (systems) of practice principles can be complete or incomplete.

5) The book, “The Foundations of Geometry” by David Hilbert, represents a methodologically incorrect work. It does not satisfy the dialectical principle of cognition, “practice $\rightarrow$ theory $\rightarrow$ practice”, because practice is not the starting point and final point in Hilbert’s theoretical approach (analysis). Hilbert did not understand that: (a) scientific intuition must be based on practical experience; intuition that is not based on practical experience is fantasy; (b) the correct science does not exist without definitions of concepts; the definitions of geometric concepts are the genetic (technological) definitions that shows how given material objects arise (i.e., how a person creates given material objects); (c) the theory must be

constructed within the framework of the correct methodological basis: the unity of formal logic and rational dialectics; (d) the theory must satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. Therefore, Hilbert cannot prove the theorem of trisection of angle and the theorem of sum of interior angles (concluded angles) of triangle on the basis of his axioms. This fact signifies that Hilbert's system of axioms is incomplete. In essence, Hilbert's work is a superficial, tautological and logically incorrect verbal description of Figures 1-52 in his work.

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