The Exclusive Or Group X(n) Correspondence With Cayley-Dickson Algebras

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Abstract

It is impossible to fully represent a non-associative algebra using group theory due to the associativity requirement for the group operation. This precludes a full group theoretical cover of Cayley-Dickson algebras beginning with the generally non-associative Octonion Algebra doubling count. However, if we "forget" the sign attached to the result of the product of any two basis elements, every Cayley-Dickson algebra of doubling count n: CD(n) is modelled by the Exclusive Or group X(n), up to ignored sign of fixed nature or orientation choice. This paper explores the bijective correspondence between every CD(n) algebra and all of its subalgebras, and every X(n) group and all of its subgroups. A recursion relationship for determining the number of subalgebras of order 2^m for any CD(n), which is equivalent to the number of subgroups of order 2^m for the corresponding group X(n), is presented.

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For any given integer $n \geq 0$, define X(n) as an order 2^n group of binary integer members ranging from 0 to $2^n - 1$ with group operation the binary bit-wise exclusive or (xor or the operator ^) of two group member integers.

The group identity member I for X(n) is the integer 0, since for any integer k: k^0 = 0^k = k. We have for any binary number k: k^k = 0. Therefore, every group member of X(n) is its own inverse. The xor operator is commutative, meaning the group table will be symmetric, thus X(n) is Abelian. Every member of X(n) will form its own single member group conjugacy class, and thus every subgroup of X(n) will be a normal subgroup. X(n) is a Dedekind Group.

The product of two Cayley-Dickson algebra basis elements is within sign a single third basis element, and the algebra applied will set the basis element types, quantities and their product rules. The basis elements for any Cayley-Dickson algebra CD(n) may be enumerated sequentially with integer indexes ranging 0 to $2^n - 1$. Every product of two basis elements fits the algebraic expression $e_a * e_b = \pm e_c$, where a, b and c can be any combination, same or not, of the integer 0 representing the single scalar basis element index, and non-zero integers representing non-scalar basis elements that square to $-e_0$. Every CD(n) may be further structured such that for any product $e_a * e_b = \pm e_c$, the index set { a b c } is restricted to sets where the xor of all three is always zero.

For all six permutations of three chosen integers that xor to 0 mapped to a, b and c in $e_a * e_b = \pm e_c$, if all three are zero, * is defined by the algebra of real numbers. If one index is zero and the other two are the same non-zero index, * is defined by Complex Algebra. If all three are different and non-zero, * is defined by the product rules of the non-scalar triplet of Quaternion Algebra. All non-triplet basis element products for Quaternion Algebra are complex or real subalgebra products, so these three algebras fully cover the definitions for the product of any two basis elements for any Cayley-Dickson algebra. There are no other ways to get three indexes to xor to 0, therefore this xor operator correspondence spans all three of the algebras that fully specify the product of any two CD(n) basis elements in a very natural, optimal and complete fashion. However, what the xor operator cannot do is provide any guidance on which sign is appropriate in the $\pm e_c$ product result.

We will be exclusively working here on the identification of all sets { a b c } representing the indexes for every basis element product and result, and how they structurally fit with other products in Cayley-

Dickson algebras, leaving unanswered whether any particular product result basis element is scaled by +1 or –1. In doing so, we obviate the issue that a non-associative algebra cannot be completely represented by any necessarily associative group structure. The correspondence between the algebraic expression $e_a * e_b = \pm e_c$ ignoring sign and the Boolean expression a^b=c or equivalently a^b^c = 0 suggests a correspondence between Cayley-Dickson algebra CD(n) ignoring basis element product result signs, and the xor group X(n). As we shall soon see, this correspondence is complete. The correspondence between X(n) including all of its subgroups, and ignored sign CD(n) including all of its subalgebras is an isomorphism.

Every group X(n) will have some quantity of non-trivial normal subgroups exclusively with order 2^m for every integer m in the range 0 < m < n. Each of these normal subgroups will be isomorphic to X(m). Every group X(n) will therefore have the same number of non-trivial quotient groups X(n)/X(m), and these quotient groups will be isomorphic to X(n-m). It should be understood that X(0) is order 1, including just the identity member I.

The number of normal subgroups of order 2^m for X(n) will be equivalent to the number of order 2^m subalgebras for order 2^n Cayley-Dickson algebra CD(n). The subset of integers within each of these normal subgroups, when applied as indexes for algebraic basis elements, partition the basis elements showing up in its corresponding subalgebra. The subgroup group table will give the unsigned basis element product combinations for its corresponding subalgebra. Since every X(n) will have the full complement of normal subgroups of order 2^m, all subalgebras for any CD(n) algebra will be represented.

Within any single quotient group X(n)/X(m), the union of its kernel and any other coset will form a group of order 2^{m+1} isomorphic to X(m+1), in a manner of speaking, doubling the kernel subgroup and thus the corresponding subalgebra. Doubling using different kernels isomorphic to X(m) can make different and meaningful representations of the same doubled group and corresponding algebra basis set. An example of this would be using the correspondence groups for the seven Quaternion subalgebras of a given Octonion Algebra. Each of the doublings on these kernels will produce the correspondence group for the chosen Octonion Algebra. As we will soon see, the other cosets paired with the kernels in this example are all basic quad indexes specific to the particular kernel. The doubling of the full complement of kernels isomorphic to X(m) will span the full complement of subgroups of X(n) isomorphic to X(m+1).

In general, for any X(n) and integer 1 ≤ m ≤ n/2, the number of order 2^m normal subgroups of X(n) will be equal to the number of order 2^{n-m} normal subgroups of X(n). Stated another way, the number of normal subgroups of order m charted vs. m for a given n will be symmetric about n/2 for both odd and even n. This is due to the duality between X(n)/X(m) ≈ X(n-m) and X(n)/X(n-m) ≈ X(m). With the bijectivity of the correspondences between subgroups and subalgebras ignoring product sign, the same can be said for the number of subalgebras of order 2^m for CD(n).

Let's start with X(4), the Sedenion Algebra correspondence group. Its group members are the integers 0 through 15. The group table for X(4) is

^	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

X(4) has 15 normal subgroups isomorphic to X(1), the Complex Algebra correspondence group:

[0,1], [0,2], [0,3], [0,4], [0,5], [0,6], [0,7], [0,8], [0,9], [0,10], [0,11], [0,12], [0,13], [0,14], [0,15]

Using each of these, we can form 15 separate order 8 quotient groups X(4)/X(1) each isomorphic to X(3), the Octonion correspondence group. The number of straight up order 8 normal subgroups of X(4) is also 15, isomorphic to X(3). They correspond to the Octonion subalgebras of the Sedenions. The full complement of 15 order 8 normal subgroups of X(4) are itemized next by group table

^	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

^	0	1	2	3	8	9	10	11
0	0	1	2	3	8	9	10	11
1	1	0	3	2	9	8	11	10
2	2	3	0	1	10	11	8	9
3	3	2	1	0	11	10	9	8
8	8	9	10	11	0	1	2	3
9	9	8	11	10	1	0	3	2
10	10	11	8	9	2	3	0	1
11	11	10	9	8	3	2	1	0

^	0	1	2	3	12	13	14	15
0	0	1	2	3	12	13	14	15
1	1	0	3	2	13	12	15	14
2	2	3	0	1	14	15	12	13
3	3	2	1	0	15	14	13	12
12	12	13	14	15	0	1	2	3
13	13	12	15	14	1	0	3	2
14	14	15	12	13	2	3	0	1
15	15	14	13	12	3	2	1	0

^	0	1	4	5	8	9	12	13
0	0	1	4	5	8	9	12	13
1	1	0	5	4	9	8	13	12
4	4	5	0	1	12	13	8	9
5	5	4	1	0	13	12	9	8
8	8	9	12	13	0	1	4	5
9	9	8	13	12	1	0	5	4
12	12	13	8	9	4	5	0	1
13	13	12	9	8	5	4	1	0

^	0	1	4	5	10	11	14	15
0	0	1	4	5	10	11	14	15
1	1	0	5	4	11	10	15	14
4	4	5	0	1	14	15	10	11
5	5	4	1	0	15	14	11	10
10	10	11	14	15	0	1	4	5
11	11	10	15	14	1	0	5	4
14	14	15	10	11	4	5	0	1
15	15	14	11	10	5	4	1	0

^	0	1	6	7	8	9	14	15
0	0	1	6	7	8	9	14	15
1	1	0	7	6	9	8	15	14
6	6	7	0	1	14	15	8	9
7	7	6	1	0	15	14	9	8
8	8	9	14	15	0	1	6	7
9	9	8	15	14	1	0	7	6
14	14	15	8	9	6	7	0	1
15	15	14	9	8	7	6	1	0

^	0	1	6	7	10	11	12	13
0	0	1	6	7	10	11	12	13
1	1	0	7	6	11	10	13	12
6	6	7	0	1	12	13	10	11
7	7	6	1	0	13	12	11	10
10	10	11	12	13	0	1	6	7
11	11	10	13	12	1	0	7	6
12	12	13	10	11	6	7	0	1
13	13	12	11	10	7	6	1	0

^	0	2	4	6	8	10	12	14
0	0	2	4	6	8	10	12	14
2	2	0	6	4	10	8	14	12
4	4	6	0	2	12	14	8	10
6	6	4	2	0	14	12	10	8
8	8	10	12	14	0	2	4	6
10	10	8	14	12	2	0	6	4
12	12	14	8	10	4	6	0	2
14	14	12	10	8	6	4	2	0

^	0	2	4	6	9	11	13	15
0	0	2	4	6	9	11	13	15
2	2	0	6	4	11	9	15	13
4	4	6	0	2	13	15	9	11
6	6	4	2	0	15	13	11	9
9	9	11	13	15	0	2	4	6
11	11	9	15	13	2	0	6	4
13	13	15	9	11	4	6	0	2
15	15	13	11	9	6	4	2	0

^	0	2	5	7	8	10	13	15
0	0	2	5	7	8	10	13	15
2	2	0	7	5	10	8	15	13
5	5	7	0	2	13	15	8	10
7	7	5	2	0	15	13	10	8
8	8	10	13	15	0	2	5	7
10	10	8	15	13	2	0	7	5
13	13	15	8	10	5	7	0	2
15	15	13	10	8	7	5	2	0

^	0	2	5	7	9	11	12	14
0	0	2	5	7	9	11	12	14
2	2	0	7	5	11	9	14	12
5	5	7	0	2	12	14	9	11
7	7	5	2	0	14	12	11	9
9	9	11	12	14	0	2	5	7
11	11	9	14	12	2	0	7	5
12	12	14	9	11	5	7	0	2
14	14	12	11	9	7	5	2	0

^	0	3	4	7	8	11	12	15
0	0	3	4	7	8	11	12	15
3	3	0	7	4	11	8	15	12
4	4	7	0	3	12	15	8	11
7	7	4	3	0	15	12	11	8
8	8	11	12	15	0	3	4	7
11	11	8	15	12	3	0	7	4
12	12	15	8	11	4	7	0	3
15	15	12	11	8	7	4	3	0

^	0	3	4	7	9	10	13	14
0	0	3	4	7	9	10	13	14
3	3	0	7	4	10	9	14	13
4	4	7	0	3	13	14	9	10
7	7	4	3	0	14	13	10	9
9	9	10	13	14	0	3	4	7
10	10	9	14	13	3	0	7	4
13	13	14	9	10	4	7	0	3
14	14	13	10	9	7	4	3	0

^	0	3	5	6	8	11	13	14
0	0	3	5	6	8	11	13	14
3	3	0	6	5	11	8	14	13
5	5	6	0	3	13	14	8	11
6	6	5	3	0	14	13	11	8
8	8	11	13	14	0	3	5	6
11	11	8	14	13	3	0	6	5
13	13	14	8	11	5	6	0	3
14	14	13	11	8	6	5	3	0

^	0	3	5	6	9	10	12	15
0	0	3	5	6	9	10	12	15
3	3	0	6	5	10	9	15	12
5	5	6	0	3	12	15	9	10
6	6	5	3	0	15	12	10	9
9	9	10	12	15	0	3	5	6
10	10	9	15	12	3	0	6	5
12	12	15	9	10	5	6	0	3
15	15	12	10	9	6	5	3	0

The cosets for the 15 X(4)/X(1) quotient groups are the following

[0, 1], [2, 3], [4, 5], [6, 7], [8, 9], [10, 11], [12, 13], [14, 15]
[0, 2], [1, 3], [4, 6], [5, 7], [8, 10], [9, 11], [12, 14], [13, 15]
[0, 3], [1, 2], [4, 7], [5, 6], [8, 11], [9, 10], [12, 15], [13, 14]
[0, 4], [1, 5], [2, 6], [3, 7], [8, 12], [9, 13], [10, 14], [11, 15]
[0, 5], [1, 4], [2, 7], [3, 6], [8, 13], [9, 12], [10, 15], [11, 14]
[0, 6]. [1, 7], [2, 4], [3, 5], [8, 14], [9, 15], [10, 12], [11, 13]
[0, 7], [1, 6], [2, 5], [3, 4], [8, 15], [9, 14], [10, 13], [11, 12]
[0, 8], [1, 9], [2, 10], [3, 11], [4, 12], [5, 13], [6, 14], [7, 15]
[0, 9], [1, 8], [2, 11], [3, 10], [4, 13], [5, 12], [6, 15], [7, 14]
[0, 10], [1, 11], [2, 8], [3, 9], [4, 14], [5, 15], [6, 12], [7, 13]
[0, 11], [1, 10], [2, 9], [3, 8], [4, 15], [5, 14], [6, 13], [7, 12]
[0, 12], [1, 13], [2, 14], [3, 15], [4, 8], [5, 9], [6, 10], [7, 11]
[0, 13], [1, 12], [2, 15], [3, 14], [4, 9], [5, 8], [6, 11], [7, 10]
[0, 14], [1, 15], [2, 12], [3, 13], [4, 10], [5, 11], [6, 8], [7, 9]

Each of the coset pairs of integers in these 15 quotient groups xor to the same integer, spanning the full complement 15 of non-identity integers. Each of their coset product tables are isomorphic to X(3). The union of a kernel and one other coset member forms a group isomorphic to X(2) .

Look now at the quotient groups X(4)/X(2). These will be order four coset product groups of four member cosets, and each of these quotient groups will be isomorphic to X(2), the Quaternion correspondence group. X(4) has 35 order 4 normal subgroups isomorphic to X(2) to use for the kernel here, matching the fact that Sedenions have 35 Quaternion subalgebras. The basis element members for all 35 Quaternion subalgebras for Sedenion Algebra are specified by index using the kernel member of each of the 35 quotient group cosets below, and the kernel group tables indicate the unsigned product pairings for their corresponding Quaternion subalgebra.

The cosets for the 35 order four X(4)/X(2) quotient groups are as follows:

[0, 1, 2, 3], [4, 5, 6, 7], [8, 9, 10, 11], [12, 13, 14, 15]
[0, 1, 4, 5], [2, 3, 6, 7], [8, 9, 12, 13], [10, 11, 14, 15]
[0, 1, 6, 7], [2, 3, 4, 5], [8, 9, 14, 15], [10, 11, 12, 13]
[0, 1, 8, 9], [2, 3, 10, 11], [4, 5, 12, 13], [6, 7, 14, 15]
[0, 1, 10, 11], [2, 3, 8, 9], [4, 5, 14, 15], [6, 7, 12, 13]
[0, 1, 12, 13], [2, 3, 14, 15], [4, 5, 8, 9], [6, 7, 10, 11]
[0, 1, 14, 15], [2, 3, 12, 13], [4, 5, 10, 11], [6, 7, 8, 9]
[0, 2, 4, 6], [1, 3, 5, 7], [8, 10, 12, 14], [9, 11, 13, 15]
[0, 2, 5, 7], [1, 3, 4, 6], [8, 10, 13, 15], [9, 11, 12, 14]
[0, 2, 8, 10], [1, 3, 9, 11], [4, 6, 12, 14], [5, 7, 13, 15]
[0, 2, 9, 11], [1, 3, 8, 10], [4, 6, 13, 15], [5, 7, 12, 14]
[0, 2, 12, 14], [1, 3, 13, 15], [4, 6, 8, 10], [5, 7, 9, 11]
[0, 2, 13, 15], [1, 3, 12, 14], [4, 6, 9, 11], [5, 7, 8, 10]
[0, 3, 4, 7], [1, 2, 5, 6], [8, 11, 12, 15], [9, 10, 13, 14]
[0, 3, 5, 6], [1, 2, 4, 7], [8, 11, 13, 14], [9, 10, 12, 15]
[0, 3, 8, 11], [1, 2, 9, 10], [4, 7, 12, 15], [5, 6, 13, 14]
[0, 3, 9, 10], [1, 2, 8, 11], [4, 7, 13, 14], [5, 6, 12, 15]
[0, 3, 12, 15], [1, 2, 13, 14], [4, 7, 8, 11], [5, 6, 9, 10]
[0, 3, 13, 14], [1, 2, 12, 15], [4, 7, 9, 10], [5, 6, 8, 11]
[0, 4, 8, 12], [1, 5, 9, 13], [2, 6, 10, 14], [3, 7, 11, 15]
[0, 4, 9, 13], [1, 5, 8, 12], [2, 6, 11, 15], [3, 7, 10, 14]
[0, 4, 10, 14], [1, 5, 11, 15], [2, 6, 8, 12], [3, 7, 9, 13]
[0, 4, 11, 15], [1, 5, 10, 14], [2, 6, 9, 13], [3, 7, 8, 12]
[0, 5, 8, 13], [1, 4, 9, 12], [2, 7, 10, 15], [3, 6, 11, 14]
[0, 5, 9, 12], [1, 4, 8, 13], [2, 7, 11, 14], [3, 6, 10, 15]
[0, 5, 10, 15], [1, 4, 11, 14], [2, 7, 8, 13], [3, 6, 9, 12]
[0, 5, 11, 14], [1, 4, 10, 15], [2, 7, 9, 12], [3, 6, 8, 13]
[0, 6, 8, 14], [1, 7, 9, 15], [2, 4, 10, 12], [3, 5, 11, 13]
[0, 6, 9, 15], [1, 7, 8, 14], [2, 4, 11, 13], [3, 5, 10, 12]
[0, 6, 10, 12], [1, 7, 11, 13], [2, 4, 8, 14], [3, 5, 9, 15]
[0, 6, 11, 13], [1, 7, 10, 12], [2, 4, 9, 15], [3, 5, 8, 14]
[0, 7, 8, 15], [1, 6, 9, 14], [2, 5, 10, 13], [3, 4, 11, 12]
[0, 7, 9, 14], [1, 6, 8, 15], [2, 5, 11, 12], [3, 4, 10, 13]
[0, 7, 10, 13], [1, 6, 11, 12], [2, 5, 8, 15], [3, 4, 9, 14]
[0, 7, 11, 12], [1, 6, 10, 13], [2, 5, 9, 14], [3, 4, 8, 15]

Looking closely at the cosets, the non-kernel members are the indexes for three different basic quad basis element sets appropriate for the Quaternion triplet in the kernel. The union of the kernel and one of the other cosets in turn builds three separate groups isomorphic to X(3) corresponding to three separate Octonion subalgebra candidates for the Sedenions, validating the fact that each Quaternion subalgebra triplet will appear in three separate Octonion subalgebra candidates for the Sedenions. Each Octonion subalgebra candidate requires seven Quaternion triplets, so these 35 sets of cosets cover correspondences for the full set of 35*3/7 = 15 Octonion subalgebras for Sedenion Algebra.

Rather than drilling down further on the Sedenion Algebra correspondence, let's look closer at the Octonion Algebra correspondence X(3). It has seven order two normal subgroups isomorphic to X(1) corresponding to the seven Complex Algebra subalgebras for Octonion Algebra. These produce seven quotient groups. It also has seven order 4 normal subgroups isomorphic to X(2) to provide the Quaternion correspondences to each of the seven Quaternion subalgebras for Octonion Algebra. The cosets for the seven X(3)/X(1) quotient groups are as follows:

[0, 1], [2, 3], [4, 5], [6, 7]
[0, 2], [1, 3], [4, 6], [5, 7]
[0, 3], [1, 2], [4, 7], [5, 6]
[0, 4], [1, 5], [2, 6], [3, 7]
[0, 5], [1, 4], [2, 7], [3, 6]
[0, 6], [1, 7], [2, 4], [3, 5]
[0, 7], [1, 6], [2, 5], [3, 4]

The coset product table group for each of these is isomorphic to the group X(2). The union of kernel and one other coset forms a group also isomorphic to X(2). The integer members of the seven order 4 normal subgroups set the basis partitions for the seven Quaternion subalgebras and their group table sets the unsigned product combinations. For completeness their group tables are as follows:

^	0	2	4	6
0	0	2	4	6
2	2	0	6	4
4	4	6	0	2
6	6	4	2	0

^	0	1	4	5
0	0	1	4	5
1	1	0	5	4
4	4	5	0	1
5	5	4	1	0

^	0	3	4	7
0	0	3	4	7
3	3	0	7	4
4	4	7	0	3
7	7	4	3	0

^	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

^	0	2	5	7
0	0	2	5	7
2	2	0	7	5
5	5	7	0	2
7	7	5	2	0

^	0	1	6	7
0	0	1	6	7
1	1	0	7	6
6	6	7	0	1
7	7	6	1	0

^	0	3	5	6
0	0	3	5	6
3	3	0	6	5
5	5	6	0	3
6	6	5	3	0

It is worth noting that X(2) is a Klein 4-group. We can use each of these seven normal subgroups isomorphic to X(2) to form seven quotient groups X(3)/X(2). The seven cosets for these quotient groups are as follows:

[0, 2, 4, 6], [1, 3, 5, 7]
[0, 1, 4, 5], [2, 3, 6, 7]

[0, 3, 4, 7], [1, 2, 5, 6]
[0, 1, 2, 3], [4, 5, 6, 7]
[0, 2, 5, 7], [1, 3, 4, 6]
[0, 1, 6, 7], [2, 3, 4, 5]
[0, 3, 5, 6], [1, 2, 4, 7]

These pairs are the full complement of basis element indexes for the seven Quaternion subalgebras for a single Octonion Algebra followed by the indexes for their basic quad basis element sets.

X(2) has three order two normal subgroups isomorphic to X(1). They correspond to the three Complex Algebra subalgebras for the Quaternions. X(1) has only the trivial normal subgroup [I], and X(0) is the correspondence group for the algebra of real numbers.

From the group structure for X(n) and thus correspondence with subalgebras of the nth doubled Cayley-Dickson algebra CD(n), the following table formulas can be derived with stated restrictions for X(n) and its subgroups, and therefore CD(n) and its subalgebras.

Features derived from X(n)	Formula	Restrictions
Order of full X(n) group and CD(n) algebra	2^n	
Number of non-trivial order 2 subgroups of X(n) = number of CD(n) Complex subalgebras	$2^n – 1$	$n \geq 2$
Number of non-trivial order 4 subgroups of X(n) = number of CD(n) Quaternion subalgebras	$(2^{n-1} – 1) (2^n – 1) / 3$	$n \geq 3$
Number of non-trivial order 8 subgroups of X(n) = number of CD(n) Octonion subalgebras	$(2^{n-2} – 1)(2^{n-1} – 1) (2^n – 1) / 21$	$n \geq 4$
Recursion for number of non-trivial order 2^m subgroups of X(n) = number of CD(n) order 2^m subalgebras	$N_m = N_{m-1} (2^{n-m+1} – 1) / (2^m – 1)$	N_x = number of subgroups of order 2^x $n \geq m+1$

The following is a table of the number of normal subgroups and subalgebras of order 2^m for group order 2^n X(n) and algebra order 2^n CD(n) for $2 \leq n \leq 8$

	n=2	3	4	5	6	7	8
m=1	3	7	15	31	63	127	255
2		7	35	155	651	2667	10795
3			15	155	1395	11811	97155
4				31	651	11811	200787
5					63	2667	97155
6						127	10795
7							255

We have significantly simplified the task of understanding Cayley-Dickson algebras and their subalgebras by permitting the tools of group theory to be applied by first forgetting which sign is appropriate in the $\pm e_c$ product result for the basis element product $e_a * e_b = \pm e_c$, building correspondence groups of positive integers representing the basis element indexes and using xor as the group operation. With the correspondence groups in hand, which specifies most of the internal structure of a Cayley-Dickson algebra and its subalgebras, we must "unforget" by assignment where the negations show up in the algebra and subalgebra basis element product tables.

The first step is to replace integers in the desired group table with the label used to represent a basis element, like "e", with the cell content for a subscript. The task is complete when all cells requiring negation have been identified and negated. The group table will then be converted to the algebra basis element product table, fully defining the algebra. Some like negations will be required for any CD(n) algebra or subalgebra, others will be choices possibly with restrictions. Real number products have no basis element negations involved. The square of any non-scalar basis element is always $-e_0$ so these negations must be done in any group table conversion. The remaining negations come from the non-scalar triplet basis element product rule for a Quaternion (sub)algebra, and these are defined by orientation choices. Of course, this is only relevant to CD(n) for $n \geq 2$.

W. R. Hamilton's discovery of Quaternion Algebra came about after the breakthroughs first of giving up on extending 2D complex numbers to 3D, trying the next increase to 4D with one scalar basis element, and three non-scalar basis elements that square to $-e_0$ equivalent to -1. The final breakthrough was coming to the realization the three non-scalar basis element products anti-commute, and the product of any two is within sign the third with a specific cyclic pattern. The six basis element product combinations between these three elements are covered by the following ordered triplet (e_a e_b e_c) with the product rule the first *second = +third when traversing through this triplet cyclically left to right, and results in –third when doing the commuted products traversing cyclically right to left. If we were to perform any odd number of transpositions of two basis elements in (e_a e_b e_c) we would come up with a different rule where all six products are negated. Each odd transposition scheme is equivalent to flipping the order to (e_c e_b e_a). This negated rule produces a different appearing Quaternion Algebra. There are no other options, so we have a choice between two orientations for Quaternion Algebra, the basis element product rule (e_a e_b e_c) or the basis element product rule (e_c e_b e_a).

The orientation of every CD(n) for $n \geq 2$ is fully, succinctly and exclusively defined by the orientations assigned to each of its Quaternion subalgebras. The importance of Quaternion Algebra cannot be overstated. This also shines a light on the importance of the group correspondence process above, it simplifies the task of identifying all Quaternion subalgebras as well as higher order subalgebras that may impose their own orientation limitations as we will see next.

If we are determining CD(2), the single Quaternion triplet orientation choice is free, choose one and done. This is not the case when we move up to CD(3), Octonion Algebra. Here, we have seven separate Quaternion subalgebras, and all $2^7 = 128$ possible orientation choice permutations will not produce a valid Octonion Algebra, only 16 will. This is where the restrictions on orientation choices first come in. For Octonion Algebra, the effort remembering the consistent Quaternion subalgebra triplet partitions easily determined by the xor to 0 rule, their orientations for one of 16 proper Octonion Algebra orientations, and a simple rule to morph one of 16 proper Octonion Algebras to another, is orders of magnitude simpler than the perhaps impossible task of memorizing 16 separate 8x8 multiplication tables, let alone one table. The CD(3+) basis element product tables are obtuse, the triplet orientation cover condense the tables to something that can be visualized or at least, simply stated.

When $n \geq 4$, we have another issue, there will be multiple appearances of any single Quaternion subalgebra in subalgebras of higher order, and each occurrence orientation must be singularly defined. This will restrict the orientation choices for these higher order subalgebras. We have already seen this when the Sedenion correspondence was drilled down on. Each of its Quaternion subalgebras will appear in three separate Octonion subalgebra candidates, and the multiplicity restriction is quite severe as it prevents all 15 Octonion subalgebras from being proper Octonion as stated above. This is why I say Octonion subalgebra *candidate*. The best that can be done is proper orientations for the seven Octonion subalgebra candidates that share a common basis element, and one more from any of the remaining eight. The Quaternion subalgebra multiplicity will force the remaining seven Octonion subalgebra candidates to broken status, one triplet rule negation off of isomorphism with one of the select 16 proper permutations. The single improper triplet in a given broken Octonion subalgebra candidate will be the intersection of that candidate and the proper Octonion subalgebra that does not include the common basis element. In a compatible manner, the worst that can be done is select sets of five Octonion subalgebra candidates where only four can be properly oriented. This is the essence of Sedenion Algebra not being a division algebra, every fundamental zero divisor involves products using the Octonion subalgebra-wise inconsistent triplet rules. The eight proper Octonion subalgebras produce no zero divisors as one might expect.

All of this is covered in some detail within Reference [1].

References

[1] Richard D. Lockyer, December 2020 *An Algebraic proof Sedenions are not a division algebra and other consequences of Cayley-Dickson Algebra definition variation*
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