# Generalizing [Un]Even Series-Sums toward an Eventual Demonstration for the Riemann Hypothesis & Implied Extensions

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ABSTRACT

The paper proposes an elegant generalization straddling beyond arithmetic series summation and functional integration/analysis alike. Based on computationally efficient averaging without necessarily invoking any classic criteria/filters/cut-offs, notably continuity or convergence checks, it leads just naturally up to applications to RH (boasting a single-line demonstration). Extra implications for primes and, more broadly, numbers-theoretic pursuits as well as functional operators will be forthcoming as part of a full-fledged roundup, *Orduale*, with a summary glimpse allowed herewith.

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## Part One: Generalized Series Summation & Further [Super]Unifying

Has it ever occurred to you that the conventional series-sums could be extended alongside their flipside filters or otherwise cost being effectively waived or somehow mitigated?

Consider a hypothetic series as characterized by *evenly distributed* increments—in the indices and the values alike. It is straightforward to show (see *Appendix*) that the underlying cumbersome summation could be cut short, with the shortcut building on the kind of *characteristic averaging* that can always be defied as well as gauged constructively with respect to a mode-like metric. The latter will show to be either coinciding identically with the actually observed term (the case of N odd on hand) or otherwise implied (N even). In either event, the characteristic can be measured as averaging over the two corners (arbitrary subset), the initial element and the final term, with *Theorem/Lemma A* proposing as below.

$$\forall\,\{X\} \equiv x_1, x_2 \ldots x_n, \forall(x_i - x_{i-1}) \equiv \Delta x = const,\;\; \Delta i = const'$$

$$\exists\bar{x} \equiv \frac{x_1 + x_n}{2}\colon \sum_{i=1}^{n} x_i = \bar{x} * n = \frac{x_1 + x_n}{2} * n \qquad (A)$$

[1] To my Mom, Ludmila Fedorovna “Mila” Shevenyonova

Evidently, (A) resembles as well as generalizes the conventional formula for a natural, arithmetic series summation:

$$1 + 2 + \cdots n \equiv \sum_{k=1}^{n} k = \frac{1+n}{2} * n$$

Even though the broader scope of qualified values could be as diverse as natural, rational, and real ones! Indeed, consider some illustrative series:

$$1 + 1.23 + 1.46 + \cdots 1.92 = \frac{1+1.92}{2} * 5 = 1.46 * 5 = 7.3$$

One helpful if near-trivial core point to be embarked on as part of the proof is adduced below:

$$\forall k = \overline{1,n}\colon x_k = x_1 + (k-1) * \Delta k \equiv x + (k-1)\Delta,\ \ \Delta = const\ \ (1a)$$

As hinted, the corners approach could be viewed as but a special meta-case of a *sub*set/sub-intervals (in the functional-analytic lingo) approach drawing upon any *interim* elements or terms hence operated as basis. In fact, one could span as much as the full-blown set corresponding to the underlying series, in which setup one may speak in terms of the characteristic factors or effective, probabilistic-like weights. Which always exist, if insofar as defined identically as calibrations.

$$\bar{x} * n = n * \sum_{i=1}^{n} (\pi_i * x_i) \equiv \sum_{i=1}^{n} x_i,\ \ e.g.\ \pi_i = \frac{1}{n}\ \ \forall i = \overline{1,n}\ \ (2)$$

At this point though, we are poised to appreciate that the early relaxation extends so far as to encompass *nearly any* [function or operator-like] nature of the objects as well as their values/levels distributions, as long as the weak prerequisite of *indices* (or, in generalized terms, *arguments*) being evenly distributed (which is subsumed under the convention of *smooth* integration and differentiation, with the *infinitesimal* increments amounting to constant-like or comparably zero-like *differentials*). In actuality, even so much as this kind of *symmetry* would seem idle/superfluous or over-restricting/non-binding in the presently attempted framework.

In other words, suppose for simplicity's sake, a *function* generalizes a *series*. As long as *any interim* choice or subset works, so too would sustained opting for the *corners*. The conjectured factors or modifiers could be inferred from,

$$\exists\{\pi\}\colon \widehat{\pi_1} * f(x_1) + \widehat{\pi_n} * f(x_n) \equiv f(x_1) + \cdots f(x_n) \equiv \sum f \equiv F$$
$$\equiv \widehat{\pi_{k_1}} * f\left(x_{k_1}\right) + \widehat{\pi_{k_2}} * f\left(x_{k_2}\right) + \ldots \widehat{\pi_{k_m}} * f\left(x_{k_m}\right)\ \ \forall m \le n$$

$$\forall j = \overline{1, m}: \widehat{\pi_{k_j}} = \frac{F}{m * f\left(x_{k_j}\right)} \quad (3)$$

Effective as these are, the proposed factor-*level* estimates might still appear inconclusive, if only due to their recursive nature with an eye toward the F-sums having yet to be measured in the first place. Needless to say, the vicious circle is broken or transformed into a virtuous one by applying factor-based *ratios* (to avail ourselves of the *orduale* paradigm), which should be most natural (and, incidentally, pertinent to RH) as per (3a) for m=2. The factors can simultaneously map into the respective probabilistic-like weights by invoking identity-based gauging residually (3b).

$$\frac{\widehat{\pi_n}}{\widehat{\pi_1}} = \frac{f(x_1)}{f(x_n)} \equiv r \quad (3a)$$

$$p_1 + p_n \equiv 1 = \frac{\widehat{\pi_1}}{\widehat{\pi_1} + \widehat{\pi_n}} + \frac{\widehat{\pi_n}}{\widehat{\pi_1} + \widehat{\pi_n}} = \frac{1}{1+r} + \frac{1}{1+\frac{1}{r}} \quad (3b)$$

**Part Two: RH Garnering a Posterior Spoil/Trophy**

The Riemann's zeta can be rethought in terms of the corner averaging:

$$\zeta(s) \equiv \sum_{k=1}^{\infty} k^{-s} = \widehat{\pi_1} * 1^{-s} + \widehat{\pi_T} * T^{-s}, \qquad T \to \infty \quad (4)$$

By the very design of the RH, the domain is confined to the singular function value, zero. Which, by invoking (3a), suggests:

$$\frac{T^{-s}}{1^{-s}} \equiv -\frac{T^{-s}}{1^{-s}} \quad (5)$$

$$\frac{0^{s}}{1^{\pm s}} = -\frac{0^{s}}{1^{\pm s}} \quad (6)$$

Now, one should feel free to either dwell on the candidate interpretations of the arcane-looking values of (5)—e.g. as in (6)—or *bypass* this interim challenge altogether. (The latter, again, stands up to the orduale tone with interim being less relevant than the ultimate, and the complete lending itself with simplicity as opposed to uncertainty arising from and traceable back to incompleteness/chaos as well as instances of prior oversimplifying, invariably costly as well as fallacious or overrated in its routinely perceived merits.)

However, the *general* solution appears as straightforward as it too proves all-comprising in its qualified branching (extending beyond the complexity-specific phenomenon). By far, the one way of ensuring *sign-variation* is to attribute it to the [absolute-value] real part of the *s*-argument being ½, in which case [the implied fudge] unity retains its square root while lending its imaginary part to the remaining [unity]. For the latter to convey no further alterations to the former [core unity], it must take on [natural/integer] values that are [even] factors of pi.

$$1^{\pm s} = -1^{\pm s} = (1 * 1)^{\pm s} = 1^{\pm Re(s)} * 1^{\pm[s+Re(s)]} = 1^{\pm 1/2} * 1^{\pm[1/2+2i\pi(l+l)]} \; \forall|l| \in N \quad (7)$$

Remarkably, not only does this solution match one postulated by the RH for the *real* part of zeta's *s*-argument, it also sheds light on the best attainable and possibly exhaustive candidate for the *imaginary residuale*, thus positing a constructive/substantive demonstration. The latter does prove all-encompassing, as it allows for the *l*-power multiple-absolute to be as large as an infinity (thus transfering to the zero-corner of the ratio in case of negative infinity) or otherwise amounting to zero in the exponent for all 'practical' purposes (thus suggesting a real-like solution interpretation without compromising its general structure). Cf. (7a):

$$1^{\pm s} = e^{\pm\left(\frac{1}{2}+2i\pi l\right)*(2i\pi k)} \rightarrow \begin{cases} e^{\pm\left(\frac{1}{2}+0\right)*(2i\pi k)} = \pm 1, \forall|k| \in N \\ e^{\pm\left(\frac{1}{2}\pm iT\right)*(2i\pi k)} = \pm 1^{\pm iT},\; T \rightarrow \infty \end{cases} \quad (7a)$$

While at it, one should not discard an alternative solution qualifying for (5) sign-reversal, namely zero on the RHS and LHS alike. In other words,

$$X = -X = 0 \leftrightarrow 0 = \begin{cases} \dfrac{T^{-s}}{1^{-s}} \\ \dfrac{1^{-s}}{T^{-s}} \end{cases} = 0^{\pm s} \quad (8)$$

Arguably, a candidate *s*-solution is real-like, yet [in absolute value] neither unity (zeta in excess of zero) nor zero (infinite series-sum). (Alternate convention-estimates being infinity versus -1/2, respectively.) This seconds the core solution without informing it substantively (unless one is to revert to *interpreting* (*beyond phenomenologically construing*) the extreme complex value based on (6) or otherwise, e.g. zeta-value *conventions*).

## Caveats: Few (if any) Conventional Strains?

It should be borne in mind that the approach being posited herewith effectively does away with the agendas of continuity or convergence, let alone their adjacent counterparts such as monotone patterns beyond 'controlled' distributions or 'well-behaved' functions. On the one hand, suffice it for any function to only really ensure an evenly distributed pattern in its *arguments* or *indices* albeit not in the resultant levels: whilst the latter is superfluous, the former is secured by convention (unless the axis/domain is itself a nonlinear function [space], a line-of-lines, etc.). For the same token, not only would it be awkward imposing any conventional 'well-behaved' structure on complex functions/series, their convergence poses no issue of immediate relevance by the very design of the method in question with an eye on the RH being a key focus. At any rate, developing a more general stance on comparing/ranking complex values would lie outside the intended scope (if at all attainable beyond conventions), with the RH agenda showing a prior reversal given that the search has been confined to a finite level—and a very specific one (zero ad hoc). It is fortunate and remarkable how all of the above strikes an orduale qualifier, with reference to higher-cost solutions boasting far lesser relevance (that which is unfeasible proves routinely irrelevant!)

Along these lines, please note in passing that the averaging approach—making heavy use of expectations or mode-like operators without the applied factors necessarily viewed as probabilistic weights when applied as a ratio (again right up the orduale alley) yet fully representable as such—need not stand up to the same scrutiny as average-value theorems do, again applied to the ever-unwinding, entangled issues of continuity and comparability. An *identity-based* method of inferring the candidate averaging factors suggest RH sufficiency that, factor ratio (orduale) applied, suggests necessity.

All things considered, one may come to wonder if it was reasonable to embark on '*super*symmetry' at the very stage of constructing the generalized, identity-based factors (referring to the implied ½-weights). This can be fixed by relaxing the *1/m* adjustments in (3) to arrive at an arbitrary, auxiliary $q$-distribution (or a set/space/distribution thereof):

$$\exists q\ \forall j = \overline{1, m}\colon\ \widehat{\pi_{k_j}}' = q_{k_j} * \frac{F}{f\left(x_{k_j}\right)} \quad (3')$$

As per the m=2 case of interest:

$$\forall q\colon \frac{\widehat{\pi_n}'}{\widehat{\pi_1}'} = \frac{f(x_1)}{f(x_n)} * \frac{1-q}{q} \equiv r * \frac{1-q}{q} \quad (3a')$$

$$\frac{T^{-s}}{1^{-s}} \equiv -\frac{T^{-s}}{1^{-s}} * \frac{1-q}{q} \quad (5')$$

It would be of importance to remark that virtually no restrictions are placed on the nature or values of the $q$-distribution, which need not even be normalized to [sub]unity, nor is forced to shun the complex domain. At this rate, whilst a sign-reversal (i.e. the -1 factor) is accounted for in the unity denominator, the $q$-factor is 'gobbled' by the [complex] zero numerator as long as this distribution is non-degenerate—which is assumed/imposed by the very nature of averaging (m>1). In effect, neither the very presence of nor any variations in the $q$-modifier affect anything in the grand $s$-solution, insofar as it is restrained to the form obtained above (postulated by RH).

## Afterthoughts: An Outlook (still further Straddling)

Any functional summation or integration can now be reworked along the averaging lines as proposed. An effective interval ($X$-$X_0$) taken in place of the set power N, it follows that:

$$\forall q, \forall dk = const\colon \sum_{k=x_0}^{x} f(k) \equiv F = [\pi(x_0) * f(x_0) + \pi(x) * f(x)] * (x - x_0)$$
$$\equiv \frac{F * q * x}{f(x_0) * (x - x_0)} * f(x_0) - \frac{F * (1-q) * x_0}{f(x) * (x - x_0)} * f(x)$$
$$= \frac{F * q * x}{(x - x_0)} - \frac{F * (1-q) * x_0}{(x - x_0)} \xrightarrow[\partial k \to 0]{} \int_{x_0}^{x} f(t)dt \equiv F(x) - F(x_0) \equiv F$$

By substituting a specific value for $x_0=0$, it obtains that:

$$\forall x\colon F(x) = \int_0^x f(t)dt = F * q \quad (9a)$$

More generally, by juxtaposing the respective terms, a functional recursion is discerned:

$$F(x) = \frac{F * q * x}{(x - x_0)} \pm const = \frac{[F(x) - F(x_0)] * q * x}{(x - x_0)} \pm const$$

$$F(x_0) = \frac{F * (1-q) * x_0}{(x - x_0)} \pm const = \frac{[F(x) - F(x_0)] * (1-q) * x_0}{(x - x_0)} \pm const \quad (9b)$$

On reconciling both the conditions around the implied [integration type] constant, one arrives at:

$$F(x) * [(1-q)x - x_0] + F(x_0) * q * x = F(x_0) * [q * x_0 + x] - F(x) * (1-q) * x_0$$

$$F(x) * [(1-q) * (x + x_0) - x_0] = F(x_0) * [q * (x_0 - x) + x]$$

$$F(x) * [(1-q) * x - q * x_0] = F(x_0) * [(1-q) * x + q * x_0]$$

$$\frac{F(x)}{F(x_0)} = \frac{[(1-q) * x + q * x_0]}{[(1-q) * x - q * x_0]} \quad (9c)$$

Evidently, a singular $q$-distribution only makes sense for a *point* estimate. That said, even a degenerate integral need not amount to zero structurally (unless either the initial function-value does and/or $q$ is assumed to make zero identically):

$$\int_{x_0}^{x} f(k)dk \xrightarrow[x \to x_0]{} F(x_0) * \frac{x_0}{(1-2q)x_0} - F(x_0) = F(x_0) * \frac{2q}{1-2q} \quad (9d)$$

Interestingly enough, this renders the *q-factor* a counterpart of a linear, *Mikusinski*-style (1953) differentiation *operator* (9e), even though in no manner does this imply linearity in or for *q* itself as such (9f).

$$dF(x_0) = \int_{x_0}^{x_0} f(t)dt = \frac{2q}{1-2q} * F(x_0) = f(x_0) = L^{-1}F(x_0) \quad (9e)$$

$$\frac{2q}{1-2q} \sim L^{-1} \quad \leftrightarrow \quad q = \frac{1}{2} * \frac{1}{1+L} = \frac{1}{2} * \frac{1/L}{1+1/L} = \frac{1}{2} * \left\{\frac{t^{-2}}{\Gamma(-1)}\right\}\{e^{-t}\} \quad (9f)$$

A characteristic composition of the sort again amounts to a resultant *interval* integral (convolution) as per the *q* in contrast to, or as implied from, the *q-factor* acting as *point*-integration, or a differential. Noteworthy, with the *t*-argument/parametrization taken to infinity amid lambda unitary, the *q*-resultant cancels out back to 1/2-symmetry (9g). It should for one come as no surprise that *q* generalizes the gamma/factorial function, given a somewhat *combinatorial/[quasi]probabilistic* nature of both (as, again, generalized by floating-basis invariance, or irrelevance on margin).

$$\{t^{-2}\}\{e^{-t}\} = \int_0^t (t-\tau)^{\lambda-1} e^{-t} d\tau \xrightarrow[\lambda\to -1, t\to\infty]{} \Gamma(-1) \quad (9g)$$

Other than the above, the scheme could apply to *primes*, e.g. when seen as *roughly* homogeneous/even distributions showing a *p(n)=2n+1* (odds-only) structure, in which event their sum is distributed as the square of N or its affine extensions. Alternatively, a prime could be formalized as a structure allowing for *no* [strongly] even distributions (additive decompositions) *within* one—the same naturally carrying over *between/across* these as an implied series. Which is another way of saying, any prime is representable as but a [flat/singular] series of none other than a null delta-step: *p=[1+0*(p-1)]*p*.

Qualifications aside, whilst it had been surmised that "almost any object-nature" would do when deployed or construed for the *(1-q)/q* construct, one might stay cautious about the very 'plus' operations, as present in a *set algebra* (logic-arithmetic procedure/axiomatization with *0='true'*) *a la Stoll* (1960). Ironically, the very pattern (*5'*), as argued from the outset, could build on any subset apart from the extreme corners of unity and infinity, e.g. as in (*5''*) suggesting an *m(k)* basis of a discretionary, floating nature. It is all the more remarkable that the general solution accommodating and elucidating RH proves *m(k)-invariant* and so unique as to posit its parts—*Re* versus *Im*—being closely intertwined. It happens, any finite factor like *(1-q)/q* would end up unitary effectively as a matter of one's failing to distinguish between |*Im(s)*|*=2ik*pi* versus 0 (if only as a *phenomenological convention* overlooking particular specifications or special narrowings of an otherwise complete ontological solution-structure), followed by this very unity being taken to the power of |*Re(s)*|*=1/2* thus implying a (-1) in the general (9h) pattern matching (8), *inter alia* corresponding to 'trivial' self-identity. This being a *single-shot* action, the latter pattern re-validates Stoll's logic while allowing for at least a *q-universe* of alternates (9i). Per q=1/2, this collapses to Stoll's axiomatization as a special case, even though, ironically, the generalized one would appear stemming from s=1 (accommodating (8) as an alternative or phenomenologically supportive scenario).

$$(\frac{k}{k \pm \Delta k})^{-s} \rightarrow \left(\frac{1-q}{q}\right)^{\pm\left(2ik\pi+\frac{1}{2}\right)\sim\pm\left(0+\frac{1}{2}\right)} * (\frac{k}{k \pm \Delta k})^{-s} \rightarrow -(\frac{k}{k \pm \Delta k})^{-s} \quad (5'')$$

$$X = -X, \qquad X + X = 0, \qquad X \equiv (\frac{k}{k \pm \Delta k})^{-s} \quad (9h)$$

$$0 = X + \frac{1-q}{q} X = \frac{X}{q} \quad (9i)$$

The central method strikes a logic balance between intuitionist/constructive versus decidability demonstrations. That said, it clearly shows promise of reaching far beyond that while bridging/revisiting areas as distant and diverse as: calculus foundations, numbers theory, logic, and probabilism.

## APPENDIX: Proving *Theorem/Lemma A*

Based on (1a), it is easy to see that,

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} [x + (k-1)\Delta] = nx + \Delta * \frac{(n-1)*n}{2} = \frac{x + [x + \Delta(n-1)]}{2} * n \equiv \frac{x_1 + x_n}{2} * n$$

QED.

## REFERENCES

Mikusinski, Jan. (1953). *Rachunek operatorow*. Warsaw.

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