Uncertainty in Multi-layer Radical Calculations and Inaccurate Numbers

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Abstract: In this paper, it is found that there is an error in the calculation of multi-layer radicals by solving a higher degree algebraic equation of one variable, and the order of power roots is higher and the number of layers of the power roots is larger, the error is bigger. The analysis shows that this phenomenon is uncertain, and hence the multi-layer radical number is defined as inaccurate numbers. The discovery and definition of inaccurate numbers after irrational and imaginary numbers enriches people's cognition of numbers.

Using the experimental observation of the established mathematical models, the calculation error of the inaccurate numbers is -0.0004~5.57‰ under the model conditions.

Keywords: multi-layer radicals, error, uncertainty, inaccurate number

Solving higher degree algebraic equations in one variable usually obtains equation solutions of multi-layer radicals[1][2][3], and calculating its numerical value finds that there are certain errors in the obtained results[3]. In order to find out its source, the errors of different radical solutions are compared, and it is found that the multi-layer radical form produces errors, and the order of power roots is higher and the number of layers of the power roots is larger, the error is bigger. The analysis shows that this phenomenon is of uncertainty and the multiple radical number is defined as inaccurate number. The results reported below.

1. Errors in solving algebraic equations

In order to observe the relationship between the source of error and the number of power root and the value in the power root, the following experiments were carried out in the form of two-layer and three-layer power roots (the data below are all values calculated by floating point 10-bits).

The equations that can be solved by the Cardan method are selected for discussion [4], and mathematical models are established:

$$x^3 + px + q = 0$$

$$x^5 + px^3 + \frac{1}{5}p^2x + q = 0$$

$$x^7 + px^5 + \frac{1}{7}p^2x^3 + \frac{1}{49}p^3x + q = 0$$

Let

$x = u + v$

A real general solution of the above mathematical model equations can be represented by two-layer radicals as:

$$x = \sqrt[n]{\frac{-q + \sqrt{q^2 - 4(\frac{p}{n})^n}}{2}} + \sqrt[n]{\frac{-q - \sqrt{q^2 - 4(\frac{p}{n})^n}}{2}}$$

Where: n is the degree of the equations.

1.1. The relationship between the power root and the error

The actual models are:

$$x^3 - 3x - 154451 = 0$$

$$x^5 - 5x^3 + 5x - 154451 = 0$$

$$x^7 - 7x^5 + 14x^3 - 7x - 154451 = 0$$

The real solutions obtained by the Cardan method are:

$$\sqrt[3]{\frac{154451 + \sqrt{154451^2 - 4(\frac{3}{3})^3}}{2}} + \sqrt[3]{\frac{154451 - \sqrt{154451^2 - 4(\frac{3}{3})^3}}{2}} = 53.68050216$$

$$\sqrt[5]{\frac{154451 + \sqrt{154451^2 - 4(\frac{5}{5})^5}}{2}} + \sqrt[5]{\frac{154451 - \sqrt{154451^2 - 4(\frac{5}{5})^5}}{2}} = 11.02319675$$

$$\sqrt[7]{\frac{154451 + \sqrt{154451^2 - 4(\frac{7}{7})^7}}{2}} + \sqrt[7]{\frac{154451 - \sqrt{154451^2 - 4(\frac{7}{7})^7}}{2}} = 5.724488722$$

The accurate solutions of the real roots of the three equations are 53.67199616, 11 and 5.69276706590322, respectively. The relative error rates for calculating the true value are:

$$\frac{(53.67199616 - 53.68050216)}{53.67199616} \cdot 10000 = -1.584811561$$

$$\frac{(11 - 11.02319675)}{11} \cdot 10000 = -21.08795455$$

$$\frac{(5.69276706590322-5.724488722)}{5.69276706590322}\cdot 10000=-55.72273664$$

1.2. The relationship between the q value in the radical and the error

The actual models are

$$x^5-5x^3+5x-3160100=0$$

$$x^5-5x^3+5x-154451=0$$

$$x^5-5x^3+5x-95050=0$$

The real solutions obtained by the Cardan method are:

$$\sqrt[5]{\frac{3160100+\sqrt{3160100^2-4(\frac{5}{5})^5}}{2}}+\sqrt[5]{\frac{3160100-\sqrt{3160100^2-4(\frac{5}{5})^5}}{2}}=19.94987437$$

$$\sqrt[5]{\frac{154451+\sqrt{154451^2-4(\frac{5}{5})^5}}{2}}+\sqrt[5]{\frac{154451-\sqrt{154451^2-4(\frac{5}{5})^5}}{2}}=11.02319675$$

$$\sqrt[5]{\frac{95050+\sqrt{95050^2-4(\frac{5}{5})^5}}{2}}+\sqrt[5]{\frac{95050-\sqrt{95050^2-4(\frac{5}{5})^5}}{2}}=9.998979486$$

The accurate solutions of the real roots of the three equations are 20, 11 and 10, respectively. The error rates of the true values are calculated as:

$$\frac{(20-19.94987437)}{20}\cdot 10000=25.06281500$$

$$\frac{(11-11.02319675)}{11}\cdot 10000=-21.08795455$$

$$\frac{(10-9.998979486)}{10}\cdot 10000=1.020514000$$

1.3. The relationship between p value and error within the radical

The actual models are

$$x^3-3x-240252=0$$

$$x^3-10x-240252=0$$

$$x^3-30x-240252=0$$

The real solutions obtained by the Cardan method are:

$$\sqrt[3]{\frac{240252+\sqrt{240252^2-4(\frac{3}{3})^3}}{2}}+\sqrt[3]{\frac{240252-\sqrt{240252^2-4(\frac{3}{3})^3}}{2}}=62.16639314$$

$$\sqrt[3]{\frac{240252+\sqrt{240252^2-4(\frac{10}{3})^3}}{2}}+\sqrt[3]{\frac{240252-\sqrt{240252^2-4(\frac{10}{3})^3}}{2}}=62.22487347$$

$$\sqrt[3]{\frac{240252+\sqrt{240252^2-4(\frac{30}{3})^3}}{2}}+\sqrt[3]{\frac{240252-\sqrt{240252^2-4(\frac{30}{3})^3}}{2}}=62.32773564$$

The accurate solutions of the real roots of the equations are 62.18247900, 62.22001267, and 62.32725140, respectively.

The error rates for calculating the true values are:

$$\frac{(62.18247900-62.16639314)}{62.18247900}\cdot 10000=2.586879819$$

$$\frac{(62.22001267-62.22487347)}{62.22001267}\cdot 10000=-0.7812277419$$

$$\frac{(62.32725140-62.32773564)}{62.32725140}\cdot 10000=-0.07769314210$$

1.4. Comparison of error multiples of the difference between the solutions of cubic, quintic and seventh equations and the accurate values

The actual models are:

$$x^3-3x-18=0$$

$$x^5-5x^3+5x-123=0$$

$$x^7-7x^5+14x^3-7x-843=0$$

The accurate value of a real root of all three equations is 3.

The error multiple between the difference of the quintic solution to its accurate value and the difference of the cubic solution to its accurate value is:

$$\frac{\sqrt[5]{\frac{123+\sqrt{123^2-4(\frac{5}{5})^5}}{2}}+\sqrt[5]{\frac{123-\sqrt{123^2-4(\frac{5}{5})^5}}{2}}-3}{\sqrt[3]{\frac{18+\sqrt{18^2-4(\frac{3}{3})^3}}{2}}+\sqrt[3]{\frac{18-\sqrt{18^2-4(\frac{3}{3})^3}}{2}}-3}=21.20000$$

The error multiple between the difference of the seventh solution to its accurate value and the difference of the cubic solution to its accurate value is:

$$\frac{\sqrt[7]{\frac{843+\sqrt{843^2-4(\frac{7}{7})^7}}{2}}+\sqrt[7]{\frac{843-\sqrt{843^2-4(\frac{7}{7})^7}}{2}}-3}{\sqrt[3]{\frac{18+\sqrt{18^2-4(\frac{3}{3})^3}}{2}}+\sqrt[3]{\frac{18-\sqrt{18^2-4(\frac{3}{3})^3}}{2}}-3}=540.2$$

1.5. Error comparisons of different radical layers

For quintic equations that can be solved by radicals

$$x^5+px+q=0$$

It is easy to know that one of its real roots is [1]:

$$\sqrt[5]{\frac{-q+\sqrt{q^2-4\cdot(\frac{5\cdot H^2-\sqrt{25\cdot H^4+4\cdot 5p}}{10})^5}}{2}}+\sqrt[5]{\frac{-q-\sqrt{q^2-4\cdot(\frac{5\cdot H^2-\sqrt{25\cdot H^4+4\cdot 5p}}{10})^5}}{2}}$$

where: H is a specific value, equivalent in this paper to a known real root.

Let the actual models are:

$$x^5-5x^3+5x-123=0$$

$$x^5-5x-228=0$$

The real solutions obtained by the Cardan method are:

$$\sqrt[5]{\frac{123+\sqrt{123^2-4(\frac{5}{5})^5}}{2}}+\sqrt[5]{\frac{123-\sqrt{123^2-4(\frac{5}{5})^5}}{2}}=3.000000106$$

$$\sqrt[5]{\frac{228+\sqrt{228^2-4(\frac{5\cdot 3^2-\sqrt{25\cdot 3^4-4\cdot 5\cdot 5}}{10})}}{2}}+\sqrt[5]{\frac{228-\sqrt{228^2-4(\frac{5\cdot 3^2-\sqrt{25\cdot 3^4-4\cdot 5\cdot 5}}{10})}}{2}}=3.001823779$$

The accurate solution to the real roots of the equations is 3.
The calculated error rates are:

$$\frac{(3-3.000000106)}{3}\cdot 10000=-0.0003533333333$$

$$\frac{(3-3.001823779)}{3}\cdot 10000=-6.079263333$$

2.Results

From the above results, it is clear that under the conditions of the established mathematical models, the larger the number of power roots, the bigger the error; the larger the q value, the bigger the error; the larger the p value, the smaller the error; the larger the number of power roots, the larger the error multiple; and the larger the number of radical layers, the bigger the error.

3.Conclusion

From the above discussion it can be concluded that:
Calculations in the form of multiple radicals with two or more layers are inaccurate.

This conclusion is called the uncertainty of the calculation of multiple radicals, and the number represented by it can be defined as inaccurate number.

4.Discussions

There are many discussions on the calculation accuracy of the solution of the radical solvable equations, such as the discussion on the root of a quartic equation [3], but there is not much error analysis on the root result. Zhou analyzed the accuracy of the two radical solutions of the quartic equation in single variable, and believed that the numerical accuracy obtained by solving the quartic equation in single variable by the matching square method was higher [3]. In this paper, it is found that the fundamental cause of the calculation error is caused by the multi-layer radical calculations. Under the conditions of the data model created in this paper, the error range is: -0.0004~5.57‰, which varies with the p, q values in the power root, the order of power root and the number of layers.

This paper discovers and defines a new class of numbers after irrational numbers and imaginary numbers—inaccurate numbers by discovering the uncertainty of multi-layer radical calculation, which enriches people's cognition of numbers. Inaccurate numbers are well worth further research to improve the reliability of data applications.

5.References

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