THE AREA INDUCED BY CIRCLES OF PARTITION AND APPLICATIONS

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Abstract. In this paper we continue with the development of the circles of partitions by introducing the notion of the area induced by circles of partitions and explore some applications.

1. Introduction

In [1] we have developed a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of $\mathbb{N}$. The method is very elementary in nature and has parallels with configurations of points on the geometric circle. The method operates in the following geometric sense: Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u+v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then the method associate each of this summands to points on the circle generated in a certain manner generated by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as the results obtained in this setting are then transformed back to results concerning the partition of integers.
In this paper we continue the development by delving into the geometric structure of circles of partitions and explore some applications. We introduce the notion of the **area** induced by circles of partitions and count the number points in a typical circle of partition.

2. The Circle of Partition

In this section we introduce the concept of the circle of partition. We study some elementary properties of this structure in the following sequel.

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subset \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the Circle of Partition generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$.

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then it is their only point which is not

Date: January 16, 2022.
2010 *Mathematics Subject Classification.* Primary 11Pxx, 11Bxx; Secondary 11Axx, 11Gxx.
Key words and phrases. area, length, chord.

an axis point. The line joining any two arbitrary point which are not axes partners on the CoP will be referred to as a **chord** of the CoP. The length of the chord $\mathcal{L}_{[x],[y]}$ joining the points $[x],[y] \in \mathcal{C}(n,\mathbb{M})$, denoted as $\Gamma([x],[y])$ is given by

$$\Gamma([x],[y]) = |x-y|.$$

We denote by

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\}$$

the **sequence** of the first n natural numbers. Further we will denote

$$\|[x]\| := x$$

as the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n,\mathbb{M})$ as $||\mathcal{C}(n,\mathbb{M})||$.

Proposition 2.3. *Each axis is uniquely determined by points $[x] \in \mathcal{C}(n,\mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n,\mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. ◻

Corollary 2.4. *Each point of a CoP $\mathcal{C}(n,\mathbb{M})$ except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n,\mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$\|[x]\| + \|[y]\| \neq n.$$

This violates Definition 2.1. Due to Proposition 2.3 the case of more than one axis partners is impossible. This completes the proof. ◻

Let us denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ with } x+y=n$$

and the number of axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \mid x<y\}. \tag{2.1}$$

Obviously holds

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n,\mathbb{M})| = k.$$

It is important to point out that the **median** of the weights of each co-axis point coincides with the center of the underlying CoP if it exists. That is to say, given all the real axes of the CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[u_1],[v_1]}, \mathbb{L}_{[u_2],[v_2]}, \ldots, \mathbb{L}_{[u_k],[v_k]}$$

then the following relations hold

$$\frac{u_1+v_1}{2} = \frac{u_2+v_2}{2} = \cdots = \frac{u_k+v_k}{2} = \frac{n}{2}$$

which is equivalent to the conditions for any of the pair of real axes $\mathbb{L}_{[u_i],[v_i]}, \mathbb{L}_{[u_j],[v_j]}$ for $1 \leq i, j \leq k$

$$\Gamma([u_i],[u_j]) = \Gamma([v_i],[v_j])$$

and

$$\Gamma([v_j],[u_i]) = \Gamma([u_j],[v_i]).$$

Definition 2.5. Let $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ be an arbitrary axis of the CoP. Then we denote the length of the axis $\mathbb{L}_{[x],[y]}$ with $|\mathbb{L}_{[x],[y]}|$.

Proposition 2.6. *Let $\mathbb{L}_{[x_1],[y_1]}, \mathbb{L}_{[x_2],[y_2]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ such that $[x_1],[x_2]$ are lower axes points. Then the identity holds*

$$|\mathbb{L}_{[x_1],[y_1]}| = \sqrt{\Gamma([x_1],[y_2])^2 + \Gamma([y_2],[y_1])^2}.$$

Proof. This follows from the variant of the rules of geometry of a circle. □

Remark 2.7. Next we formalize the notion of the axes of a CoP $\mathcal{C}(n,\mathbb{M})$ coinciding with the diameter of a circle.

Proposition 2.8. *Let $\mathbb{L}_{[x_1],[y_1]}, \mathbb{L}_{[x_2],[y_2]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ be arbitrary axes of the CoP. Then the equality holds*

$$|\mathbb{L}_{[x_1],[y_1]}| = |\mathbb{L}_{[x_2],[y_2]}|.$$

Proof. Without loss of generality we assume that $[x_1],[x_2]$ are lower axes points of their corresponding axes. Appealing to the identity in Proposition 2.6, we can write

$$\begin{aligned} |\mathbb{L}_{[x_1],[y_1]}| &= \sqrt{\Gamma([x_1],[y_2])^2 + \Gamma([y_2],[y_1])^2} \\ &= \sqrt{\Gamma([x_2],[y_1])^2 + \Gamma([y_2],[y_1])^2} \\ &= |\mathbb{L}_{[x_2],[y_2]}| \end{aligned}$$

since $\Gamma([x_1],[y_2]) = \Gamma([x_2],[y_1])$. □

Remark 2.9. Next we show that the length of any axis in a typical CoP is mostly controlled by the generator of the underlying CoP.

Proposition 2.10. *Let $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ with $\nu(n,\mathbb{M}) \geq 2$. Then $|\mathbb{L}_{[x],[y]}| < n$.*

Proof. Since $\nu(n,\mathbb{M}) \geq 2$, we let $\mathbb{L}_{[u],[v]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ be another axis of the CoP. Without loss of generality we let $[x],[u]$ be lower axis points on the CoP so that by appealing to the identity in Proposition 2.6, we have

$$\begin{aligned} |\mathbb{L}_{[x],[y]}| &= \sqrt{\Gamma([x],[v])^2 + \Gamma([y],[v])^2} \\ &= \sqrt{|x-v|^2 + |y-v|^2} \\ &\leq \sqrt{x^2+y^2} < n. \end{aligned}$$

□

Remark 2.11. It is worth noting that the length of each axis $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ coincides with the length of the diameters of the CoP $\mathcal{C}(n,\mathbb{M})$. Next we launch the notion of the area induced by a CoP.

Definition 2.12. Let $\mathcal{C}(n,\mathbb{M})$ be a CoP with $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$. Then by the **area** induced we mean the amount of space occupied by $\mathcal{C}(n,\mathbb{M})$. We denote the area induced with

$$\mathcal{A}_{\mathcal{C}(n,\mathbb{M})} = \frac{\pi}{4} \times |\mathbb{L}_{[x],[y]}|^2$$

where $\pi \approx \frac{22}{7}$.

Proposition 2.13 (Area formula)**.** *Let $\mathcal{C}(n,\mathbb{M})$ be a CoP with $\nu(n,\mathbb{M}) \geq 2$. Then we have*

$$\mathcal{A}_{\mathcal{C}(n,\mathbb{M})} \approx \frac{1}{4}\sum_{i=1}^{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor -1} \Gamma([x_i],[x_{i+1}]) \times (|\mathbb{L}_{[x_i],[y_i]}|^2 - \Gamma([x_i],[x_{i+1}])^2)^{\frac{1}{2}}$$
$$+ \frac{1}{4}\Gamma([x_{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor}],[y_1]) \times (|\mathbb{L}_{[x_{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor}],[y_{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor}]}|^2 - \Gamma([x_{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor}],[y_1])^2)^{\frac{1}{2}},$$

where $[x_i],[x_{i+1}]$ are adjacent points on the CoP.

Proof. Under the requirement $\nu(n,\mathbb{M}) \geq 2$ the space within the CoP $\mathcal{C}(n,\mathbb{M})$ is partitioned into sectors induced by the axes. The resulting approximation formula emanates from summing the area of each of the induced isosceles triangle. □

Theorem 2.14. *Let $\mathcal{C}(n,\mathbb{M})$ be a CoP with $\nu(n,\mathbb{M}) \geq 2$. Then we have*

$$\sum_{i=1}^{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor -1} \Gamma([x_i],[x_{i+1}]) + \Gamma([x_{\lfloor \frac{|\mathcal{C}(n,\mathbb{M})|}{2}\rfloor}],[y_1]) \approx |\mathbb{L}_{[x],[y]}| \times \pi,$$

where $[x_i],[x_{i+1}]$ are adjacent points on the CoP and $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$.

Proof. The inequality follows by applying the approximation formula in Proposition 2.13 and noting that for an arbitrary axis $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M})$ the inequality holds

$$|\mathbb{L}_{[x],[y]}| > \Gamma([u],[v])$$

where $\Gamma([u],[v])$ is the length of any chord in $\mathcal{C}(n,\mathbb{M})$. □

The above approximation formula in Proposition 2.13 becomes more accurate in the very large if the limiting law holds for the CoP $\mathcal{C}(n,\mathbb{M})$

$$\lim_{n\longrightarrow\infty} \nu(n,\mathbb{M}) = \infty.$$

It turns out that we can apply this formula to estimate the size of any CoP with base set an arithmetic progression. By taking $\mathbb{M}_{a,d}$ to be an arithmetic progression with first term a and common difference d, we now launch the following result:

Theorem 2.15 (The cardinality law)**.** *Let $\mathcal{C}(n,\mathbb{M}_{a,d})$ be a CoP with $n \in 2a \pmod d$ and $\nu(n,\mathbb{M}_{a,d}) \geq 2$. Then we have*

$$|\mathcal{C}(n,\mathbb{M}_{a,d})| \approx \frac{2\pi \times |\mathbb{L}_{[x],[y]}|}{d}.$$

for any $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}_{a,d})$.

Proof. This follows by comparing the notion of the area of CoPs induced by axes in Definition 2.12 with the approximation in Proposition 2.13 and exploiting the inequality $|\mathbb{L}_{[x],[y]}| > \Gamma([u],[v])$, where $\Gamma([u],[v])$ is an arbitrary chord in the CoP $\mathcal{C}(n,\mathbb{M}_{a,d})$. □

Corollary 2.16. *Let* $\mathcal{C}(n, \mathbb{M}_{a,d})$ *be a CoP with* $n \in 2a \pmod{d}$ *and* $\nu(n, \mathbb{M}_{a,d}) \geq 2$. *Then for any* $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}_{a,d})$, *we have*

$$|\mathbb{L}_{[x],[y]}| \approx \frac{d+n-2a}{2\pi}.$$

Proof. This follows from the counting function

$$|\mathcal{C}(n, \mathbb{M}_{a,d})| = 1 + \frac{n-2a}{d}.$$

□

References

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