Probabilistic and Deterministic Approaches to some Problems of Number Theory

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Abstract. I suggest a probabilistic approach that helps to address some classical questions and problems of Number Theory, like the Goldbach Conjecture [1], distributions of twin- and d-primes, prime numbers among arithmetic sequences and others.
The concepts of ‘randomness’ and ‘independence’ relevant to number-theoretic problems are discussed here, and the basic concepts of divisibility of natural number are interpreted in terms of probability spaces and appropriate probability distributions on classes of congruence.
I analyze and demonstrate the importance of Zeta probability distribution and prove theorems stating the equivalence of probabilistic independence of divisibility of random integers by coprime factors, and the fact that random variables with the property of independence of coprime factors must have Zeta probability distribution. The idea to use Zeta distribution is motivated by the fact that it provides the validity of the probabilistic Cramér’s model for asymptotic prime number distribution, in full agreement with the Prime Number Theorem. Multiplicative and additive models with recurrent equations for generating sequences of prime numbers are derived based on the reduced Sieve of Eratosthenes Algorithm. This allows to interpret such sequences as realizations of random walks on set $\mathbb{N}$ of natural numbers and on multiplicative semigroups $S(\mathbb{P})$ generated by sets of prime numbers $\mathbb{P}$, representing paths of stochastic dynamical systems. The H. Cramér’s model for probability distribution of primes is modified as a *generalized predictable non-stationary Bernoulli process with unequally distributed terms* that are *asymptotically pairwise independent*. This model is applied then to analyze the sequences of primes generated by appropriate random walks. With intense use of Zeta probability distribution, it seems possible by using the modified Cramér’s model to approximate probability distribution of various arithmetic function.
Since probabilistic approach meets certain skepticism and even disbelief from a part of mathematicians working in traditional manner in Number Theory, I decided to attack the problem of Strong Goldbach Conjecture (SGC) from pure deterministic point of view.

As a result, I derived a recursive formula $\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P}+2\cdot(m-k)\right)\cap S_m\right]=G_m\mathbb{P}\neq\varnothing,\ m>3$

which generates a sequence of consecutive nonempty Goldbach sets $\left\{G_m\mathbb{P}\right\}_{m\geq 3}$,

where $\left\{G_m\mathbb{P}\right\}_{m\geq 3}$ denotes set of all prime numbers. The recursive formula justifies SGC by mathematical induction. Thus, this work represents two independent proofs of validity for Strong Goldbach Conjecture.

> *"...Mathematics is the art of giving the same name to different things...The only facts worthy of our attention are those which introduce order into this complexity and so make it accessible to us".*
>
> Henry Poincaré, The Value of Science, Random House, Inc., 2001.

1. Stochastic Predictable Sequences, Prime Numbers and Zeta Probability Distribution

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{P}$ the set of all primes. Our major assumption follows the amazing Cramér's idea [9] to represent a deterministic sequence of prime numbers as realizations of binary random variables ξ_k in the sequence $\left(\xi_k\right)_{k\in\mathbb{N}}$ with an appropriate choice of their probability distributions. Pursuing this goal, we address two problems:

1) the choice of an adequate probability distributions P_k for each ξ_k;

2) stochastic relationship among all ξ_k in the sequence $\left(\xi_k\right)_{k\in\mathbb{N}}$.

We need several definitions [7].

Definition 1.1

Let $\left\{v_n \mid n\in\mathbb{N}\right\}$ be random variables $v_n:\Omega\to\mathbb{N}$ defined on probablity space $\left(\Omega,\mathcal{F},P\right)$ and $\mathcal{F}_n=\sigma\left\{v_k \mid 1\leq k\leq n\right\}$ a σ-algebra generated by all events created by random variables $\left\{v_k \mid 1\leq k\leq n\right\}$. We have: $\mathcal{F}_1\subseteq\cdots\mathcal{F}_n\subseteq\mathcal{F}_{n+1}\subseteq\cdots\subseteq\mathcal{F}$, and for each $n\in\mathbb{N}$ random variable

ν_n is $\mathcal{F}_n$ -measurable. Then, the sequence $(\nu_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ is called a *stochastic sequence*.

A stochastic sequence $(\nu_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ is called *predictable* if for each $n \in \mathbb{N}$ there exists $k = k(n) < n$ such that ν_n is $\mathcal{F}_{k(n)}$-measurable. A pedictable sequence we can write as $(\nu_n, \mathcal{F}_{k(n)})_{n\in\mathbb{N}}$.

Predictability of a stochastic sequence $(\nu_n, \mathcal{F}_n)_{n\in\mathbb{N}} = (\nu_n, \mathcal{F}_{k(n)})_{n\in\mathbb{N}}$ means that for each $n \in \mathbb{N}$ the probability distribution P_n of ν_n given the entire prehistory $\mathcal{F}_{n-1}$ is compeletly determined by the condition $\mathcal{F}_{k(n)}$, that is it depends on values taken by some (or all) variables $\nu_1, \nu_2, \ldots, \nu_{k(n)}$, where $k(n) < n$. So, in terms of conditional probabilities,

$$P_n\{\nu_n \in A \mid \mathcal{F}_n\} = P_n\{\nu_n \in A \mid \mathcal{F}_{k(n)}\} \text{ for all } A \in \mathcal{F}_n \ . \tag{1.1}$$

Notice that general stochastic sequences include classes of sequences of independent as well as dependent random variables like martingales, Markov chains, etc.

A sequence of mutually independent random variables $(\nu_n, \mathcal{F}_n)_{n\in\mathbb{N}}$ is *unpredictable* since probability distribution of each ν_n is determined only by events from $\mathcal{F}_n$ and does not depend on condition given by 'previous' events from $\mathcal{F}_k$ $(k < n)$. Markov chains and martingales are examples of predictable stochastic sequences.

In Number Theory we are interested in recursively defined sequences of numbers, generated by certain recurrent relations, mostly nonlinear. From probabilistic point of view, such recurrent relations generate sequences of dependent random variables. The problem of dependence of events and random variables in the framework of Number Theory had been dicussed in some detail in the monograph of Mark Kac [4]. As M. Kac underlined in [4], the concept of independence "though of central importance in probability theory, is not a purely mathematical notion", and it appears quite naturally in Statistical Physics. He mentioned that "the rule of multiplication of probabilities of independent events is an attempt to formalize this notion and to build a calculus arount it". By using informal language, the concept of independence is stated in [14] as follows: "Two events are said to be independent if they have 'nothing to do' with each other". To decide whether a 'randomly choosen' (odd) integer $\nu > 2$ is a prime number, we subject ν to divisibility

test, by using the Eratosthenes algorithm. If event $A=\{p\setminus\nu\}$ (' p divides ν ') does not tell us anything about event $B=\{q\setminus\nu\}$ (' q divides ν ') for $p\neq q$, we can say that A and B do not depend on each either logically or statistically, and should be considered as independent events for a 'reasonable' choice of probability distribution of random variable ν. Meantime, events $\{\nu\in\mathbb{P}\}$ and $\{(\nu+1)\in\mathbb{P}\}$ are dependent events since they exclude each other for $\nu>2$, because only one of them holds true at a time.
More sofisticated example of dependent events represent $\{\nu\in\mathbb{P}\}$ and $\{(\nu+2)\in\mathbb{P}\}$, which are both true for twin primes, and false otherwise.
We demonstrate below that with an appropriate choice of probability distribution for random variable ν events $A=\{p_i\setminus\nu\}$ and $B=\{p_j\setminus\nu\}$ are independent for any choice of prime numbers $p_i\neq p_j$. Such a choice is provided by Zeta probability distribution

$$P\{\nu_m=n\}=\frac{n^{-s}}{\zeta(s)} \quad (s>1), \quad n\in\mathbb{N} \tag{1.2}$$

Both dependence and independence of 'events' in Number Theory are results of complicated recurrent nonlinear relations between terms of numeric sequences, which can generate 'dynamical chaos', imitating pseudo-randomness in a long run behavior of such deterministic sequences. The precise prediction of behavior of terms in the sequences demands for 'big' numbers almost infeasible calculations caused by the expanding memory of prehistory of their evolution. To make a study feasible and overcome "the curse of dependence" researchers in this area typically suggest heuristic assumptions that terms in $(\nu_n)_{n\in\mathbb{N}}$ are independent, or asymptotically independent, or uncorrelated, or 'weakly' dependent, in certain sense.

Proposition

The basic fact is that the set of prime numbers $\mathbb{P}$ is a *recursive set* [17].

Proof.

We can prove this by using an indicator function $I_{\mathbb{P}}:\mathbb{N}\to\{0,1\}$ of set $\mathbb{P}$. We need to show that the function $I_{\mathbb{P}}$ is *recursively defined*.

(1) *Initial step*: let $I_{\mathbb{P}}(2)=1$, $I_{\mathbb{P}}(3)=1$.

(2) *Inductive step*: if $n>3$ is the smallest number such that $k \nmid n$ for each $k \leq \sqrt{n}$ (symbol $\nmid$ means 'does not divide'), then $I_{\mathbb{P}}(n)=1$, otherwise $I_{\mathbb{P}}(n)=0$.

Notice that such number n exists since $\mathbb{N}$ is a well-ordered set, so that any nonempty subset of $\mathbb{N}$ has the least element (the smallest number).

(3) *Closure step*: Only numbers n obtained in steps (1) and (2) satisfy condition $I_{\mathbb{P}}(n)=1$.

It holds true that if a function is recursively defined then it is unique [17].

This means that set $\mathbb{P}$ of prime numbers recursively defined above is uniquely defined.

We can explain the above statement concerning the recursive definition of prime numbers as follows. Occurrence of a prime number $n=p\in\mathbb{P}$ in the sequence of consecutive natural numbers $n=\{2,3,4,\ldots\}$ depends on the values of reminders $r=\mathrm{mod}(n,p)$ for all primes $p\leq n$, due to the Sieve of Eratosphenes Algorithm [5]. This requirement can be relaxed: we need to consider only divisibility of n by all primes $p\leq\sqrt{n}$.

The proof of this statement (attributed to Fibonacci) is given below.

Lemma 1.1

A natural number $n\geq 5$ is prime if and only if n is not divisible by of any prime numbers $p\leq\sqrt{n}$, or, equivalently, if $r=\mathrm{mod}(n,p)\neq 0$ for all primes $p\leq\sqrt{n}$.

Proof.

If we assume that n is a composite number with no primes $p\leq\sqrt{n}$ that divide n, then n should be divided by primes p_1' and p_2' both greater than $\sqrt{n}$, and therefore also divided by their product $p_1'\cdot p_2'$. But this would imply that $p_1'\cdot p_2'>n$, which is impossible. This means that if n is not divisible by any of prime numbers $p\leq\sqrt{n}$, then n itself must be a prime number.

Q.E.D.

The above discussion implies that sequence of consecutive primes can be considered as a realization of a predictable stochastic sequence $\left(v_n,\mathcal{F}_{k(n)}\right)_{n\in\mathbb{N}}$, where $k(n)=\left[\sqrt{n}\right]$ for all $n>3$

($\left[x\right]$ stands for integer part of x).

One of the most challenging problems of Number Theory is the distribution of primes in the set $\mathbb{N}$ of natural numbers. The sequence of consecutive odd prime numbers $(3,5,7,11,\ldots)$ may look like a path of sporadic walks $\omega:\mathbb{N}\to\mathbb{P}$ given by a random sequence of natural numbers $\omega=\left(\nu_k(\omega)\,|\,k\in\mathbb{N}\right)$ where randomness of each term ν_j is determined by the choice of elementary event $\omega\in\Omega$ due to a probability distribution P defined by a probability space $\left(\Omega,\mathcal{F},P\right)$.

Primes in $\omega=\left(\nu_1,\nu_2\ldots,\nu_j,\ldots\right)$ for each $\nu_k=k$ can be represented by the indicator function $I_{\mathbb{P}}(k)=\xi_k$ as a sequence of binary-valued variables $\xi_k=\begin{cases}1, \text{ if } \nu_k(\omega)=k\in\mathbb{P}\\ 0, \text{ otherwise}\end{cases}$,

This can be directly observed in the sequence of prime numbers below 100:

$$\left(2\ 3\ 5\ 7\ 11\ 13\ 17\ 19\ 23\ 29\ 31\ 37\ 41\ 43\ 47\ 53\ 59\ 61\ 67\ 71\ 73\ 79\ 83\ 89\ 97\right)$$

Table 1.1

The sequence $\left(\xi(n)\,|\,1\le n\le 100\right)$ *of sequential primes among natural numbers from 1 to 100 represented by values of* n *such that* $\xi_k=1$ if k is prime*:*

0 1 1 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 1 0 0 0

0 0 1 0 0 0 0 0 1 0 1 0 0 0 0 0 1 0 0 0 1 0 1 0 0 0 0 0 1 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0

In Number Theory we are interested in recursive sequences of numbers, generated by certain recurrent relations, mostly nonlinear. Let $\left(\Omega,\mathcal{F},P\right)$ be a probability space, where Ω is a set of all $\mathbb{N}$-valued sequences, $\mathcal{F}$ is a σ-algebra generated by the algebra of cylinder sets in Ω, and P is a probability measure on $\left(\Omega,\mathcal{F}\right)$. Unsurmountable challenge is to describe probability distributions on a set $\Omega=\mathbb{N}^{\mathbb{N}}$ of all $\mathbb{N}$-valued sequences $\left(\nu_k\right)_{k\in\mathbb{N}}$ that include all recursively generated sequences of positive integers with all possible dependences between their terms. A sequence of integers in their natural increasing order $\left\{ v_k = k \middle| k \in \mathbb{N} \right\}$ is our main concern.

In the framework of Probability Theory, we consider basic sequences $\left(\xi(n) \mid n \in \mathbb{N}\right)$ as realizations of $\left(0,1\right)$-valued random variables traditionally called *Bernoulli variables*.

To avoid pure heuristic justification of probabilistic conclusions, we try to conduct our discourse entirely in the framework of Probability Theory. This means that, prior to discussion of dependence issues related to sequences like $\omega = \left(v_1, v_2 \ldots, v_j, \ldots\right)$, we should introduce random variables $v_j : \Omega \to \mathbb{N}$ with the corresponding probability distribution P defined on σ-algebra $\mathcal{F}$ of events $v_j^{-1}(A)$ (generated in our context by all finite subsets $A \subseteq \mathbb{N}$).

We assume that a binary-valued sequence $\left(\xi_k \mid k \in \mathbb{N}\right)$, where $\xi_k(\omega) = \begin{cases} 1, \text{ if } v_k(\omega) = k \in \mathbb{P} \\ 0, \text{ otherwise} \end{cases}$,

representing primes, is a realization of a non-stationary sequence of possibly *dependent* Bernoulli variables, by postulating probabilities

$$P\left\{\xi_k = 1\right\} = P_k, \quad P\left\{\xi_k = 0\right\} = Q_k = 1 - P_k, \text{ where } 0 \le P_k \le 1. \tag{1.3}$$

The major challenges in the study of such sequences are evaluation of P_n in (1.3) and analysis of dependence of random variables $\left(\xi_k \mid k = 1,2,3,\ldots\right)$ included in the sequence. As we mentioned above, problem of dependence of events and random variables in the framework of Number Theory had been discussed in some detail in the monograph of Mark Kac [4].

In number of works authors tried to avoid a standard probabilistic approach based on the concept of sigma-additive probability measures and the corresponding probability spaces, and considered so-called density measures, which are additive but not σ-additive. The notions of statistical (probabilistic) independence and dependence of events have been sometimes confused with functional or logical dependence. Both dependence and independence of "events" in Number Theory are results of complicated recursive nonlinear relations between terms of numeric sequences, which can generate a 'dynamical chaos', imitating pseudo-randomness in the long run behavior of purely 'deterministic' sequences. The precise prediction of behavior on a 'long run' for terms in such sequences demanding tremendous calculations requires expanding memory of prehistory of their evolution. To make a study feasible and overcome 'the curse of dependence', a typically suggested heuristic assumption is that terms in $\left(\xi_k\right)_{k\in\mathbb{N}}$ are 'asymptotically independent', or 'uncorrelated', or 'weakly' dependent in a certain sense.

In the framework of modified H. Cramér's model we show that the sequence of dependent not identically distributed random variables $\left(\xi_k\right)_{k\in\mathbb{N}}$ is *asymptotically pairwise independent* in a sense that we are going to discuss below.

Surprisingly, in many discussions of probabilistic interpretations of Number Theory problems, some authors use 'by default' a naive approach as in the following sentence:

"Assume that we choose number X at random from 1 to n. Then $\text{Prob}(X \text{ is prime}) = \dfrac{\pi(n)}{n}$ …".

The above sentence, due to its ambiguity, raises the following critical comments.

When one chooses number X "at random" in the sense of Probability Theory, it is presumed that the probability distribution of X exists and is known (at least theoretically).

The formula $\text{Prob}(X \text{ is prime}) = \dfrac{\pi(n)}{n}$ cited above tells us that the probability distribution is assumed to be uniform on the sequence of integers $\{1,2,3,\ldots,n\}$.

Here $\pi(n) = \#\{p\in\mathbb{P} \mid p\le n\}$ denotes a counting function of number of primes not exceeding n. If the probability distribution of X is not uniform on the interval of integers $[1,n] = \{1,2,\ldots,n\}$, then, in a statistical framework, $\dfrac{\pi(n)}{n}$ can be interpreted not as a probability but rather as an observed relative frequency of occurrences of prime numbers in the interval $[1,n]$.

One of goals in our study is to construct a probabilistic model for the "statistical" distribution of primes given by the observed frequencies $\dfrac{\pi(n)}{n}$. Notice here the obvious fact that a *discrete uniform probability distribution* does not exist on an infinite support, that is on infinite subsets of $\mathbb{N}$ (including $\mathbb{N}$ itself). The following analysis is about divisibility of ν by a prime $p\le n$.

Denote $p\cdot\mathbb{N}$ a set of all multiples of number p. As mentioned above, the probability $P\{\nu\in p\cdot\mathbb{N}\}$ does not exists if ν is evenly distributed on $\mathbb{N}$. But the problem can be easily resolved if one assigns the probability $P\{\nu \in p\cdot\mathbb{N}\}$ to the class $C_{p,0}=\{n \mid n=k\cdot p,\ k\in\mathbb{N}\}$ of integers in $\mathbb{N}$ congruent 0 modulo p. There are exactly p congruent classes modulo p:

$$C_{p,r}=\{n \mid n=k\cdot p+r;\, 0\le r\le p-1;\, k\in\mathbb{N}\cup\{0\}\},$$

which make a partition of $\mathbb{N}$. Then, we can define a probability distribution

$P(C_{p,r})=q_{p,r}$ $(r=0,1,2,\ldots,p-1)$ on $\{C_{p,0},C_{p,1},\ldots,C_{p,p-1}\}$ such that $\sum_{r=0}^{p-1} q_{p,r}=1.$

Then, $P\{\nu\in p\cdot\mathbb{N}\}=P(C_{p,0})=q_{p,0}$. By assuming equal probabilities to randomly choose a class of congruence for a number ν given by $P(C_{p,r})=q_p$ for all r: $0\le r\le p-1$, we have $P(C_{p,r})=\frac{1}{p}$,

where $C_{p,0}=p\cdot\mathbb{N}$. Then, $P\{\nu\in p\cdot\mathbb{N}\}=P(C_{p,0})=\frac{1}{p}$. Considering $P\{\nu\in p\cdot\mathbb{N}\}$, we have nothing but to assume that random variable ν can take any (unknown and unpredictable) value within a congruence class $p\cdot\mathbb{N}$. The value of probability can be different from $P\{\nu\in p\cdot\mathbb{N}\}=\frac{1}{p}$ if we impose some limitations on ν, say, if we assume that $\nu\le n$. For arbitrary $n\in\mathbb{N}$ and a given probability distribution of ν, an event $\{\nu\le n\}$ may not belong, in general, to the algebra of events created by the partition of $\mathbb{N}$ into p congruence classes $\{C_{p,r} \mid r=0,1,2,\ldots,p-1\}$, therefore, it would be impossible, in general, to assign probability to the event $\{\nu\in C_{p,o}\cap[1,n]\}$, where we denote $[1,n]=\{k \mid k=1,2,\ldots,n\}$. Since $[1,n]$ is a finite set, we can define a uniform probability distribution on this set, but the agreement of uniform distribution with the assumption $P\{C_{p,0}\cap[1,n]\}=\frac{1}{p}$ would depend on the choice of n, specifically, on divisibility of n by p. For example, if $p=3$ and $n=20$, we have $P\{C_{3,0}\cap[1,20]\}=\frac{6}{20}=0.3\ne\frac{1}{3}$.

For $p=3$ and $n=21$ we have:

$$C_{3,0}\cap[1,21]=\{3,6,9,12,15,18,21\}, \text{ and } P\{C_{3,0}\cap[1,21]\}=\frac{7}{21}=\frac{1}{3}\approx 0.333\ldots$$

Independence of divisibility of random number by different primes is determined by choice of probability distribution of ν. As it had been noticed by Mark Kac in [4], "primes play a game of chance". He pointed out to the obvious fact that ν to be divisible by both different primes p and q is equivalent of being divisible by $p \cdot q$. This mean that if $P\{C_{m,0}\} = \frac{1}{m}$ for any positive integer m, then, since $C_{p \cdot q,0} = C_{p,0} \cap C_{q,0}$, we have

$$P\left\{C_{pq,0}\right\} = P\left\{C_{p,0}\right\} \cdot P\left\{C_{q,0}\right\} \text{ because } \frac{1}{p \cdot q} = \frac{1}{p} \cdot \frac{1}{q}. \tag{1.4}$$

Mark Kac was not able to establish and use the independence of divisibility events in terms of probability theory since he used a density set functions $d\left(A\right) = \lim_{n \to \infty} \frac{A(n)}{n}$, where $A(n) = A \cap [0,n],\ A \subset \mathbb{N}$, which is not a probability measure as it is additive but not σ-additive.

Definition. 1.1

We call a probability distribution P_f on $\mathbb{N}$ of a random variable ν *multiplicative or completely multiplicative* if for all $A \subseteq \mathbb{N}$ we have:

$$P_f\left\{\nu \in A\right\} = \frac{1}{Z}\sum_{n \in A} f(n) \text{, where } f : \mathbb{N} \to (0,1], \tag{1.5}$$

is a *multiplicative*, *or respectively, completely multiplicative function,* such that $Z = \sum_{n \in \mathbb{N}} f(n)$ is a convergent series.

As we show below, independence of divisibility of random number ν by different primes can be guaranteed if ν has a multiplicative probability distribution defined above. Each prime number p determines a partition of the set $\mathbb{N}$ into p classes of congruence modulo p: $C_{p,r}$, where $r \in \left\{0,1,2,\ldots,p-1\right\}$. We show below that a randomly chosen value ν with multiplicative distribution P_f is divisible by natural m with probability $f(m)$.

For $f(n) = \frac{1}{n^s}$ $(s > 1)$ and $Z = \zeta(s)$ (where $\zeta(s)$ is Zeta function), the probability P_f on $\mathbb{N}$ is *Zeta probability distribution*

$$P_{\zeta(s)}\left\{\nu=n\right\}=\frac{n^{-s}}{\zeta(s)},\ n\in\mathbb{N},\text{ for any choice of } s>1$$

and random ν with Zeta distribution is divisible by a prime number p with probability

$f(p)=\dfrac{1}{p^s}$, so that for each $p\in\mathbb{P}$,

$$P_s\left\{\nu\in C_{p,0}\right\}=\frac{1}{p^s}\ ,\ P_s\left\{\nu\notin C_{p,0}\right\}=1-\frac{1}{p^s} \tag{1.6}$$

Each natural n, due to the Fundamental Theorem of Arithmetic, can be represented in the unique form

$$n=p_1^{a_1}\cdot p_2^{a_2}\cdots p_k^{a_k}=\prod_{i=1}^{k}p_i^{a_i} \tag{1.7}$$

where $p_1,p_2,\ldots,p_k$ are distinct primes, and $a_1,a_2,\ldots,a_k$ are natural numbers.

The formula (1.7) is called a *canonical representation* of n, where $a_1,a_2,\ldots,a_k$ are called *multiplicities* of prime factors of n.

If n is a realization of a random variable ν, that is $\nu(\omega)=n$, then (1.7) can be written in the form

$$\nu=\prod_{p\in\mathbb{P}}p^{\alpha(\nu,p)}=\prod_{k=1}^{\kappa(\nu)}p_k^{\alpha_k(\nu)} \tag{1.8}$$

as a canonical presentation of a random variable ν . Here $\alpha(\nu,p_k)=\alpha_k(\nu)$.

Thus, the probability that p does not divide ν equals $P_s\left\{\alpha(\nu,p)=0\right\}=1-\dfrac{1}{p^s}$.

In general, the event $\left\{\alpha(\nu,p)=k\right\}$ in (1.8) means that p^k divides ν but p^{k+1} does not divide ν :

$$P_s\left\{\alpha(\nu,p)=k\right\}=\left(\frac{1}{p^s}\right)^k\cdot\left(1-\frac{1}{p^s}\right),\ k=0,1,2,3,\ldots, \tag{1.9}$$

This shows that each $\alpha(p_j,\nu)$ has a *geometric distribution* with parameter $\left(\dfrac{1}{p^s}\right)$, and we have $E\alpha(\nu,p)=\dfrac{p^{-s}}{1-p^{-s}}=\dfrac{1}{p^{s}-1}$; $Var\left(\alpha(\nu,p)\right)=\dfrac{p^{-s}}{\left(1-p^{-s}\right)^{2}}=\left(\dfrac{p^{s}}{p^{s}-1}\right)\cdot\left(\dfrac{1}{p^{s}-1}\right)$.

Sum $\varphi(\nu)=\sum_{p\in\mathbb{P}}\alpha(\nu,p)$ counts the total number of prime factors (with their multiplicities) in the prime factorization of $\nu=\prod_{p\in\mathbb{P}}p^{\alpha(\nu,p)}$. Here are parameters of $\varphi(\nu)$:

$$E_{s}\left[\varphi(\nu)\right]=\sum_{p\in\mathbb{R}}\frac{1}{p^{s}-1}\,,\ Var_{s}\left[\varphi(\nu)\right]=\sum_{p\in\mathbb{R}}\left(\frac{p^{s}}{p^{s}-1}\right)\cdot\left(\frac{1}{p^{s}-1}\right).$$

Assume now that there is a vector $\vec{p}=\left(p_1,p_2,\ldots,p_N\right)$, which components are N different consecutive prime numbers, and we consider a multiplicative semigroup $S(\vec{p})$ with unity, generated by components of vector $\vec{p}$ and number 1.

For any $n\in S(\vec{p})$ we have $n=\prod_{i=1}^{k}p_i^{a_i}$ where $a_i>0$ for all i $(1\le i\le k)$, $k\le N$.

Notice that by using computer simulation, we can generate N pseudo-random variables $\alpha_j=\alpha(p_j,\nu), 1\le j\le N$, where each $\alpha(p_j,\nu)$ has a geometric distribution with parameter $\left(\dfrac{1}{p^{s}}\right)$ and then, simulate a 'pseudo-random' number $\nu=\prod_{j=1}^{k}p_j^{\alpha_j}$ with $k=k(\nu)\ \le\ N$.

Further we consider a multiplicative semigroup $S(\mathbb{P}_N)$ generated by all primes $\mathbb{P}_N$ not exceeding $N\in\mathbb{N}$, that is $\mathbb{P}_N=\left\{p\le N\mid p\in\mathbb{P}\right\}$.

THEOREM 1.1.

If P_f is a *multiplicative probability distribution* on $\mathbb{N}$ and ν is a random variable such that

$$P_f\left\{\nu\in A\right\}=\frac{1}{Z}\cdot\sum_{n\in A}f(n)\ \text{where}\ A\subseteq\mathbb{N},\ f:\mathbb{N}\to(0,1]\ ,$$

then

1) For any natural $m\ge 2$ random event E of occurrence of a random number ν divisible

by m has probability $P_f(E) = P_f(C_{m,0}) = f(m)$.

2) for any two mutually prime numbers m_1 and m_2, random events E_1 and E_2 of occurrence of ν divisible by both m_1 and by m_2, respectively, are P_f-independent events:

$$P_f(E_1 \cap E_2) = P_f(E_1) \cdot P_f(E_2).$$

Since $E_1 = C_{m_1,0}$, $E_2 = C_{m_2,0}$ and $E_1 \cap E_2 = C_{m_1 \cdot m_2,0}$ we have, equivalently,

$$P_f(C_{m_1,0} \cap C_{m_2,0}) = P_f(C_{m_1,0}) \cdot P_f(C_{m_2,0})$$

Proof.

For $m = m_1 \cdot m_2$ we have:

$$P_f\left(C_{m,0}\right) = \frac{1}{Z}\sum_{k\in\mathbb{N}} f(m\cdot k) = \frac{1}{Z}\sum_{k\in\mathbb{N}} f(m)\cdot f(k) = f(m) = f(m_1)\cdot f(m_2) \quad \text{since } \frac{1}{Z}\sum_{k\in\mathbb{N}} f(k) = 1,$$

and $P_f\left(C_{m_i,0}\right) = \frac{1}{Z}\sum_{k\in\mathbb{N}} f(m_i \cdot k) = f(m_i)$ $(i = 1,2)$. Then, $C_{m_1 \cdot m_2} = C_{m_1} \cap C_{m_2}$ implies

$$P_f(C_{m_1,0} \cap C_{m_2,0}) = P_f(C_{m_1 \cdot m_2,0}) = P_f(C_{m_1,0}) \cdot P_f(C_{m_2,0})$$

Q.E.D.

The following theorem states that the assumption that the probability distribution P_f on $\mathbb{N}$ is 'complete multiplicative' (with an appropriate choice of function f) is a necessary and sufficient condition for such distribution P_f to be Zeta probability distribution.

THEOREM 1.2.

Let ν be a random variable with values in $\mathbb{N}$ that follows probability distribution

$$P_f\left\{\nu \in A\right\} = \frac{1}{Z}\sum_{n\in A} f(n) \ , \tag{1.10}$$

where $f : \mathbb{N} \to [0,1]$, $A \subseteq \mathbb{N}$ and $Z = \sum_{n=1}^{\infty} f(n)$ is a convergent series.

The series $Z = \sum_{n=1}^{\infty} f(n)$ takes a form of the 'Euler product of the series' [12, p.230]:

1) if f in (1.5) is multiplicative, then $Z=\sum_{n=1}^{\infty} f(n)=\prod_{p\in\mathbb{P}}\left[1+f(p)+f(p^2)+\cdots\right]$;

2) if f in (1.5) is a completely multiplicative function such that $0<f(p)<1$ for all $p\in\mathbb{P}$, then

$$Z=\sum_{n=1}^{\infty} f(n)=\prod_{p\in P}\frac{1}{1-f(p)};$$

3) the probability distribution P_f is a Riemann Zeta distribution

$P_{\zeta(s)}\left\{v=n\right\}=\frac{n^{-s}}{\zeta(s)}$, $n\in\mathbb{N}$, for any choice of $s>1$. Further we denote $P_{\zeta(s)}=P_s$.

Proof.

1) Let $S(\mathbb{P}_N)$ be a semigroup of all integers generated by $\mathbb{P}_N\cup\{1\}$: $\mathbb{P}_N=\{p\mid p\le N, p\in\mathbb{P}\}$.

Due to the Fundamental Theorem of Arithmetic,

$$n=\prod_{p\in\mathbb{P}^*} p^{\alpha(n,p)},\ \text{where } \alpha(n,p)\ge 0,\ \alpha(n,p)=\begin{cases} a_j>0 \text{ if } p^{a_j}\mid n \text{ and } p^{a_j+1}\nmid n\\ 0, \text{ otherwise}\end{cases}$$

Then, if f is a multiplicative function, we have

$$Z=\sum_{n=1}^{\infty} f(n)=\sum_{n=1}^{\infty}\left[\prod_{p\in\mathbb{P}} f(p^{\alpha(n,p)})\right]=\prod_{p\in\mathbb{P}}\left[\sum_{k=0}^{\infty} f(p^k)\right]=\prod_{p\in\mathbb{P}}\left[1+f(p)+f(p^2)+\cdots\right].$$

2) In the proof above we have used the multiplicative property of function f.

If f is completely multiplicative, we have $f(p^k)=\left(f(p)\right)^k$. Then, we can write

$1+f(p)+\left(f(p)\right)^2+\left(f(p)\right)^3+\cdots=\frac{1}{1-f(p)}$ and the above equality takes a form:

$$Z=\sum_{n=1}^{\infty} f(n)==\prod_{p\in\mathbb{P}}\left[\sum_{k=0}^{\infty}\left(f(p)\right)^k\right]=\prod_{p\in\mathbb{P}}\frac{1}{1-f(p)}$$

Notice that the right-hand sides of the above equalities are convergent infinite products, since the left-hand side is given by the convergent series.

3) Notice that for any $n\in S(\mathbb{P}_N)$ we have $n^s=\prod_{p\in\mathbb{P}} p^{a(n,p)\cdot s}=\prod_{p\in\mathbb{P}} p^{a(p)\cdot s}$,

where $a(n,p)=a(p)\in\mathbb{N}\cup\{0\}$.

Since $\dfrac{1}{1-\dfrac{1}{p^s}}=\sum_{k=0}^{\infty}\left(\dfrac{1}{p^s}\right)^k$, we have $\zeta_{\mathbb{P}_N}(s)=\prod_{p\le N}\left[\sum_{k=0}^{\infty}p^{-sk}\right]=\sum_{\{a(p),p\le N\}}\prod_{p\le N}p^{-s\cdot a(p)}=\sum_{n\in S(\mathbb{P}_N)}n^{-s}$

Denote $\xi_p(\nu)=p^{\alpha(\nu,p)}$, $\xi=p$ .Then, $P\left\{\xi_{\nu,p}=p^{\alpha(\nu,p)}\right\}=\left[P\left\{\xi=p\right\}\right]^{\alpha(\nu,p)}$

For any natural m we write the event “ m divides ν” as $E=\left\{m\setminus\nu\right\}$ and the opposite event “ m does not divide ν“ as $\bar{E}=\left\{m\nmid\nu\right\}$. The probability that a prime number p divides ν is $P\{p\setminus\nu\}=f(p)$ and the probability that p does not divide ν is $P\{p\nmid\nu\}=1-f(p)$. The probability that the number ν divides p^k and does not divide p^{k+1} is given by the formula

$$P\left\{\left(p^k\setminus\nu\right)\cap\left(p^{k+1}\nmid\nu\right)\right\}=\left(f(p)\right)^k\cdot\left(1-f(p)\right)$$

Then, by virtue of Theorem 1.1 and the canonical factorization of n, we have

$$\begin{aligned}P\left\{\nu=n\right\}&=\prod_{p\in\mathbb{P}}P\left\{\left(p^{a(n,p)}\mid\nu\right)\cap\left(p^{a(n,p)+1}\nmid\nu\right)\right\}\\&=\prod_{p\in\mathbb{P}}\left[\left(f(p^{a(n,p)})\right)\cdot\left(1-f(p)\right)\right]=\prod_{p\in\mathbb{P}}\left[f(p)\right]^{a(\nu,p)}\cdot\prod_{p\in\mathbb{P}}\left[1-f(p)\right]\end{aligned}\qquad(1.11)$$

Summation of both sides of (1.11) results in the formula:

$$1=\sum_{n\in\mathbb{N}}P\left\{\nu=n\right\}=\prod_{p\in\mathbb{P}}\left(1-f(p)\right)\cdot\sum_{n\in\mathbb{N}}\prod_{p\in\mathbb{P}}\left[f(p)\right]^{\alpha(n,p)}$$, which implies:

$$\prod_{\nu\in\mathbb{N}}\frac{1}{1-f(p)}=\sum_{\nu\in\mathbb{N}}\prod_{p\in\mathbb{P}}f(p^{\alpha(n,p)})=\sum_{\nu\in\mathbb{N}}f\left(\prod_{p\in\mathbb{P}}p^{\alpha(n,p)}\right)=\sum_{\nu\in\mathbb{N}}f(n)=Z$$

$$1=\sum_{n\in\mathbb{N}}P\left\{\nu=n\right\}=\prod_{p\in\mathbb{P}}\left(1-f(p)\right)\cdot\sum_{n\in\mathbb{N}}\prod_{p\in\mathbb{P}}\left[f(p)\right]^{\alpha(n,p)}\qquad(1.12)$$

$P_s\left\{\nu=n\in\mathbb{P}\right\}$ $p'=p+d$ $p+p'$

provided that $f(n)$ is such that the infinite product and the infinite sum in the above formulas are both convergent. Completely multiplicative function $f:\mathbb{N}\to(0,1]$ satisfies the functional equation $f(x\cdot y)=f(x)\cdot f(y)$, known as one of ‘fundamental’ Cauchy functional equations. Due to Theorem 3, p.41 in [19] for positive x,y, it has the most general solution of the form $f(x)=e^{c\cdot\ln x}=x^c$. Obviously, in our context $f(n)=n^{-s}(s>1)$ is a completely multiplicative arithmetic function and for this choice of f, $Z(f)=\zeta(s)$ is Zeta function which generates Zeta probability distribution

$$P_{\zeta(s)}\left\{\nu = n\right\} = \frac{1}{n^s \cdot \zeta(s)},\ n \in \mathbb{N}.$$

Q.E.D.

Remark 1.1.

The problem with the choice $f(n) = \dfrac{1}{n}$ for $s = 1$ is that it leads to the divergent harmonic series $\zeta(1) = \sum_{n=1}^{\infty} \dfrac{1}{n}.$ To avoid the situation with the series divergence, we follow the steps of Euler [3] by restricting values of s to $s > 1$. Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \dfrac{1}{n^s}$ is well known to be directly related to the probability distribution of prime numbers. This motivates the choice of Zeta distribution. Due to the property of independence of divisibility for Zeta distribution, if p divides ν, then $\nu = p \cdot \nu'$ while the quotient $\nu' = \dfrac{\nu}{p}$ is again distributed over p classes of congruence $C_{p,r}$, and so on. A number $\nu = n$ is prime if and only if it does not divide all primes less than or equal to $\sqrt{n}$:

$$P_s\left\{\nu = n \in \mathbb{P}\right\} = P_s\left\{\bigcap_{p \le \sqrt{n}} \left[\alpha(\nu, p) = 0\right] \mid \nu = n\right\} = \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p^s}\right) \qquad (1.13)$$

For $p \in \mathbb{P}$ we have $P_s\left\{\alpha(\nu, p) = k\right\} = p^{-sk} \cdot \left(1 - p^{-s}\right)$ for all $k = 0, 1, 2, \ldots$.

In particular, $P_s\left\{\alpha(\nu, p) = 1\right\} = p^{-s} \cdot \left(1 - p^{-s}\right)$, and $P_s\left\{\alpha(\nu, p) = 0\right\} = 1 - p^{-s}$.

Then, the probability of $\left\{\nu = p_j \in \mathbb{P}\right\}$ is calculated as

$$P\left\{\nu = p_j\right\} = P_s\left\{\alpha(\nu, p_1) = 0, \ldots, \alpha(\nu, p_{j-1}) = 0, \alpha(\nu, p_j) = 1, \alpha(\nu, p_{j+1}) = 0, \ldots\right\}$$

$$= \left(\frac{1}{p_j^s}\right) \cdot \left(1 - \frac{1}{p_j^s}\right) \cdot \prod_{k \ne j} P_s\left\{\alpha(\nu, p_k) = 0\right\} = \left(\frac{1}{p_j^s}\right) \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_j^s}\right) = \frac{p_j^{-s}}{\zeta(s)}$$

Probability of $\left\{\nu = 1\right\}$, due to the canonical presentation (1.8), can be expressed as

$$P_s\{\nu=1\}=P_s\left\{\bigcap_{p\in\mathbb{P}}\{\alpha(1,p)=0\}\right\}=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)=\frac{1}{\zeta(s)}$$

In general, for any natural number $\nu=n=\prod_{p\in\mathbb{P}}p^{\xi_p}=p_1^{k_1}\cdot p_2^{k_2}\cdots p_m^{k_m}\cdots$, we have

$$P_s\{\nu=n\}=\prod_{p\in\mathrm{P}}\left[\left(\frac{1}{p^s}\right)^{\alpha(n,p)}\cdot\left(1-\frac{1}{p^s}\right)\right]=\prod_{p\in\mathrm{P}}\left(\frac{1}{p^s}\right)^{\alpha(n,p)}\cdot\prod_{p\in\mathrm{P}}\left(1-\frac{1}{p^s}\right)=\frac{1}{n^s}\cdot\zeta^{-1}(s),$$

that is

$$P_s\{\nu=n\}=\frac{n^{-s}}{\zeta(s)} \tag{1.14}$$

Formula (1.14) may provide some probabilistic interpretations of Riemann Zeta function. If ν has Zeta probability distribution, then the probability that $\nu(\omega)$ for certain ω results in a prime number is evaluated as

$$P_s\{\nu\in\mathbb{P}\}=\frac{1}{\zeta(s)}\sum_{p\in\mathbb{P}}p^{-s} \tag{1.15}$$

Notice that formula (1.13) does not provide 'reasonable' values of probabilities for specific realizations of ν. For example, it is not equal to zero for any composite value of ν, say for even ν. Actually, as we show further, formula (1.13) gives satisfactory predictions of asymptotic values of probability $P_s\{\nu=n\in\mathbb{P}\}$ as $n\to\infty$.

Since $\{\nu\le n\}=\bigcup_{i=1}^{n}\{\nu=i\}$, we have

$$P_s\{\nu\le n\}=\frac{\sum_{k=1}^{n}k^{-s}}{\zeta(s)} \text{ and } P_s\{(\nu\in\mathbb{P})\cap(\nu\le n)\}=P_s\{\nu=p\in\mathbb{P}_n\}=\frac{\sum_{p\in\mathbb{P}_n}p^{-s}}{\zeta(s)},.$$

We compare (Table 3.1) frequency estimate $\frac{\pi(n)}{n}$ with probability in (1.13) and with the Cramér's model prediction $\frac{1}{\ln n}$, though, dependence of probability P_s on parameter $s>1$ makes the above formulas harder to interpret.

We know that one can circumvent divergence of $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $s \le 1$ by using the analytic continuation of $\zeta(z)$ on the complex plane $\mathbb{C}$, as suggested by B. Riemann.

Meanwhile, as we have mentioned above, the use of *Incomplete Product Zeta* function (*IPZ*) $\zeta_{\mathbb{P}_N}(s)$ defined as a partial product of $\zeta(s)$, $\zeta_{\mathbb{P}_N}(s)=\prod_{p\le N}\frac{1}{1-\frac{1}{p^s}}$,

provides another opportunity to deal with the divergence of $\zeta(1)=\prod_{p\in\mathbb{P}}\frac{1}{1-\frac{1}{p^s}}$ for $s=1$.

Lemma 1.2

Let $S(\mathbb{P}_N)$ be a semigroup of all integers generated by $\mathbb{P}_N \cup \{1\}$,

$\mathbb{P}_N=\{p \mid p\le N, p\in\mathbb{P}\}$.

Then,

$$\zeta_{\mathbb{P}_N}(s)=\sum_{n\in S(\mathbb{P}_N)} n^{-s}.$$

Proof.

Notice that for any $n\in S(\mathbb{P}_N)$ we have $n^s=\prod_{p\in\mathbb{P}} p^{\alpha(p)\cdot s}$, where $\alpha(p)\in\mathbb{N}\cup\{0\}$.

Since $\frac{1}{1-\frac{1}{p^s}}=\sum_{k=0}^{\infty}\left(\frac{1}{p^s}\right)^k$, we have $\zeta_{\mathbb{P}_N}(s)=\prod_{p\le N}\left[\sum_{k=0}^{\infty}p^{-sk}\right]=\sum_{\{a(p),p\le N\}}\prod_{p\le N}p^{-s\cdot a(p)}=\sum_{n\in S(\mathbb{P}_N)}n^{-s}$.

Q.E.D.

Lemma 1.2

If ν follows Zeta distribution P_s, then

$$P_s\{\nu\in\mathbb{P}\}=\prod_{p<\sqrt{\nu}}\left(1-\frac{1}{p^s}\right). \tag{1.16}$$

Proof.

By using the recursive property of the sequence of prime numbers $p\le\nu$ with the memory size $\sqrt{\nu}$ and the property of independence of divisibility for Zeta distribution,

we have $P\left\{\nu \in \mathbb{P} \mid \nu = n\right\} = P\left\{\bigcap_{p \le \sqrt{n}} \left\{p \nmid n\right\}\right\} = \prod_{p \le \sqrt{n}} P_s\left\{p \nmid n\right\} = \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p^s}\right)$,

which means that $\nu = n$ is prime if and only if any prime $p \le \sqrt{n}$

does not divide n. Formally, $\nu = n$ is prime if and only $r = \mathrm{mod}(n, p') \ne 0$ for all primes $p' \le \sqrt{n}$.

This implies that if $\nu > 5$ follows Zeta probability distribution then

$$P_s\left\{\, \nu \text{ is prime } \right\} = \prod_{p \le \sqrt{\nu}} \left(1 - \frac{1}{p^s}\right).$$

Q.E.D.

Lemma 1.3

Let $S(\mathbb{P}_N)$ be a semigroup of all integers generated by $\mathbb{P}_N \cup \left\{1\right\}$,

$\mathbb{P}_N = \left\{p \mid p \le N, p \in \mathbb{P}\right\}$.

Then,

$$\zeta_{\mathbb{P}_N}(s) = \sum_{n \in S(\mathbb{P}_N)} n^{-s}.$$

Proof.

Notice that for any $n \in S(\mathbb{P}_N)$ we have $n^s = \prod_{p \in \mathbb{P}} p^{\alpha(p) \cdot s}$, where $\alpha(p) \in \mathbb{N} \cup \left\{0\right\}$.

Since $\dfrac{1}{1 - \dfrac{1}{p^s}} = \sum_{k=0}^{\infty} \left(\dfrac{1}{p^s}\right)^k$, we have $\zeta_{\mathbb{P}_N}(s) = \prod_{p \le N} \left[\sum_{k=0}^{\infty} p^{-sk}\right] = \sum_{\left\{a(p), p \le N\right\}} \prod_{p \le N} p^{-s \cdot a(p)} = \sum_{n \in S(\mathbb{P}_N)} n^{-s}$.

Q.E.D.

2. Multiplicative and Additive Recurrent models for Primes

The famous Harald Cramér's model [2,3] describes the occurrence of prime numbers as a *sequence of independent Bernoulli variables* with probabilities

$$P\left\{\xi(n) = 1\right\} = \frac{1}{\ln n},\ \ P\left\{\xi(n) = 0\right\} = 1 - \frac{1}{\ln n},\ \text{ where } n \ge 2. \tag{2.1}$$

Notice that similar to (1.16), formula (2.1) is valuable only asymptotically for distribution of primes. In what follows, we provide rigorous arguments in support of Cramér's model, related to the values of probabilities $q_n = \frac{1}{\ln n}$, and then analyze dependence of $\xi(n)$ in the sequence $\left(\xi(n) \mid n = 1,2,\ldots\right)$. As we have discussed above, appearance of prime $n = \nu(k)$ in the sequence $\left\{\nu(k) = k \mid k \in \mathbb{N}\right\}$ are dependent events determined by the prehistory $\mathcal{F}_{\sqrt{n}} = \sigma\left\{\nu(k) \mid 1 \le k \le \sqrt{n}\right\}$. Obviously, if $\nu(n) = p \in \mathbb{P}$, then $\nu(n+1) = p+1 \notin \mathbb{P}$ since $p+1$ is an even number. Even if we restrict values of $\nu(n)$ to odd numbers $2k+1$, still divisibility of $\nu(2k+1) = n$ by the previously occurred primes would depend on the prehistory $\mathcal{F}_{\sqrt{n}}$. Therefore, the sequence of consecutive primes and the corresponding Bernoulli variables $\xi(n)$ cannot be interpreted as occurrence of independent events in the sequence, or as a realization of a Markov chain with a constant size of memory, because for each $\nu(k) = n$ the size $[\sqrt{n}]$ of the memory $\mathcal{F}_{\sqrt{n}}$ increases in the sequence with n.

We analyze the sequence of prime numbers $\left\{\nu(k) = p \mid p \in \mathbb{P}, k \in \mathbb{N}\right\}$ by using *multiplicative and additive models*. In any kind of a model, we will be using the equivalent canonical realizations

$$\left(\Omega, \mathcal{F}, P\right) = \left(X^T, \mathcal{B}^T, P_X\right) \text{ so that } \nu(\omega, t) = \nu(t) \ .$$

The transformations $\theta_t : X^T \to X^T$, $t \in T$, are $\mathcal{B}^T / \mathcal{B}^T$ -measurable.

We define the transformations by $\theta_s \nu(t) = \nu(t+s)$, for $s,t \in T$.

A *multiplicative model* is based on the *canonical representation of primes* [5, p.18]:

$$n = \prod_{p \in \mathbb{P}} p^{\alpha(n,p)} \text{ where } \alpha(n,p) = \begin{cases} \alpha_p > 0 \text{ if } p \text{ divides } n \\ 0, \text{ otherwise} \end{cases} \tag{2.2}$$

and is concerned with the questions of divisibility of integer-valued random variables by integers, and with their connection to Zeta probability distribution:

$$P_s\left\{\nu \in A\right\} = \frac{1}{\zeta(n)} \cdot \sum_{n \in A} \frac{1}{n^s} \text{ , for any subset } A \subseteq \mathbb{N} . \tag{2.3}$$

For the multiplicative model of the dynamical system representing (2.2), where $\nu = n$, we define

$$\theta_i \nu = \nu(i);\ \nu(0) = 1, \theta_{i+1}\nu = \theta_i \nu \cdot \eta(i+1),$$

where $\eta(n+1) = p_{n+1}^{\alpha_{n+1}(\nu)} (i = 0,1,2,\ldots,\kappa(\nu)-1)$.

Additive models are useful in problems related to counting of various types of integers in $\mathbb{N}$.

In additive models dynamical systems are defined by the equations:

$$\theta_i \nu = \nu(i); \nu(0) = 0, \theta_{i+1}\nu = \theta_i \nu + \xi(i+1),$$

where definition of the 'updating' term $\xi(n+1)$ determines the specifics of the model, as illustrated below. First, we consider the function $\pi(x)$, counting the number of primes less than or equal to x. Second, for all $m \geq 3$ we consider the number $G(2m)$ of *Goldbach m-primes,* or G_m*-primes*,

which are such primes p that a difference $2m - p$ is again a prime number.

In the first situation we use recurrent equations:

$$\begin{cases} \pi(1) = 0 \\ \pi(n+1) = \pi(n) + \xi(n+1),\ n \in \mathbb{N} \end{cases} \tag{2.7}$$

It is well-known that the connections between additive and multiplicative properties of numbers are extraordinarily complicated, and this leads to various difficult problems in Number Theory. We start from the *division algorithm* [5, p.19]. Given integer n and $m > 0$ there exists a unique pair of integers k and r such that $n = mk + r$, with $0 \leq r \leq m$. In this equation, $r = 0$ if and only if m divides n. We derive here a recursive formula generating a sequence of prime numbers: $2, 3, 5, 7, \ldots$ For any prime number $p \in \mathbb{P}$ and a natural number $n \geq 2$, consider a function $\operatorname{mod}(n, p) = r$ of residuals (remainders) such that $n = m \cdot p + r,\ 0 \leq r < p$, where $m \in \mathbb{N} \cup \{0\}$. Consider a vector of consecutive prime numbers $\vec{p}(n) = (p_1, p_2, \ldots, p_k)$ such that $p_k \leq n$ and $p_{k+1} > n$. Index k determines here the value $\pi(n) = k$ for the number of primes less than or equal to n so that $\vec{p}(n) = \left(p_1, p_2, \ldots, p_{\pi(n)}\right)$. For each coordinate p_i of vector $\vec{p}(n)$ we determine the residual value $r_i = \operatorname{mod}(n, p_i),\ i = 1, 2, \ldots, \pi(n)$, and consider vector of residuals $\vec{r}(n) = \left(r_1, r_2, \ldots, r_{\pi(n)}\right)$. Notice that, due to the Sieve Algorithm, for an integer $n > 2$ to be prime it is necessary and sufficient that all coordinates r_i of the 'reduced' vector of residuals $\vec{r}(n)$ such that $1 \le i \le \pi\left(\sqrt{n}\right)$ be different from zero. Thus, the events

$$\left\{ \min_{i \le \pi(\sqrt{\nu})} \left\{ r_i \mid r_i = \mathrm{mod}(\nu, p_i) \right\} > 0 \right\} \text{ and } \left\{ \nu \in \mathbb{P} \right\} \text{are equivalent.}$$

See calculations below in the Table 2.1.

Table 2.1. *The recursive sequence of primes driven by their residuals*

n	$\pi(n)$	$\vec{p}(n) = \left(p_1, p_2, \ldots, p_{\pi(n)}\right)$	$\vec{r}(n) = \mathrm{mod}\left(n, \vec{p}(n)\right) = \left(r_1, r_2, \ldots, r_{\pi(n)}\right)$
2	1	(2)	(0)
3	2	$(2,3)$	$(1,0)$
4	2	$(2,3)$	$(0,1)$
5	3	$(2,3,5)$	$(1,2,0)$
6	3	$(2,3,5)$	$(0,0,1)$
7	4	$(2,3,5,7)$	$(1,1,2,0)$
8	4	$(2,3,5,7)$	$(0,2,3,1)$
9	4	$(2,3,5,7)$	$(1,0,4,2)$
10	4	$(2,3,5,7)$	$(0,1,0,3)$
11	5	$(2,3,5,7,11)$	$(1,2,1,4,0)$
12	5	$(2,3,5,7,11)$	$(0,0,2,5,1)$
13	6	$(2,3,5,7,11,13)$	$(1,1,3,6,2,0)$
…	…	…	…
30	10	$(2,3,5,7,11,13,17,19,23,29)$	$(0,0,0,2,8,4,13,11,7,1)$
31	11	$(2,3,5,7,11,13,17,19,23,29,31)$	$(1,1,1,3,9,5,14,12,8,2,0)$

We evaluate $P\left\{\nu \in \mathbb{P} \mid \nu = n\right\}$ assuming that a random integer ν follows Zeta probability distribution.

To assign a probability value to a set $m \cdot \mathbb{N}$ ("all multiples of number m"), we should refer it to the class $C_{m,0} = \{n \mid n = k \cdot m,\, k \in \mathbb{N}\}$ of integers in $\mathbb{N}$ congruent 0 modulo m so that $C_{m,0} = m \cdot \mathbb{N}$. There are exactly m congruent classes modulo m: $C_{m,r} = \{n \mid n = r + k \cdot n,\, k \in \mathbb{N} \cup \{0\}\},\, 0 \le r \le m-1$, which make a finite partition of $\mathbb{N}$. Then, for each integer $m > 1$ we can define a probability distribution on $\{C_{m,0}, C_{m,1}, \ldots, C_{m,m-1}\}$:

$$P\{\nu \in C_{m,r}\} = q_{m,r} \ge 0,\, 0 \le r \le m-1 \quad \text{and} \quad \sum_{r=0}^{m-1} q_{m,r} = 1,\, m = 2,3,4\ldots$$

Theorem 2.1

Let ν be a random variable with Zeta probability distribution P_s and

$$\nu = \prod_{p \in \mathbb{P}} p^{\alpha(\nu,p)} = \prod_{k=1}^{\kappa(\nu)} p_k^{\alpha_k(\nu)} \tag{2.8}$$

its canonical representation. Then, each random variable $\alpha(\nu, p)$ in (2.8) has geometric probability distribution with a parameter $u = \dfrac{1}{p^s}$ $(0 < u < 1)$:

$$P_s\{\alpha(\nu,p) = a\} = u^a \cdot (1-u) = \left(\frac{1}{p^s}\right)^a \cdot \left(1 - \frac{1}{p^s}\right), \quad a = 0,1,2,\ldots \tag{2.9}$$

We have then,

$$E[\alpha(\nu,p)] = \frac{1-u}{u} = p^s - 1,\; Var[\alpha(\nu,p)] = \frac{1-u}{u^2} = p^{2s} - p^s .$$

Variables $\alpha_k(\nu) = \alpha(\nu, p_k)$ are independent for all primes p_k $(k = 1,2,\ldots,\kappa(\nu))$ as well as factors $p_k^{\alpha(\nu,p_k)}$ and $p_j^{\alpha(\nu,p_j)}$ for all $k \ne j$ in the canonical factorization $\nu = \prod_{p \in \mathbb{P}} p^{\alpha(\nu,p)}$.

Proof.

Denote $a \backslash b$ and $a \nmid b$ events 'a divides b' and 'a does not divide b', respectively.

We have:

$$P_s\{(p^k \backslash \nu) \cap (p^{k+1} \nmid \nu)\} = P_s\{p^k \backslash \nu\} - P_s\{p^{k+1} \backslash \nu\} \;\text{ since }\; \{p^{k+1} \backslash \nu\} \subset \{p^k \backslash \nu\}.$$

Notice that $P_s\left\{p^k \setminus \nu\right\} = P_s\left\{\nu \in p^k \cdot \mathbb{N}\right\} = \frac{1}{\zeta(s)} \cdot \sum_{m\in\mathbb{N}} \frac{1}{\left(p^k \cdot m\right)^s} = \left(\frac{1}{p^s}\right)^k \cdot \frac{1}{\zeta(s)} \cdot \sum_{m\in\mathbb{N}} \frac{1}{m^s} = \left(\frac{1}{p^s}\right)^k$.

Therefore,

$$P\left\{\left(p^{\alpha(\nu,p))} \setminus \nu\right) \cap \left(p \nmid \frac{\nu}{p^{\alpha(\nu,p)}}\right)\right\} = P\left\{\left(p^{\alpha(\nu,p)} \setminus \nu\right) \cap \left(p^{\alpha(\nu,p)+1} \nmid \nu\right)\right\} = \left(\frac{1}{p^s}\right)^{\alpha(\nu,p)} \cdot \left(1 - \frac{1}{p^s}\right).$$

Denote $E_m = C_{m,0}$ the event $\left\{m \setminus \nu\right\}$ ('m divides ν'). Then, for $m = m_1 \cdot m_2$ we have

$$P_{\zeta(s)}\left(E_m\right) = P_{\zeta(s)}\left(C_{m,0}\right) = \sum_{k\ge 11} \frac{\left(m \cdot k\right)^{-s}}{\zeta(s)} = \frac{1}{m^s} \sum_{k\ge 1} \frac{k^{-s}}{\zeta(s)} = \frac{1}{m^s} = \frac{1}{m_1^{\,s} \cdot m_2^{\,s}},$$

$$P_{\zeta(s)}\left(C_{m_i,0}\right) = \sum_{k\ge 11} \frac{\left(m_i \cdot k\right)^{-s}}{\zeta(s)} = \frac{1}{m_i^s} \sum_{k\ge 1} \frac{k^{-s}}{\zeta(s)} = \frac{1}{m_i^s} \ (i = 1, 2).$$

Similar,

$$P_{\zeta(s)}\left(C_{m_1 \cdot m_2,0}\right) = \sum_{k\ge 11} \frac{\left(m_1 \cdot m_2 \cdot k\right)^{-s}}{\zeta(s)} = \sum_{k\ge 1} \frac{k^{-s}}{\zeta(s)} \cdot \frac{1}{m_1^s \cdot m_2^s} = \frac{1}{m_1^s \cdot m_2^s}.$$
$$= P_{\zeta(s)}\left(C_{m_1,0}\right) \cdot P_{\zeta(s)}\left(C_{m_2,0}\right)$$

If m_1 and m_2 are co-prime numbers, then $C_{m_1 \cdot m_2,0} = C_{m_1} \cap C_{m_2}$, that is $E_{m_1 \cdot m_2} = E_{m_1} \cap E_{m_2}$,

and $P_s\left(E_{m_1} \cap E_{m_2}\right) = P_s\left(E_{m_1}\right) \cdot P_s\left(E_{m_2}\right)$, which holds true for any two different primes

$m_1 = p_1$ and $m_2 = p_2$. This proves independence of $\alpha(p,\nu)$ for different primes p,

as well as independence of factors $p_i^{\alpha(\nu,p_i)}$ and $p_j^{\alpha(\nu,p_i)}$ for all $i \ne j$ in the canonical

factorization $\nu = \prod_{p\in\mathbb{P}} p^{\alpha(p,\nu)}$.

Q.E.D.

Theorem 2.2

Random variables $\nu(i),\ i = 0,1,2,...$ with Zeta distribution

$$P_s\left\{\nu(i) = n\right\} = \frac{n^{-s}}{\zeta(s)},\ s > 0,\ \ n \in \mathbb{N}$$

represents a random walk $\left\{\nu(i)\,|\,0\le i\le\kappa(\nu)\right\}$ on a multiplicative semigroup

$S(\mathbb{P}^*)$ generated by the extended set of primes $\mathbb{P}^*=\mathbb{P}\cup\left\{1\right\}$.

The walk on $\mathbb{P}^*$ is defined recursively as follows:

$$\begin{cases}\nu(1)=\nu(0)\cdot\eta(1), \text{ where } \nu(0)=1,\ \eta(1)=p_1^{\alpha_1(\nu)}\\ \nu(i+1)=\nu(i)\cdot\eta(i+1), \text{ where } \eta(i+1)=p_{i+1}^{\alpha_{i+1}(\nu)}(i=0,1,2,\ldots,\kappa(\nu)-1)\end{cases} \tag{2.10}$$

The sequence $\left\{\nu(i)\,|\,0\le i\le\kappa(\nu)\right\}$ is a finite walk on $S(\mathbb{P}^*)$ with independent

multiplicative increments $\eta(i)=p_i^{\alpha_i(\nu)}$ such that $P\left\{\eta(i)=p_i^{a_i}\right\}=\left(\frac{1}{p_i^s}\right)^{a_i}\cdot\left(1-\frac{1}{p_i^s}\right)$,

and $\kappa(\nu)\le\log_{p_{\min}}\nu=\dfrac{\ln\nu}{\ln p_{\min}}$, where $p_{\min}$ is the least prime number that divides ν.

Proof.

Formulas (1.7) and (1.9) imply: $\nu=\prod_{p\in\mathbb{P}}p^{\alpha(\nu,p)}=\left(\prod_{p:\,\alpha(\nu,p)=0}1\right)\cdot\left(\prod_{p:\,\alpha(\nu,p)>0}p^{\alpha(\nu,p)}\right)=\prod_{k=1}^{\kappa(\nu)}p_i^{\alpha_i}$

Since $\xi(i)=p_i^{\alpha_i}$ and all $\alpha_i=\alpha(\nu,p_i)$, due to Theorem 1, are independent random variables each with

geometric distribution, we have $P\left\{\eta(i)=p_i^{a_i}\right\}=\left(\frac{1}{p_i^s}\right)^{a_i}\cdot\left(1-\frac{1}{p_i^s}\right)$,

were $i=1,2,\ldots,n,$ so that $\nu(n)=\prod_{i=1}^{n}\eta(i)$ for all $n:\ 1\le n\le\kappa(\nu)$ and $\nu(n)=\nu$ if $n=\kappa(\nu)$.

Thus, $P\left\{\nu=m\right\}=\prod_{i=1}^{\kappa(m)}\left(\frac{1}{p_i^s}\right)^{\alpha_i}\cdot\prod_{i=1}^{\infty}\left(1-\frac{1}{p_i^s}\right)=\frac{1}{m^s}\cdot\frac{1}{\zeta(s)}$ since $m=\prod_{i=1}^{\kappa(m)}p_i^{\alpha_i}$.

Since $m=\prod_{i=1}^{\kappa(m)}p_i^{\alpha_i}\ge\left(p_{\min}\right)^{\kappa(m)}$, where $p_{\min}\le p_i$ for all $i:\ 1\le i\le\kappa(m)$, we have: $\kappa(m)\le\log_{p_{\min}}m$

Q.E.D

Theorem 2.3

Let $h:R\to\left\{0,1\right\}$ be the Heaviside function $h(x)=\begin{cases}1 \text{ if } x>0\\0 \text{ if } x\le 0\end{cases}$, $\vec{r}(\nu)=\left(r(\nu_i)\,|\,1\le i\le\pi\left(\sqrt{\nu}\right)\right)$

a vector of residuals $r(\nu_i) = \mathrm{mod}(\nu, p_i)$, and $\rho(\nu) = \min\left(\vec{r}(\nu)\right) = \min_i \left(r(\nu_i) \,|\, 1 \le i \le \pi\left(\sqrt{\nu}\right)\right)$.

If a random variable ν has Zeta probability distribution and $\xi(n) = h\left(\rho(n)\right)$, then for each $n \in \mathbb{N}$ the following statements hold true:

$$(1)\quad P_s\left\{\nu \in \mathbb{P} \middle| \nu = n\right\} = P_s\left\{h\left(\rho(\nu)\right) = 1 \middle| \nu = n\right\} = \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p^s}\right) \qquad (2.11)$$

$$(2)\quad P_s\left\{\xi(n+1) = \pi(n+1) - \pi(n) = 1\right\} = P_s\left\{h(\rho(\nu) = 1 \middle| \nu = n+1\right\} = \prod_{p \le \sqrt{n+1}} \left(1 - \frac{1}{p^s}\right)$$

Proof.

Theorem 1 implies

$$P_s\left\{\nu \in \mathbb{P} \middle| \nu = n\right\} = P_s\left\{\bigcap_{p \le \sqrt{n}} \{p \nmid \nu\} \middle| \nu = n\right\} = \prod_{p \le \sqrt{n}} P_s\left\{\{p \nmid \nu\} \,|\, \nu = n\right\} = \prod_{i=1}^{\pi(\sqrt{n})} \left(1 - \frac{1}{p_i}\right)$$

Notice that the event $\left\{\bigcap_{p \le \sqrt{n}} \{p \nmid \nu\} \middle| \nu = n\right\}$ can be expressed in the form of conditions

$$\left\{\bigcap_{p \le \sqrt{n}} \left\{\left[\mathrm{mod}(\nu, p) > 0 \middle| p \in \mathbb{P}, \nu = n\right]\right\}\right\} = \left\{\bigcap_{1 \le i \le \sqrt{\pi(n)}} \{r_i > 0\}\right\} = \left\{\min\left[r_i \middle| 1 \le i \le \sqrt{\pi(n)}\right] > 0\right\}. \qquad (2.12)$$

By using the Heaviside function $h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$, we can write the recursive equation

for $\pi(n)$ in the form: $\pi(n+1) = \pi(n) + h\left(\min_{p \le \sqrt{n+1}} \left\{\mathrm{mod}(n+1, p) \,|\, p \in \mathbb{P}\right\}\right)$

or, equivalently,

$$\pi(n+1) = \pi(n) + h\left(\min_{i \le \sqrt{n}} \left\{r_i \,|\, r_i = \mathrm{mod}(n+1, p_i)\right\}\right) = \pi(n) + h\left(\min(\vec{r}(n+1)\right) \qquad (2.13)$$

which controls the occurrence of prime numbers in the sequence of all integers $n = 3, 4, 5, 6, \ldots$ For a random number ν with Zeta probability distribution, vector of residuals $\vec{r}(\nu) = \left(r_1(\nu), r_2(\nu), \ldots, r_{\kappa(\nu)}(\nu)\right)$ is a vector with independent random components $r_k(\nu) = \mathrm{mod}(\nu, p_k)$ distributed within congruence classes $C_{p_k, r_k(\nu)}$ for all $k:\ 1 \le k \le \pi(\nu)$. For ν to be prime is necessary and sufficient that ν

should not be divisible by all of primes $p \le \sqrt{\nu}$, which means that

$\rho(\nu) = \min\left\{r_i(\nu) \mid 1 \le i \le \pi\left(\sqrt{\nu}\right)\right\} > 0$. Denoting $\xi(n) = h\left(\rho(n)\right)$ $(n = 1,2,3,\ldots)$,

we have:

$$P_s\left\{\xi(n+1) = \pi(n+1) - \pi(n) = 1 \mid \pi(1) = 0\right\} = P\left\{h(\rho(n+1) = 1\right\}$$

$$= P_s\left\{\min\left\{(\overline{r}(n) > 0\right\} \middle| \pi(1) = 0\right\} = \frac{\frac{1}{\zeta(s)} \cdot \prod_{p \le \sqrt{n+1}} \left(1 - \frac{1}{p^s}\right)}{\frac{1}{\zeta(s)}} = \prod_{p \le \sqrt{n+1}} \left(1 - \frac{1}{p^s}\right), \tag{2.14}$$

since $P_s\left\{\pi(1) = 0\right\} = \dfrac{1}{\zeta(s)}$. Therefore, by letting $s \to 1$, we obtain

$$P_s\left\{\xi(n) = 1 \middle| \pi(1) = 0\right\} \to \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p}\right) \tag{2.15}$$

Probability of random variable ν with Zeta distribution to be a prime number in the interval $[2,\ n]$ for all $n \ge 5$ is given by the formulas:

$$P\left\{\nu \in \mathbb{P} \middle| \nu = n\right\} = P\left\{h\left(\rho(\nu)\right) = 1\right\} = \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p}\right),$$

$$P\left\{\xi(n) = 1 \middle| \min(r_i \mid 1 \le i \le \pi(\sqrt{n}) > 0\right\} = \prod_{p \le \sqrt{n}} \left(1 - \frac{1}{p}\right) \tag{2.16}$$

Examples.

1) $\nu = 108 = 1 \cdot 2^2 \cdot 3^3 \cdot 5^0 \cdot 7^0 \cdots$ with $\alpha(108, p) = 0$ for all $p > 3$.

We have: $\alpha(108,2) = 2, \alpha(108,3) = 3; \kappa(108) = 2$

2) ν=110=$2 \cdot 3^0 \cdot 5 \cdot 7^0 \cdot 11 \cdot 13^0 \cdot 17^0 \cdots$ with $\alpha(110, p) = 0$ for $p = 3, 7,$ and all $p > 11$

We have: $\alpha(110,2) = 1, \alpha(110,5) = 1, \alpha(110,11) = 1;$ $\kappa(110) = 3$.

In the above setting, the number $108 = \prod_{i=0}^{\infty} \xi(i)$ in example 1) represents the path:

$$1 \to 2^2 \to 3^3 \to 5^0 \to 7^0 \to \cdots$$

The number $110 = \prod_{i=0}^{\infty} \xi(i)$ in example 2) represents the path:

$$1 \to 2 \to 3^0 \to 5 \to 7^0 \to 11 \to 13^0 \to 17^0 \to \ldots$$

By setting $P\left\{\xi(j)=p_j^{\alpha_j}\right\}=\left(\frac{1}{p_j^s}\right)^{\alpha_j}$ for all $p_j\in\mathbb{P}$, we can calculate probability $P\{\nu=n\}$ of any given value $n\in\mathbb{N}$.

In example 1):

$$P_s\{\nu=108\}=\frac{1}{2^{2s}}\cdot\left(1-\frac{1}{2^s}\right)\cdot\frac{1}{3^{3s}}\cdot\left(1-\frac{1}{3^s}\right)\cdot\left(1-\frac{1}{7^s}\right)\cdot\left(1-\frac{1}{11^s}\right)\cdots\left(1-\frac{1}{p_j^s}\right)\cdots$$

$$=\frac{1}{2^{2s}\cdot 3^{3s}}\cdot\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)=\frac{1}{108^s}\cdot\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)$$

In example 2):

$$P_s\{\nu=110\}=\frac{1}{2^s}\cdot\left(1-\frac{1}{2^s}\right)\cdot\left(1-\frac{1}{3^s}\right)\cdot\frac{1}{5^s}\cdot\left(1-\frac{1}{5^s}\right)\cdot\left(1-\frac{1}{7^s}\right)\cdot\frac{1}{11^s}\cdot\left(1-\frac{1}{11^s}\right)\cdot\left(1-\frac{1}{13^s}\right)\cdots\left(1-\frac{1}{p_j^s}\right)\cdots$$

$$=\frac{1}{2^s\cdot 5^s\cdot 11^s}\cdot\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)=\frac{1}{110^s}\cdot\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)$$

Notice that, in general, in the formal expression $P_s\{\nu=n\}=\frac{1}{n^s}\cdot\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)$ the product involves a set of all prime numbers. In the above expressions the ‘probability’ $P_s\{\nu=n\}$ depends on a parameter s:

$$P_s\{\nu=n\}=\frac{1}{n^s}\cdot\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j^s}\right)=\frac{1}{n^s\cdot\zeta(s)},\ n\in\mathbb{N}, s>1 \qquad (2.17)$$

To cope with the divergence of the infinite product $\prod_{j=1}^{\infty}\left(1-\frac{1}{p_j}\right)=\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p}\right)=\zeta(1)$,

we consider $\zeta(s)=\sum_{n\geq 1}\frac{1}{n^s}$ for $s>1$, and define the probability P_s as a function of parameter s.

Meanwhile, there is another way to cope with divergence of $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ for $s\leq 1$. We can do so by introducing a sequence of *incomplete* (or *partial*) *Riemann Zeta* functions. We define the *incomplete product Zeta function* $\zeta_{\mathbb{P}_N}(s)$ as a partial product in the multiplicative presentation of $\zeta(s)$ for $s\geq 1$:

$$\zeta_{\mathbb{P}_N}(s) = \prod_{p \le N} \frac{1}{1 - \frac{1}{p^s}} \qquad (2.18)$$

Remark 2.1.

Since $\frac{1}{1-\frac{1}{p^s}} = \sum_{k=0}^{\infty} \left(\frac{1}{p^s} \right)^k$, we have a convergent additive partial presentation of $\zeta(s)$:

$$\zeta_{\mathbb{P}_N}(s) = \prod_{p \le N} \left[\sum_{k=0}^{\infty} p^{-sk} \right] = \sum_{n \in S(\mathbb{P}_N)} n^{-s}. \qquad (2.19)$$

Here $S(\mathbb{P}_N)$ is a multiplicative semigroup of all integers generated by $\mathbb{P}_N^{\,*} = \mathbb{P}_N \cup \{1\}$, where $\mathbb{P}_N = \{ p \mid p \le N, p \in \mathbb{P} \}$. Notice that $S(\mathbb{P}_N)$is an infinite set generated by a finite set $\mathbb{P}_N^{\,*}$. Then, we consider the corresponding probability distribution $P_{s,N}$, $s > 1$, given by the formula:

$$P_{s,N}\{\nu = n\} = \frac{1}{n^s \cdot \zeta_{\mathbb{P}_N}(s)}, \; n \in S(\mathbb{P}_N), \; s > 0, \, N \in \mathbb{N} \qquad (2.20)$$

Since $\zeta_{\mathbb{P}_N}(s) = \prod_{p \le N} \left(1 - \frac{1}{p^s} \right)^{-1} = \sum_{n \in S(\mathbb{P}_N)} \frac{1}{n^s}$, we have $\sum_{n \in S(\mathbb{P}_N)} P_{s,N}\{\nu = n\} = 1$.

The probability $P_{s,N}$ of ν to be a prime number in the set of numbers $S(\mathbb{P}_N)$, generated by primes not exceeding N, can be calculated by the formula:

$$P_{s,N}\{\nu \text{ is prime} | \nu \in S(\mathbb{P}_N)\} = \frac{\sum_{p \in \mathbb{P}_N} p^{-s}}{\zeta_{\mathbb{P}_N}(s)} = \frac{\sum_{p \le N} p^{-s}}{\prod_{p \le N} \frac{1}{1 - p^{-s}}} = \left(\sum_{p \le N} p^{-s} \right) \cdot \prod_{p \le N} \left(1 - \frac{1}{p^s} \right) \qquad (2.21)$$

The convergence of the infinite series $\zeta_{\mathbb{P}_N}(z) = \sum_{n \in S(\mathbb{P}_N)} n^{-s}$ is guaranteed by (2.18) and (2.19). In general, from the probabilistic point of view, every finite path on the monoid set. $S(\mathbb{P}^*) = \mathbb{N}$ can be identified with a randomly chosen natural number ν by assuming that it has a probability distribution $P\{\nu = n\}$, $n \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} P\{\nu = n\} = 1 \ .$$

3. Asymptotics of a generalized Bernoulli process and the Cramér's model of prime numbers distribution

Definition 3.1

A sequence of $\{0,1\}$-valued random variables $(\xi_k)_{k\in\mathbb{N}}$ defined on probability space $(\Omega, \mathcal{F}, P)$ which terms are not in general independent and identically distributed we call *a generalized Bernoulli process*. We have:

$$P\{\xi_k(\omega) = 1\} = P_k, P\{\xi_k(\omega) = 0\} = Q_k, \ P_k + Q_k = 1, \ k \in \mathbb{N}.$$

Probabilitstic approach to distribution of prime numbers in $\mathbb{N}$ is addresed in the Harald Cramér's model [2,3]. The sequence of random variables $(\nu_k)_{k\in\mathbb{N}}$ on the main probability space $(\Omega, \mathcal{F}, P)$, such that $\nu_n : \Omega \to \mathbb{N}$, for some $\omega \in \Omega$ has realizations resulted in prime numbers: $\nu_k(\omega) = n \in \mathbb{P}$. The assignments of probabilities

$$P\{\nu_k(\omega) = n \in \mathbb{P}\} = P\{\xi(n) = 1\} = \frac{1}{\ln n}, \ P\{\nu_k(\omega) = n \notin \mathbb{P}\} = P\{\xi(n) = 0\} = 1 - \frac{1}{\ln n},$$

in the Cramér's model was originally motivated by the Prime Number Theorem [10, p.133],

where the counting function of primes on $\mathbb{N}$ is given by the asymptotic formula

$$\pi(x) = \sum_{p\in\mathbb{P}\cap[2,x]} 1 \sim Li(x) = \int_2^x \frac{dt}{\ln t},$$

which leads to the heuristic assumption about the probability $P\{p \in [x-1, x]\} \sim \int_{x-1}^{x} \frac{dt}{\ln t} \sim \frac{1}{\ln x}.$

The Cramér's model describes the occurrence of prime numbers as a special case of a Bernoulli process given by a *sequence of independent Bernoulli variables* $(\xi(n) \mid n \in \mathbb{N})$, where $\xi(n) = \xi(\nu_n)$, with probabilities

$$P\{\nu_n \in \mathbb{P}\} = \frac{1}{\ln n},\ \ P\{\nu_n \notin \mathbb{P}\} = 1 - \frac{1}{\ln n},$$

or equivalently,

$$P\{\xi(n) = 1\} = \frac{1}{\ln n},\ \ P\{\xi(n) = 0\} = 1 - \frac{1}{\ln n}\ ,\ \text{where } n \geq 2.$$

The above formulas (2.1) and (2.16) , due to the Merten's 1st and 2nd theorems [9, p.22], have the asymptotic expression:

$$P\{\xi(n) = 1\} = \prod_{p \leq \sqrt{n}} \left(1 - \frac{1}{p}\right) \approx \frac{e^{-\gamma}}{\frac{1}{2}\ln(n)}\left[1 + O\left(\frac{1}{\ln(n)}\right)\right] = \frac{c}{\ln(n)}\left[1 + O\left(\frac{1}{\ln(n)}\right)\right], \tag{3.1}$$

where $c = \frac{2}{e^{\gamma}} \approx 1.122918968$. In both models A and B we consider all values of $n > N$ in (3.1) by choosing an arbitrary large natural N. As we pointed above, the Cramér's assumption about *independence of terms* in the sequence $(\xi(n) \mid n = 1, 2, \ldots)$ is not accurate for any finite subset of $\mathbb{N}$. The more adequate approach would be to consider the sequence of consecutive primes represented by $(\nu_n)_{n \in \mathbb{N}}$ and $(\xi(n) \mid n = 1, 2, \ldots)$, respectively, as *stochastic predictable sequences of dependent random variables.*

Actually the sequence of random variables in the updated Cramér's model is *asymptotically Bernoullian* (and asymptotically pairwise independent) in a sense of Definition 3.1 given below. Meanwhile, the demand for idependence of terms in the sequence $(\nu_n)_{n \in \mathbb{N}}$ and in $(\xi(n))_{n \in \mathbb{N}}$ could be be relaxed for the Cramér's model, due to the version of the *Central Limit Theorem (CLT) for dependent random variables* in sequences with a sort of '*asymptotically forgetful memory*' [7]. This version of CLT tracks back to the S.N. Bernstein's ideas [22]. One of the most general forms of the Central Limit Theoems for dependent variables has been proved for sequences of random walks on differentiable manifolds and Lie groups by the author [24,25].

Never the less, in what follows we use the assumption of independent terms in sequences of random variables $\{\nu_k\}_{k \in \mathbb{N}}$ as the most adequate for the goals of this article and apply here the classical form of the CLT [23].

Let discuss now, following M. Loèv [18], *asymptotic behavior of generalized Bernoulli processes.*

We have for ξ_k mathematical expectation $E\left\{\xi_k\right\} = P_k$ and variance $V\left\{\xi_k\right\} = P_k \cdot Q_k$

Denote $X_n = \frac{1}{n}\sum_{k=1}^{n}\xi_n$. Then $E\left\{X_n\right\} = \frac{1}{n}\sum_{k=1}^{n}P_k$. Since $\left(\xi_k\right)^2 = \xi_k$, we have

$E\left\{\left(\xi_k\right)^2\right\} = E\left\{\xi_k\right\}$, $E\left\{\xi_k \cdot \xi_l\right\} = P\left\{\xi_k \cdot \xi_l = 1\right\} = P_{kl}$. Then, $\left(E\left\{X_n\right\}\right)^2 = \frac{1}{n^2}\left(\sum_{k=1}^{n}P_k^2 + 2\sum_{1\le k<l<]\le n}P_k \cdot P_l\right)$

and $E\left\{\left(X_n\right)^2\right\} = \frac{1}{n^2}\left(\sum_{k=1}^{n}P_k + 2\sum_{k<l}P_{kl}\right)$.

This implies:

$$V\left\{X_n\right\} = E\left\{(X_n)^2\right\} - \left(E\left\{X_n\right\}\right)^2 = \frac{1}{n^2}\left(\sum_{k=1}^{n}P_kQ_k + 2\sum_{1\le k<l\le n}(P_{kl} - P_k \cdot P_l)\right) = \frac{1}{n^2}\sum_{k=1}^{n}V(\xi_k) + D_n\,, \tag{3.2}$$

where $D_n = \frac{2}{n^2}\sum_{1\le k<l\le n}(P_{kl} - P_k \cdot P_l) = \frac{n(n-1)}{2n^2}\left(\frac{2}{n(n-1)}\sum_{1\le k<l\le n}P_{kl} - \frac{2}{n(n-1)}\sum_{1\le k<l\le n}P_k \cdot P_l\right)$.

If terms in $\left(\xi(n)\right)_{n\in\mathbb{N}}$ are pairwise independent, then $P_{kl} = E\left\{\xi_k \cdot \xi_l\right\} = E\left\{\xi_k\right\} \cdot E\left\{\xi_l\right\} = P_k \cdot P_l$

and $D_n = 0$, which implies $V\left\{X_n\right\} = \frac{1}{n^2}\sum_{k=1}^{n}V\left\{\xi_k\right\}$.

Thus, D_n can be viewed as a *cummulative measure of pairwise independence* of terms in Bernoulli process $\left(\xi(n)\right)_{n\in\mathbb{N}}$. Denote:

$$\overline{P}_1(n) = \frac{1}{n}\sum_{k=1}^{n}P_k \text{ and } \overline{P}_2(n) = \frac{2}{n(n-1)}\sum_{1\le k<l\le n}P_{kl}\,.$$

Notice that

$$D_n = \frac{n-1}{2n}\left(\overline{P}_2 - \overline{P}_{1,2}\right) \text{ where } \overline{P}_{12} = \frac{2}{n(n-1)}\sum_{1\le k<l\le n}P_k \cdot P_l\,.$$

We consider below a slightly different measure d_n that shows how close a Bernoulli process $\left(\xi(n)\right)_{n\in\mathbb{N}}$ is to a classical Bernoulli sequence of independent equally distributed random variables.

Then, $E\left\{\left(X_n\right)^2\right\} = \frac{1}{n^2}\left(\sum_{k=1}^{n} P_k + 2\sum_{k<l} P_{kl}\right) = \frac{\overline{P}_1 - \overline{P}_2}{n} + \overline{P}_2$ and $\left(E\left\{X_n\right\}\right)^2 = \left(\overline{P}_1\right)^2$

Since $V\left\{X_n\right\} = E\left\{(X_n)^2\right\} - \left(E\left\{X_n\right\}\right)^2 = \frac{\overline{P}_1 - \overline{P}_2}{n} + \overline{P}_2 - \left(\overline{P}_1\right)^2$, we have

$$V\left(X_n\right) = \frac{\overline{P}_1 - \overline{P}_2}{n} + \overline{P}_2 - \left(\overline{P}_1\right)^2 = \frac{\overline{P}_1 - \overline{P}_2}{n} + d_n \tag{3.3}$$

where $d_n = \overline{P}_2 - \left(\overline{P}_1\right)^2$.

In the classical Bernoulli scheme witn independent identically distributed terms $\left(\xi_k\right)_{k\in\mathbb{N}}$

we have $P_{kl} = E\left\{\xi_k \cdot \xi_l\right\} = P\left\{\xi_k \cdot \xi_l = 1\right\} = P\left\{\xi_k = 1\right\} \cdot P\left\{\xi_l = 1\right\} = P_k \cdot P_l = P^2$, due to independence

and equal distribution of terms in the sequence $\left(\xi_k\right)_{k\in\mathbb{N}}$, so that $d_n = \overline{P}_2 - \left(\overline{P}_1\right)^2 = P^2 - P^2 = 0$.

This implies $d_n = 0$ and $V\left\{X_n\right\} = \frac{1}{n^2}\sum_{k=1}^{n} V\left\{\xi_k\right\}$.

This means that the value of d_n is a measure of a deviation of the sequence $\left(\xi_k\right)_{k\in\mathbb{N}}$

from a classical Bernoulli scheme with identically distributed terms.

Definition 3.1

We call a sequence $\left(\xi_n\right)_{n\in\mathbb{N}}$ of $\left\{0,1\right\}$-valued random variables defined on probability space

$\left(\Omega, \mathcal{F}, P\right)$ *asymptotically pairwise Bernoullian* if $\max_{N<k<l}\left|P_{kl} - P_k \cdot P_l\right| \to 0$ as $N \to \infty$. This means

that for sufficiently large N variables ξ_k, ξ_l are asymptotically independent for all $l > k > N$.

Lemma 3.1

For *asymptotically Bernoullian sequence* $\left(\xi_n\right)_{n\in\mathbb{N}}$ we have $D_n \to 0$ so that

$$\left|V\left\{X_n\right\} - \frac{1}{n^2}\sum_{k=1}^{n} V\left\{\xi_k\right\}\right| \to 0 \text{ as } n \to \infty .$$

Proof.

Due to (3.2), $V\left\{X_n\right\} - \frac{1}{n^2}\sum_{k=1}^{n} V\left\{\xi_k\right\} = D_n$.

Since $D_n = \frac{2}{n^2} \sum_{k<l\le n} \left(P_{kl} - P_k \cdot P_l\right)$, and $\left|\sum_{k<l\le n} \left(P_{kl} - P_k \cdot P_l\right)\right| \le \frac{n(n-1)}{2} \max_{N<k<l} \left|P_{kl} - P_k \cdot P_l\right|$,

we have $\left|D_n\right| \le \frac{2}{n^2} \cdot \frac{n(n-1)}{2} \cdot \max_{N<k<l} \left|P_{kl} - P_k \cdot P_l\right| \to 0$.

This implies $\left|V\left\{X_n\right\} - \frac{1}{n^2} \sum_{k=1}^{n} V\left\{\xi_k\right\}\right| \to 0$.

Q.E.D.

Keeping in mind approximation (3.1), we restrict the sequence $\left(\xi_k\right)_{k\in\mathbb{N}}$ by considering its 'tail' $\left(\xi_k\right)_{k>N}$ of the original seqience for sufficiently large N.

Theorem 3.1

The sequence $\left(\xi_k\right)_{k\in\mathbb{N}}$ in the modified Cramér's model is *asymptotically pairwise Bernoullian,*

that is $\max_{N<k<l} \left|P_{kl} - P_k \cdot P_l\right| = O\left(\frac{1}{\ln N}\right)$, where $P\left\{\xi_k = 1\right\} = P_k$, $P\left\{\xi_k \cdot \xi_l = 1\right\} = P_{kl}$,

and $\left|V\left\{X_n\right\} - \frac{1}{n^2} \sum_{k=1}^{n} V\left\{\xi_k\right\}\right| = O\left(\frac{1}{\ln N}\right)$ as $D_n = O\left(\frac{1}{\ln N}\right)$ for all $n > N$. (3.4)

Proof.

Indeed, $P\left\{\xi_k = 1\right\} = P_k$, $P\left\{\xi_k \cdot \xi_l = 1\right\} = P_{kl}$. Then, since $\left|P_{kl} - P_k \cdot P_l\right| < P_k \le \frac{1}{\ln N}$

for all $N < k < l \le n$, we have $\max_{N<k<l} \left|P_{kl} - P_k \cdot P_l\right| \le \frac{c}{\ln N} \to 0$ and $D_n = O\left(\frac{1}{\ln N}\right)$ as $N \to \infty$.

This implies $\left|V\left\{X_n\right\} - \frac{1}{n^2} \sum_{k=1}^{n} V\left\{\xi_k\right\}\right| = O\left(\frac{1}{\ln N}\right)$ for all $n > N$.

Q.E.D.

In the Cramér's model $\pi_N\left(n\right) = \sum_{k=N}^{N+n} \xi(k)$ represents the number of primes among n terrms

in the interval $(N, N+n]$ of the sequence and $\frac{\pi_N(n)}{n} = \frac{1}{n} \sum_{k=N}^{N+n} \xi(k) = \frac{\hat{\pi}_N(n)}{n}$ is a relative freqiency

of primes for these terms. predicted by the improved model based on Zeta probability distribution.

In the Table 4 below, we demonstrate how well $E\left\{\frac{\pi(n)}{n}\right\}$ approximates relative frequencies of primes $\frac{\pi(n)}{n}$ in the Zeta distribution model for $(\xi_k)_{k\geq 3}$ as n increses from 10^1 to 10^9.

Table 3.1. *Comparison of probabilities* $P\{\nu\in\mathbb{P}|\nu=n\}$ *and frequencies* $\frac{\pi(n)}{n}$ *of primes in intervals* $[1,\ n]$

Natural n	$P\{\nu\in\mathbb{P}\|\nu=n\}=\prod_{p\leq\sqrt{n}}\left(1-\frac{1}{p}\right)$	$\frac{\pi(n)}{n}$
10^1	0.33333333	0.400000 00
10^2	0.22857143	0.25000000
10^3	0.15285215	0.16800000
10^4	0.12031729	0.12290000
10^5	0.09621491	0.09592000
10^6	0.08096526	0.07849800
10^7	0.06957939	0.06645790
10^8	0.06088469	0.05761455
10^9	0.05416682	0.05084753

Consider now the *Generalized Law of Large Numbers* for a general Bernoulli process as it stated in [18] and apply it then to Zeta distribution model for $(\xi_k)_{k\in\mathbb{N}}$.

Theorem 3.2

Let $\xi(k)=\begin{cases}1 \text{ if } k\in\mathbb{P}\\ 0 \text{ otherwise}\end{cases}$ and $\frac{\hat{\pi}_N(n)}{n}=\frac{1}{n}\sum_{k=N+1}^{N+n}\xi(k)$ be a relative freqiency of primes the interval $[N,N+n]$. Then, the *Generalized Law of Large Numbers* holds true:

$$P\left\{\left|\frac{\hat{\pi}_N(n)}{n}-E\left\{\frac{\hat{\pi}_N(n)}{n}\right\}\right|>\varepsilon\right\}\to 0 \text{ as } N,n\to\infty. \tag{3.5}$$

If $d_{n.N}=\frac{2}{n(n-1)}\sum_{N\le k<l\le N+n}P\left\{(k\in\mathbb{P})\cap(l\in\mathbb{P})\right\}-\left(\frac{1}{n}\sum_{k=N}^{N+n}P\left\{k\in\mathbb{P}\right\}\right)^2=O\left(\frac{1}{n}\right)$,

then the *Generalized Strong Law of Large Numbers* holds true:

$$P\left\{\left|\frac{\hat{\pi}_N(n)}{n}-E\left\{\frac{\hat{\pi}_N(n)}{n}\right\}\right|\to 0\right\}=1 \text{ as } N,n\to\infty \tag{3.6}$$

Proof.

Due to [25], we apply the following *Propositions:*

1. Generalized Bernoulli Theorem that for every $\varepsilon>0$: $P\left\{\left|X_n-E\left\{X_n\right\}\right|>\varepsilon\right\}\to 0$

holds true for a Bernoulli process $\left(\xi_k\right)_{k\in\mathbb{N}}$ if and only if $d_n=\overline{P}_2-\left(\overline{P}_1\right)^2\to 0$ as $n\to\infty$.

2. Generalized Strong form of Bernoulli Theorem that $P\left\{\left|X_n-E\left\{X_n\right\}\right|\to 0\right\}\to 1$

holds true if $d_n=O\left(\frac{1}{n}\right)$.

We show here that these propositions asymptotically hold true for tails $\left(\xi_k\right)_{k\ge N}$.

For tail $\left(\xi_k\right)_{k\ge N}$ in the framework of Cramér's model we have:

$$\overline{P}_{1,N}(n)=E\left\{\frac{\hat{\pi}_N(n)}{n}\right\}=\frac{1}{n}\sum_{k=N+1}^{N+n}E\left\{\xi(k)\right\}=\frac{1}{n}\sum_{k=N+1}^{N+n}P\left\{k\in\mathbb{P}\right\}=\frac{1}{n}\sum_{k=N+1}^{N+n}\frac{1}{\ln k}\sim\int_N^{N+n}\frac{dt}{\ln t}=Li(N+n)-Li(N),$$

$$\overline{P}_{2,N}(n)=\frac{2}{n(n-1)}\sum_{N\le k<l\le N+n}E\left\{\xi(k)\cdot\xi(l)\right\}=\frac{2}{n(n-1)}\sum_{N\le k<l\le N+n}P\left\{(k\in\mathbb{P})\cap(l\in\mathbb{P})\right\}\cdot$$

Then, $d_{n.N}=\frac{2}{n(n-1)}\sum_{N\le k<l\le N+n}P\left\{(k\in\mathbb{P})\cap(l\in\mathbb{P})\right\}-\left(\frac{1}{n}\sum_{k=N}^{N+n}P\left\{k\in\mathbb{P}\right\}\right)^2$

Notice that $d_{n.N}\le\max_{N\le k<l\le N+n}\left(P\left\{(k\in\mathbb{P})\cap(l\in\mathbb{P})\right\}\right)<\frac{1}{\ln N}$ implies $d_{n,N}\to 0$ as $n,N\to\infty$.

This implies $d_{n,N} \to 0$ as $N, n \to \infty$.

Then, $P\left\{\left|\frac{\hat{\pi}_N(n)}{n} - E\left\{\frac{\hat{\pi}_N(n)}{n}\right\}\right| > \varepsilon\right\} = O\left(\frac{1}{\ln N}\right)$ as $n, N \to \infty$ and (9) holds true.

In addition, if $d_{n,N} = O\left(\frac{1}{n}\right)$, as $n, N \to \infty$, then (10) holds true.

Q.E.D.

4. Additive Walks and Distribution of Twin- and d- Primes

Abstract. Some classical questions and problems of Number Theory are addressed here from an entirely probabilistic point of view. With certain intense use of Zeta probability distribution, we approached some of old classical problems in Number Theory like the Twin Primes problem (the last generalized to the d-primes distribution problem for consecutive prime numbers), and the distribution of prime numbers among arithmetic sequences. A list of variety of problems in Number Theory related to the gaps between consecutive primes is given in [1]. In this work sequences of natural numbers are considered as realizations of paths of multiplicative random walks with independent increments (generated by random variables ν followed Zeta distribution), while prime-valued sequences are represented as realizations of additive random walks with asymptotically independent increments. We denote here $\mathbb{N}$ set of natural numbers and $\mathbb{P}$ set of prime numbers. Better foundations for the Cramer's model in this work is provided by considering the sequence $\left(\xi(\nu_n)\right)_{n\in\mathbb{N}}$ of (0,1) - valued random variables $\xi(\nu_n) = \begin{cases} 1 \text{ if } \nu_n \in \mathbb{P} \\ 0 \text{ otherwise} \end{cases}$ as generalized predictable non-stationary Bernoulli process.

> "Using randomness to study certainty may seem somewhat surprising.
> It is, however, one of the deepest contributions of our century to mathematics
> in general and to the theory of numbers in particular."
>
> *Gérald Tenenbaum, Michel Mendès France,*
> The Prime Numbers and Their Distribution. AMS, 2000.

Consider, as an example, an additive rule to generate stochastic or deterministic sequences of positive integers:

$$\nu(1)=0,\ \nu(k+1)=\nu(k)+\xi(k+1), \tag{1}$$

where $\xi(k)=\begin{cases}k & \text{if } k\in\mathbb{P}\\ 0 & \text{otherwise}\end{cases}$ for all $k=1,2,3\ldots$

The recursively generated sequence (1) represents obviously sums of consecutive prime numbers. This approach leads to 'additive models' of random walks in the study of prime numbers distribution. Though the sequence $\nu(k)$ $(k=1,2,\ldots,)$ generated recursively is deterministic, each jump of the 'walk' (1) can result either in a prime $\nu(k)=p_k\in\mathbb{P}$ or in 0 (if k is a composite number), where that jumps that are differences ('gaps') $\xi(k+1)=p_{k+1}-p_k$ between consecutive primes look very sporadic and hard to predict. It is well known that the gaps between two consecutive primes $p_k\ge 3$ and p_{k+1} can be as small as 2 (for twin primes) or arbitrary big (see the table below). Indeed, in the sequence of $n-1$ consecutive integers $\left\{n!\ +\ k\,|\,2\le k\le n\right\}$ each integer $n!\ +\ k$ is divisible by k, and therefore this sequence does not include primes. This means that there are consecutive prime numbers p_i and p_{i+1} such that $p_i<n!\ +\ 2$ and $p_{i+1}>n!\ +\ n$, which implies that $\Delta\, p_i=p_{i+1}-p_i\ge n$.

The next definition is a generalization of the notion of twin primes.

Definition 1

We call prime numbers $p<p'$ *consecutive* if there is no prime q between them (that is no prime q such that $p<q<p'$). A prime number p we call d*-prime* if p,p' are consecutive primes and $p'=p+d$.

Notice that the number $d = p_{i+1} - p_i = g_i$ for a d-prime p_i is called a "gap between two successive primes" (see the article "Prime gap" in [8]).

For example, p is a 2-prime, if and only if p and $p+2$ are twin primes, since for $p \geq 3$ twin primes p and $p+2$ are automatically consecutive. Let us denote

$D\mathbb{P}_d = \{ p \mid p \text{ and } p+d \text{ are consecutive primes} \}$, the set of d-primes (that is prime numbers p such that p and $p+d$ are consecutive primes).

For example, $D\mathbb{P}_1 = \{2\}$; the set of twin primes is $D\mathbb{P}_2 = \{3,5,11,17,29,41,...\}$.

One of famous conjectures is that the set $D\mathbb{P}_2$ of twin primes is infinite [1].

Table 1. *d-primes for $d = 2,4,6$, among all primes $p < 200$*

$D\mathbb{P}_2$	3 5 11 17 29 41 59 71 101 107 137 149 179 191 197
$D\mathbb{P}_4$	7 13 19 37 43 67 79 97 103 109 127 163 193
$D\mathbb{P}_6$	23 31 47 53 61 73 83 131 151 157 167 173

Notice that $D\mathbb{P}_d = \varnothing$ for all odd $d > 1$ and *the first conjecture is that* $D\mathbb{P}_d \neq \varnothing$ *for all even values of* $d \geq 2$. Obviously, $D\mathbb{P}_d \cap D\mathbb{P}_{d'} = \varnothing$ for all $d \neq d'$ and $D\mathbb{P}_1 \cup \left[\bigcup_{\text{even } d=2}^{\infty} D\mathbb{P}_d \right] = \mathbb{P}$.

This implies that $\{D\mathbb{P}_d\}_{d \in 2\mathbb{N}}$ makes a partition of the set of primes.

This means that any prime number p is a d-prime for an appropriate d. Indeed, due to the Euclid theorem, there are infinitely many prime numbers, therefore for any prime p there exist the next (that is consecutive) prime $p' = p + d$, where $d = p' - p$, and $p \in D\mathbb{P}_d$.

The second conjectire is that every $D\mathbb{P}_d$ *is an infinte set for all even values of* $d \geq 2$.

Lemma 1

For a positive even integer d and a prime number p such that $p > d$, we have $p \in D\mathbb{P}_d$

if and only if $p \in C_{d,r}$ and $(p+d) \in C_{d,r}$, where r is an odd number such that $1 \le r \le d-1$ and $p = k \cdot d + r,\ p + d = \left(k+1\right) \cdot d + r$.

Proof.

Let p_i and p_{i+1} be two consecutive prime numbers, that is $p_i \in D_d\mathbb{P}$ and $p_{i+1} = p_i + d \in \mathbb{P}$. We have then,

$p_i = k_i \cdot d + r_i,\ p_{i+1} = k_{i+1} \cdot d + r_{i+1},$

where $1 \le r_i \le d-1,\ 1 \le r_{i+1} \le d-1$. Then, $p_{i+1} = p_i + d$ implies $\Delta\, p_i = \left(k_{i+1} - k_i\right) \cdot d + \left(r_{i+1} - r_i\right) = d$.

Since $\left| r_{i+1} - r_i \right| < d$, we should have $r_{i+1} = r_i = r$ and $k_{i+1} - k_i = 1$.

Thus, $p_i = k_i \cdot d + r,\ p_{i+1} = \left(k_i + 1\right) \cdot d + r$ where r is an odd number and $r \ge 1$.

Q.E.D.

Remark 1

Since for each even number d the finite number of congruence classes $\left\{ C_{d,r} \cap \mathbb{P} \,|\, 1 < even\ r < d-1 \right\}$ make a partition of the infinite set of all primes $\mathbb{P}$, then, due to the 'pigeonhole principle', at least one of classes $C_{d,r}$ must contain infinitely many prime numbers.

Prime numbers populate the sets $D\mathbb{P}_{d,N} = D\mathbb{P}_d \cap [2,N]$ not evenly for different even integers d, as illustrated by the histogram below for $N = 10^9$. Computer calculations show so far that the most frequent value of consecutive prime gaps is $d = 6$. This can be claimed as another unproven conjecture.

According to the Prime Number Theorem [10, p.133], the counting function of primes on $\mathbb{N}$ is given by the asymptotic formula

$$\pi(x) = \sum_{p \in \mathbb{P} \cap [2,x]} 1 \sim Li(x) = \int_2^x \frac{dt}{\ln t} \qquad (2)$$

This leads to the heuristic assumption about the probability

$$P\left\{ p \in [x-1,x] \right\} \sim \int_{x-1}^{x} \frac{dt}{\ln t} \sim \frac{1}{\ln x}.$$

According to the Cramér's model, occurrences of primes in $\mathbb{N}$ are represented by the sequence of *independent Bernoulli variables* $\{\xi_n\}_{n\geq 3}$ such that $\xi_n = \begin{cases} 1 \text{ if } n \in \mathbb{P} \\ 0 \text{ otherwise} \end{cases}$ and

$$P\{\xi_n = 1\} = \frac{1}{\ln n}, \; P\{\xi_n = 0\} = 1 - \frac{1}{\ln n} \text{ for all } n \in \mathbb{N} \cap \{n \geq 3\}. \tag{3}$$

Remark 2

As we know, the sequence of primes $\{p_n\}_{n\in\mathbb{N}}$ is deterministic and is recursively determined by the corresponding vectors of residual

$$\vec{r}(n) = \left(r_1, r_2, ..., r_{\pi\left(\sqrt{n}\right)} \right), \text{ where } r_i = \operatorname{mod}(n, p_i), \; i = 1,2,...,\pi\left(\sqrt{n}\right).$$

Therefore, in contrast to the Cramér's model, the terms of a more adequate sequence of random variables $\{\xi_n\}_{n\geq 3}$ should be considered as dependent, since the value of ξ_n depends on values of $\vec{\xi}(n) = \left(\xi_1, \xi_2, ..., \xi_{\pi\left(\sqrt{n}\right)} \right)$. Moreover, each ξ_n must be equal to 0 for all even n. Indeed, any prime $p > 2$ is an odd number, and all primes, except for $p_1 = 2$, belong to the set of odd numbers (that is to the congruence class $C_{2,1}$ of $\mathbb{N}$).

Dependence of terms ξ_n though is not very restrictive because, due to our results in section3, Cramér's model is asymptotically Bernoullian, so that pairwise dependence of terms in the sequence $\{\xi_n\}_{n\geq 3}$ vanishes as $n \to \infty$.

Denote $\pi_d(x) = \sum_{D\mathbb{P}_d \cap [2,x]} 1 = \sum_{p\in D\mathbb{P}_d} I_{[2,x]}(p)$ number of d-primes in the interval $[2,x]$.

Given a prime number p, the corresponding vector of residuals $\vec{r}(p) = \left(r_1, r_2, \ldots, r_{\pi\left(\sqrt{p}\right)} \right)$ must have all non-zero components $r_i = \operatorname{mod}(p, p_i), \; 0 \leq i \leq \pi\left(\sqrt{p}\right)$ and obviously the complete vector of residuals $\vec{R} = \left(r_1, r_2, \ldots, r_{\pi(p)-1} \right)$ has also all non-zero components.

One of quite reasonable questions is how frequently d-primes may occur among all prime numbers.

We can evaluate the empirical probability of d-primes by the relative frequency:

$$P\left\{ \nu \text{ is a } d\text{-prime} \mid \nu \in \mathbb{P} \cap \left[2, x\right] \right\} \approx \frac{\pi_d(x)}{\pi(x)} \tag{4}$$

Denote $\xi_d(n) = \begin{cases} 1 \text{ if } n \text{ is } d\text{-prime} \\ 0, \text{ otherwise} \end{cases}$. Then $\pi_d(x) = \sum_{n \le x} \xi_d(n)$,

where $\xi_d(\nu) = 1$ if $\nu = p_i$ and $\nu + d = p_{i+1}$ are consecutive prime numbers.

Thus,

$$\pi_d(\nu + d) = \pi_d(\nu) + \xi_d(\nu) \quad (i = 1, 2, \ldots). \tag{5}$$

Assuming the Cramer's assumption of independence of consecutive primes,
we have:

$$\begin{aligned} P\left\{\xi_d(\nu) = 1\right\} &= P\left\{\nu \text{ and } \nu + d \text{ are consecutive primes}\right\} \\ &= P\left\{\nu \text{ and } \nu + d \text{ are prime numbers with no primes in the open interval } (\nu, \nu + d)\right\} \\ &= P\left\{\nu \in \mathbb{P}\right\} \cdot P\left\{\bigcap_{i=1}^{d-1} \left\{(\nu + i) \notin \mathbb{P}\right\}\right\} \cdot P\left\{(\nu + d) \in \mathbb{P}\right\} \end{aligned}$$

Then, $P\left\{\bigcap_{i=1}^{d-1} \left\{(\nu + i) \notin \mathbb{P}\right\}\right\} = \prod_{i=1}^{d-1} \left(1 - P\left\{(\nu + i) \in \mathbb{P}\right\}\right)$.

Following the Cramér's model assumption: $P\left\{\nu \in \mathbb{P} \mid \nu = n\right\} = \dfrac{1}{\ln n}$, we obtain

$$P\left\{\xi_d(\nu) = 1 \mid \nu = n\right\} = \frac{1}{(\ln n) \cdot (\ln(n + d))} \cdot \prod_{i=1}^{d-1} \left(1 - \frac{1}{\ln(n + i)}\right) = \Psi(n, d) \tag{6}$$

Denoting $\phi(n, d) = \prod_{i=1}^{d-1} \left(1 - \dfrac{1}{\ln(n + i)}\right)$, we write the function $\Psi(n, d)$ in (6) as

$$\Psi(n, d) = \frac{\phi(n, d)}{\ln(n) \cdot \ln(n + d)} \tag{7}$$

Thus, mathematical expectation and variance of $\xi_d(\nu)$ given $\nu = n$ can be approximated as $E\left\{\xi_d(\nu)\}\,|\,\nu=n\right\}=\Psi(n,d),\;\; Var\left\{\xi_d(\nu)\}\,|\,\nu=n\right\}=\Psi(n,d)\cdot\left(1-\Psi(n,d)\right)$.

This implies:

$$\begin{aligned}
E\left\{\pi_d(x)\right\} &= \sum_{n\le x} E\left\{\xi_d(\nu)\}\,|\,\nu=n\right\} = \sum_{n\le x}\Psi(n,d) \\
Var\left\{\pi_d(x)\right\} &= \sum_{n\le x} Var\left\{\xi_d(\nu)\}\,|\,\nu=n\right\} = \sum_{n\le x}\Psi(n,d)\cdot\left(1-\Psi(n,d)\right)
\end{aligned} \tag{8}$$

Using (6.6 and 6.7), we can approximate the mathematical expectation and variance of $\xi_d(\nu)$ in the integral form:

$$\begin{aligned}
E\left\{\pi_d(x)\right\} &= \sum_{n\le x}\Psi(n,d) \sim \int_2^x \frac{\phi(t,d)}{\ln(t)\cdot\ln(t+d)}dt \\
Var\left\{\pi_d(x)\right\} &\sim \int_2^x \frac{\phi(t,d)}{\ln(t)\cdot\ln(t+d)}\cdot\left(1-\frac{\phi(t,d)}{\ln(t)\cdot\ln(t+d)}\right)dt
\end{aligned} \tag{9}$$

The comparison of $\pi_d(x)$ distribution with its mathematical expectation $E\pi_d(x)$ is given in the tables below computed for $d = 2,4,6,8,10,12$ and x changing in steps: $10^1, 10^2, \ldots,\ 10^8$.

Number $\pi_d(x)$ of d-primes for $p \le x$

x \ d:	2	4	6	8	10	12	14	16
10^2	8	7	7	1	0	0	0	0
10^3	35	40	44	15	16	7	7	0
10^4	205	202	299	101	119	105	54	33
10^5	1224	1215	1940	773	916	964	484	339
10^6	8169	8143	13549	5569	7079	8005	4233	2881
10^7	58980	58621	99987	42352	54431	65513	35394	25099
10^8	440312	440257	768752	334180	430016	538382	293201	215804

Expectation $E\pi_d(x)$ of numbers of d-primes for $p \le x$

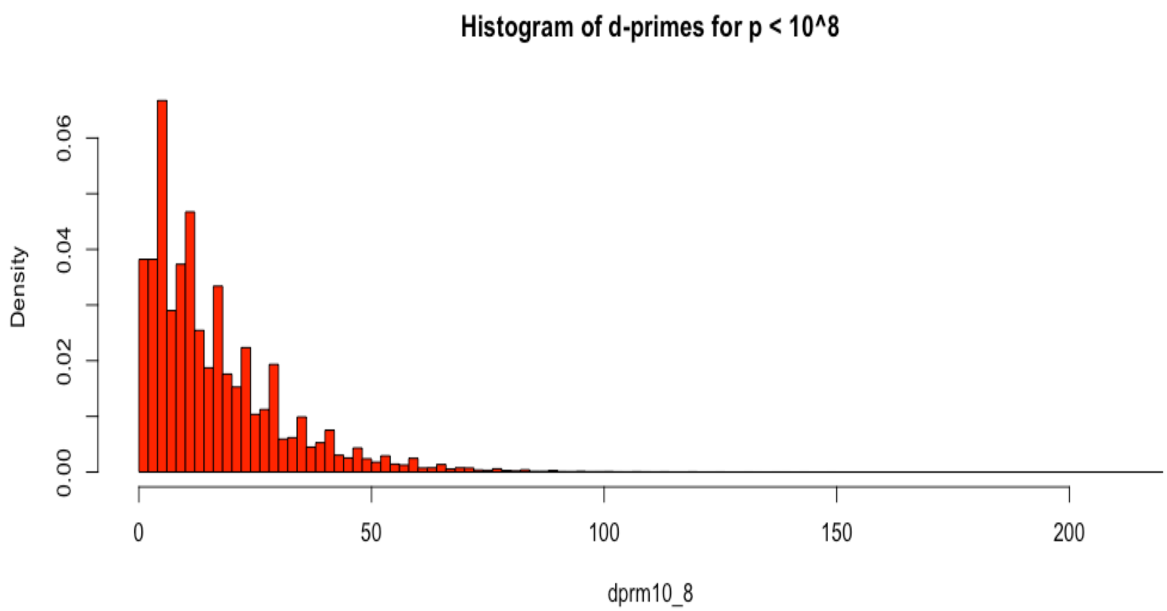
x \ d:	2	4	6	8	10	12	14	16
10^2	5	5	5	1	0	0	0	0
10^3	27	32	36	13	13	6	6	0
10^4	177	175	261	88	104	92	47	29
10^5	1100	1093	1748	697	827	871	437	307
10^6	7510	7487	12464	5124	6515	7371	3898	2653
10^7	55001	54667	93255	39505	50776	61125	33026	23422
10^8	414685	414638	724062	314770	405047	507165	276210	203311

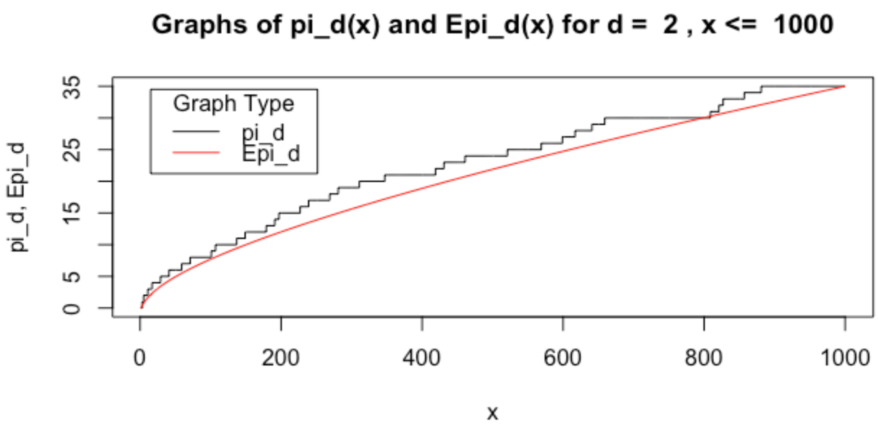
The quality of prediction of $\pi_d(x)$ by its expectation $E\pi_d(x)$ is given by the measure of

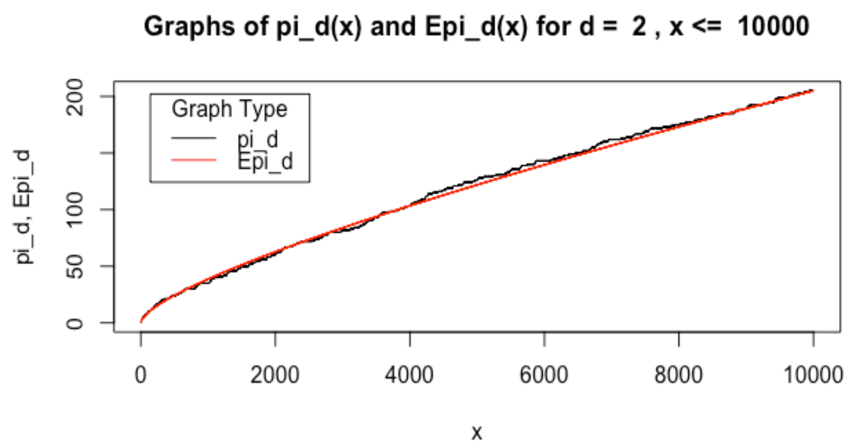
relative error $R_d(x) = \dfrac{\pi_d(x) - E\pi_d(x)}{E\pi_d(x)}$ as illustrated by the table below.

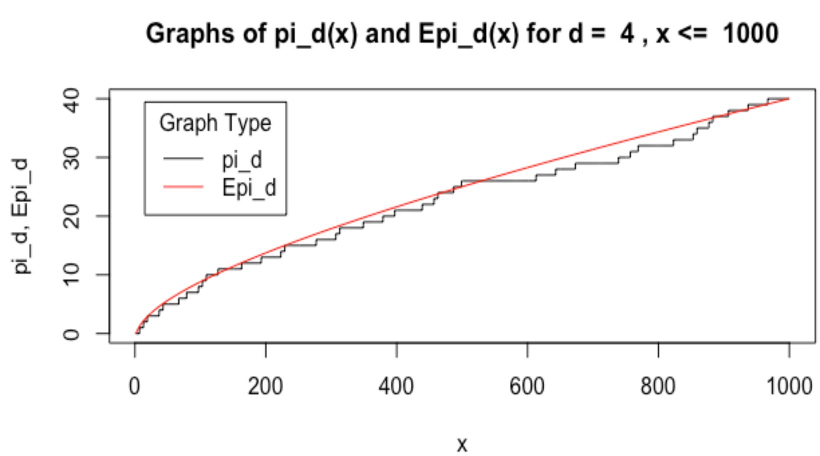
$$\text{Relative errors } \varepsilon_d(x) = \frac{\pi_d(x) - E\pi_d(x)}{E\pi_d(x)}$$

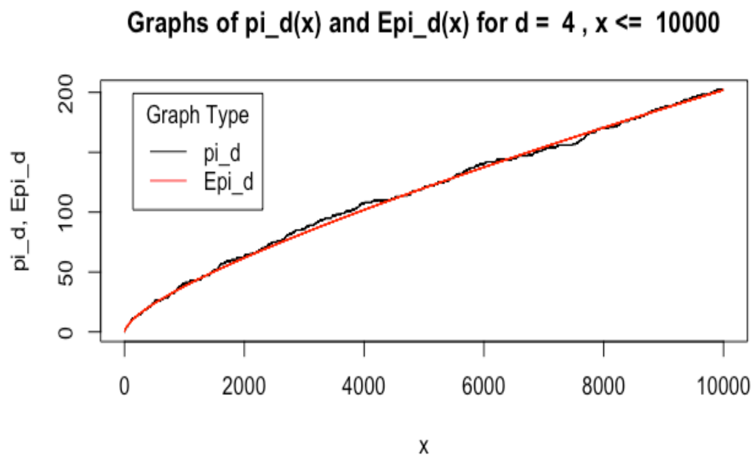
x \ d:	2	4	6	8	10	12	14	16
10^2	0.628	0.445	0.344	0.284	NaN	NaN	NaN	NaN
10^3	0.284	0.238	0.219	0.191	0.196	0.198	0.196	NaN
10^4	0.160	0.155	0.147	0.144	0.144	0.138	0.141	0.134
10^5	0.112	0.112	0.110	0.108	0.108	0.107	0.107	0.106
10^6	0.088	0.088	0.087	0.087	0.086	0.086	0.086	0.086
10^7	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072
10^8	0.062	0.062	0.062	0.062	0.062	0.062	0.062	0.061




Graphs of pi_d(x) and Epi_d(x) for d = 2 , x <= 1000
Graph Type
pi_d
Epi_d
pi_d, Epi_d
0
5
15
25
35
0
200
400
600
800
1000
x


Graphs of pi_d(x) and Epi_d(x) for d = 2 , x <= 10000
Graph Type
pi_d
Epi_d
pi_d, Epi_d
0
50
100
200
0
2000
4000
6000
8000
10000
x


Graphs of pi_d(x) and Epi_d(x) for d = 4 , x <= 1000
Graph Type
pi_d
Epi_d
pi_d, Epi_d
0
10
20
30
40
0
200
400
600
800
1000
x



Theorem 6.1

For each even value of $d \ge 2$ there are infinitely many consecutive prime numbers with a gap equal to d (so that every $D\mathbb{P}_d$ is an infinte set for all even values of $d \ge 2$) .

Proof.

This statement could be proved by using the equivalence:

$$E\left\{\pi_d(x)\right\} = \sum_{n \le x} \Psi(n,d) \sim \int_2^x \frac{\phi(t,d)}{\ln(t)\cdot \ln(t+d)} dt \text{ as } x \to \infty.$$

Indeed, if we assume that there exists $x_{\max}$ such that $\pi_d(x) = \pi_d(x_{\max})$ for all $x \ge x_{\max}$, then $\pi_d(x)$ becomes constant for sufficiently large values of x . But this contradicts the above equivalence since function $F(x) = \int_2^x \frac{\phi(t,d)}{\ln(t)\cdot \ln(t+d)} dt$ is strictly increasing for all $x > 2$ due to its derivative

$F'(x) = \frac{\phi(x,d)}{\ln(x)\cdot \ln(x+d)} > 0$ for all even $d \ge 2$ and $x > 2$.

Q.E.D.

References to section 4.

[1]. Richard K. Guy, Unsolved Problems in Number Theory, 2nd Edition. Springer-Verlag, 1994.

5. Probabilistic proof of Strong Goldbach Conjecture

According to the conjecture stated by Goldbach in his letter to Euler in 1742, "every even number $2m \geq 6$ is the sum of two odd primes" [1]. Regardless numerous attempts to prove the statement, supported in our days by computer calculations up to 4×10^{18} , it remains unproven till now.
In this part we try to solve the 'puzzle' in the framework of Probability Theory, by using the modified Cramér's probabilistic model for distribution of primes in the sequence of natural numbers $\mathbb{N}$. Strong Goldbach Conjecture (SGC), as one of the oldest notoriously known problems in Number Theory, raises a question, why it seems so difficult to decide whether the equation

$$p + p' = 2m \tag{1}$$

where p and p' are prime numbers, has at least one solution for each even number $2m \geq 6$. Indeed, occurrences of primes look very sporadic, so this is hard to oversee all possible partitions (p, p') for even numbers, like $p + p' = 2m$, especially for 'big' values of m.
One of ideas to solve such a combinatorial problem is to apply methods of Probability Theory. The first obstacle in probabilistic approach is assumed 'randomness' of occurrences of prime numbers, the second – 'independence' of their occurrences. From point of view probabilistic modeling, a sequence of natural numbers may be considered as realizations of a series of independent trials, each of which results either in a prime number or in a composite number, occurring with certain probabilities. Assume that a randomly dropped point ν on a set $\mathbb{N}$ of natural numbers appears as a prime number with probability $P\{\nu = n \in \mathbb{P}\} = g(n)$, and as a composite number with probability $P\{\nu = n \notin \mathbb{P}\} = 1 - g(n)$. Then, we are interested in an appropriate choice of function $g(n)$ that provides correct asymptotic behavior for occurrence of primes in the sequence of natural numbers. Notice that every integer solution (n, n') in primes to the equation (1) must satisfy the inequalities: $3 \leq n \leq n' \leq 2m - 3$.
For each integer $m \geq 3$ we can populate interval of integers $I_m = [3, 2m-3]$ by randomly and independently chosen numbers $(\nu, \nu') = (n, n')$ that belong to this interval, in a hope that a pair $(n, n') = (p, p') \in I_m^{\ 2} \cap \mathbb{P}^2$ would satisfy the equation (1), if such a pair exists.

Observe that if in the first throw realization $\nu = n$ in I_m occurs as a prime number with probability $g(n)$, then, independently, in the second throw realization $\nu' = n'$ in I_m may also occur as a prime number with probability $g(n')$. Therefore, given independent throws of random points ν and ν' in I_m, probability that both numbers $\nu = n$ and $\nu' = n' = 2m - k$ occur as primes is equal to the product $g(k) \cdot g(2m-k)$ of probabilities.

A well-known serious objection to this approach in solving SGC problem, pointed out by Hardy and Littlewood [26], is that 'randomly chosen' primes in a pair $(\nu, \nu') = (p, p')$ such that $p + p' = 2m$, must be dependent random variables. This means that, given m, the choice of prime number p' in the equation (1) is completely determined by the choice of p, which should occur with the same probability as p'.

Meanwhile, our probabilistic assumption is that each occurrence of a pair of primes (p, p') is a realization of an independent trial (ν, ν'), in which the value of sum $\nu + \nu'$ is unknown in advance. We are interested in an outcome which results in $p + p' = 2m$, if it occurs.

In the Hardy-Littlewood objection, instead of independently 'rolling' two 'dice' at a time for a pair of integer outcome (n, n'), it is 'rolled' just one 'die', since the sum $p + p' = 2m$ is assumed to be known in advance.

The correct resolution of this issue should be based on a reasonable definition of probability space $\Omega = \Omega_\nu \times \Omega_{\nu'}$ (a set of all possible elementary outcomes) of the 'game', generating pairs of prime numbers. The key point is that ν and ν' are considered as independent random variables in a pair (ν, ν'), and among their realizations (n, n') we are interested in those which satisfy the equation (1). If there exists a pair (ν, ν') with probability distribution that guarantees for every $m \geq 3$ occurrence of pairs (p, p') satisfying (1), then we can say that SGC is confirmed. It is an objective of this part of the paper.

Definition 1

Prime numbers $p \in \mathbb{P}, p' \in \mathbb{P}$ we call G_m*- primes* if there exist an even number $2m \geq 6$ such that $2m = p + p'$. The set of all G_m*-primes* for a given m we denote $G_m\mathbb{P}$.

For each natural $m \geq 3$ we define Goldbach function $G(2m)$ as a number of primes solving the equation $2m = p + p'$. Thus, $G(2m) = \left|G_m\mathbb{P}\right|$, where $|A|$ is a number of elements in a finite set A. As we have pointed above, we have $G_m\mathbb{P} \subseteq I_m = [3, 2m-3]$ for each $m \geq 3$.

In the context of the Strong Goldbach conjecture (SGC) we are interested in evaluation of $G(2m)$ for all even numbers $2m$ in the form $2m = p + p'$, where $(p, p') \in \mathbb{P}^2$, $m \geq 3$. Evaluation of $G(2m)$ for each natural $m \geq 3$ is a difficult combinatorial problem. Calculations show that Goldbach function $G(2m)$ asymptotically increases as m increases (though not in a monotonic way) and becomes larger for the larger values of m (see Table 4.2 and Figure 4.3), but so far there is no conclusive statements regarding behavior of $G(2m)$ as $m \to \infty$. Examples of sets $G_m\mathbb{P}$ for $2m = 10, 10^2, 10^3$ with the corresponding values of $G(2m)$ are given in the following table.

Table 1.

G_m - primes in sets G_mP and Goldbach function values

$2m$	$G_m\mathbb{P}$ sets	$G(2m)$
10	3 5 7	3
100	3 11 17 29 41 47 53 59 71 83 89 97	12
1000	3 17 23 29 47 53 59 71 89 113 137 173 179 191 227 239 257 281 317 347 353 359 383 401 431 443 479 491 509 521 557 569 599 617 641 647 653 683 719 743 761 773 809 821 827 863 887 911 929 941 947 953 971 977 983 997	56

The idea of probabilistic approach in this context is based on presentation of a naturally ordered sequence $\left(p_i\right)_{i\in\mathbb{N}}$ of prime numbers as *realizations* $\nu_k(\omega)=n$ of independent random variables in the sequence $\left(\nu_k\right)_{k\in\mathbb{N}}$ such that $P\left\{\nu_k=k\in\mathbb{P}\right\}=g(k),\ P\left\{\nu_k=k\notin\mathbb{P}\right\}=1-g(k)$.

There are two requirements for adequate presentation of primes by a sequence of random variables $\left(\nu_k\right)_{k\in\mathbb{N}}$ in an appropriate probabilistic model:

1) the choice of probability values $g(n)$ should provide an accurate asymptotic approximation to the actual distribution of prime numbers in $\mathbb{N}$ for large values of n (that is as $n\to\infty$). Meanwhile, a probabilistic model is not designed to guarantee 'intuitively correct' assignments of probability $P\left\{\nu_k(\omega)=n\in\mathbb{P}\right\}$ to concrete values of each natural number $n\in\mathbb{N}$.

2) the joint probability distribution of random variables in the sequence $\left(\nu_k\right)_{k\in\mathbb{N}}$ should objectively reproduce dependence (correlation) in occurrence of primes in $\left(p_i\right)_{i\in\mathbb{N}}$ if such probabilistic dependence exists, especially as $\nu_k=n\to\infty$.

To address the conditions mentioned above, we consider two options for the sequence $\left(\nu_k\right)_{k\in\mathbb{N}}$:

A. *The Cramér's model* for occurrences of prime numbers in the sequence of independent random variables $\left(\nu_k\right)_{k\in\mathbb{N}}$.

Recall that in the Cramér's model we consider the sequence of prime numbers as realizations of independent random variables $\nu_k,\ k\in\mathbb{N}$, on a probability space $\left(\Omega,\mathcal{F},\mathcal{P}\right)$, such that

$$\begin{aligned}P\left\{\xi(k)=1\right\}&=P\left\{\nu_k=k\in\mathbb{P}\right\}=g(k)=\frac{1}{\ln(k)},\\P\left\{\xi(k)=0\right\}&=P\left\{\nu_k=k\notin\mathbb{P}\right\}=1-g(k)=1-\frac{1}{\ln(k)}\end{aligned}\tag{4.1}$$

Here $\xi(k)=\begin{cases}1, \text{ if } \nu_k(\omega)=k\in\mathbb{P}\\0, \text{ otherwise}\end{cases}$, is an indicator function for primes in sequence of realizations $k=\nu_k(\omega)$ of random variables $\nu_k,\ k\in\mathbb{N}$.

B. *Zeta probability distribution model* considers occurrences of prime numbers in the sequence of independent random variables $\left(\nu_k\right)_{k\in\mathbb{N}}$, where each integer n is *a realization* $k=\nu_k(\omega)$ *of random variable* $\nu_k(\omega)$ on a probability space $\left(\Omega,\mathcal{F},\mathcal{P}\right)$, following Zeta probability distribution

$$P_s\{v_k = k\} = \frac{k^{-s}}{\zeta(s)}, s > 1, \tag{4.2}$$

As stated in Theorem 2.3 (formula 2.11),

$$\begin{aligned} P_s\{\xi(k)=1\} &= P_s\{v_k = k \in \mathbb{P}\} = P_k = \prod_{p\le\sqrt{k}}\left(1-\frac{1}{p^s}\right) \\ P_s\{\xi(k)=0\} &= P_s\{v_k = k \notin \mathbb{P}\} = Q_k = 1-\prod_{p\le\sqrt{k}}\left(1-\frac{1}{p^s}\right) \end{aligned} \tag{4.3}$$

Remark 1

In both models A and B we consider sample spaces Ω for sequences of random variables $(v_k)_{k\in\mathbb{N}}$ as $\Omega = \mathbb{N}^{\mathbb{N}}$, respectively, and sample spaces for the corresponding $(1,0)$-valued sequences $\{\xi(n)\}_{n\in\mathbb{N}}$ as $\Omega = \{1,0\}^{\mathbb{N}}$. In both models, each σ-algebra $\mathcal{F}$ of events is generated by all finite subsets of the corresponding sample space Ω.

This allows us to think of each pair (v_k, v_l') taken from sequence $(v_k)_{k\in\mathbb{N}}$, as of a pair of independent random variables with probabilities given in (4.1) or (4.3). Since the sequences $(v_k)_{k\in\mathbb{N}}$ in both Cramér's model and Zeta distribution model are assumed to consist of independent terms v_k, pairs (v_k, v_l') inherit the same property of independence for their terms. Notice that both models asymptotically agree with each other as stated below in Lemma 4.3.

Remark 4.2

As we mentioned above, realization of sums $v_k + v_l' = 2m$ may cause certain confusion related to possible dependence of events $\{v_k \in \mathbb{P}\}$ and $\{v_l' \in \mathbb{P}\}$. The problem of dependence for primes in the equation $v_k + v_l' = 2m$ allegedly undermines 'heuristic justification' of "a very crude probabilistic argument" [see the article 'Goldbach conjecture', Wikipedia] for evaluation of probability for a 'random' pair (v_k, v_l') as

$$P\{\{v_k = k\}\cap\{v'_{2m-k} = 2m-k\}\} = P\{\{v_k = k\}\}\cdot P\{\{v_{2m-k} = 2m-k\}\}.$$

This situation has been addressed in 1923 by Hardy and Littlewood in their *Hardy – Littlewood prime tuple conjecture* [26]. Meanwhile, the dependence of variables ν_k and ν'_l should be considered in the framework of choice for an appropriate probability space for pairs (ν_k, ν'_l) .

Our approach surmounts this obstacle: every integer n is considered as *a realization* $n = \nu_n(\omega)$ *of a random variable* $\nu_n(\omega)$ that follows *Zeta probability distribution* on probability space $(\Omega, \mathcal{F}, \mathcal{P})$:

$$P_s\{\nu_n = n\} = \frac{n^{-s}}{\zeta(s)}, s > 1.$$

We assume that *random variables in the sequence* $(\nu_k)_{k\in\mathbb{N}}$ *are* ***independent***.

Actually, ν_k and ν'_l are terms of the sequence of independent variables $(\nu_k)_{k\in\mathbb{N}}$ with realizations $n = \nu_n(\omega)$ and $n' = \nu'_{n'}(\omega)$. To evaluate $G(2m)$, define an indicator function:

$$\gamma_m(n,n') = \begin{cases} 1 \text{ if } n \in \mathbb{P} \text{ and } n' = 2m - n \in \mathbb{P} \\ 0, \text{ otherwise} \end{cases} \quad (4.4)$$

Then, $G(2m) = \sum_{n=3}^{2m-3} \gamma_m(n, 2m-n)$. Consider integers n and n' in the given interval $[3, 2m-3]$ as *realizations* $n = \nu_n(\omega)$ and $n' = \nu'_{n'}(\omega)$ *of random variable*s $\nu_n(\omega)$ and $\nu'_{n'}(\omega)$ on probability space $(\Omega, \mathcal{F}, \mathcal{P})$, which follow probability distribution according to Cramér's model A or to the model of Zeta probability distribution B.

Realizations $n = \nu_n(\omega)$and $n' = \nu'_{n'}(\omega)$ are determined by the choice of elementary events $\omega \in \Omega$ from the set Ω of all elementary events. The choice of Zeta distribution is motivated, by the fact that, due to Theorems 2.1, 2.2, 2.3 and Lemmas 4.1 and 4.3, it provides the validity of the probabilistic Cramér's model for asymptotic prime number distribution, in a full agreement with the *Prime Number Theorem*. This is especially important for SGC since we are interested in the asymptotic behavior of $G(2m)$ as $m \to \infty$.

By substituting $n = \nu_{mk}(\omega)$ and $n' = \nu'_{n'}(\omega)$ into each deterministic indicator function $\gamma_m(n)$, we obtain 'randomization' of these functions. Thus, each of the 'randomized' functions $\gamma_m(n,n') = \gamma_m(\nu_n(\omega), \nu'_{n'}(\omega))$ takes values 1 or 0 with probabilities, respectively equal to

$$P\{\gamma(n,n') = 1\} = P\{\nu_n(\omega) = n \in G_m\mathbb{P} \text{ and } \nu_{n'}(\omega) = n' = 2m - n \in G_m\mathbb{P}\} \text{ and}$$

$$P\{\gamma(n,n')=0\}=P\{\nu_n(\omega)=n\notin\mathbb{P} \text{ or } \nu'_{n'}(\omega)=n'=2m-n\notin\mathbb{P}\}$$

The combinations of 1 or 0 values of $\gamma(n,n')$, occurred for all n,n' in the interval $3\le n\le 2m-3$, determines the counts of prime numbers in each set $G_m\mathbb{P}$. We have then, $2m=\nu_k+\nu'_{2m-k}$ where $\nu_{mk}\in\mathbb{P}$, $\nu'_{2m-k}\in\mathbb{P}$. We summarize this in the following Lemma

Lemma 4.1

The Goldbach function $G(2m)$, $m\ge 3$, represents *a realization of random variable* $G(2m,\vec{\nu}_m)$ as a sum of $2m-5$ independent Bernoulian random variables

$$G\left(2m,\vec{\nu}_m(\omega)\right)=\sum_{k=3}^{2m-3}\gamma(k,2m-k) \tag{4.3}$$

where $\vec{\nu}_m=\left(\nu_{m,n}\right)_{3\le n\le 2m-3}$, and $\gamma(n,n')=\begin{cases}1 \text{ if } \nu_n=n\in\mathbb{P} \text{ and } \nu'_{n'}=n'=2m-n\in\mathbb{P}\\ 0 \text{ otherwise}\end{cases}$.

Let $\vec{\nu}_m=\left(\nu_{m,n}\right)_{3\le n\le 2m-3}$ be a subgsequence of the sequence of independent variables $\left(\nu_k\right)_{k\in\mathbb{N}}$ with $P\{\xi(\nu_n)=1\}=P\{\nu_n=n\in\mathbb{P}\}$ on a probability space $\left(\Omega,\mathcal{F},\mathcal{P}\right)$ that follows Zeta probability distribution

$$P_s\{\nu_n=n\}=\frac{n^{-s}}{\zeta(s)},\ s>1.$$

Then,

$$P\{\gamma(\nu_n,\nu'_{2m-n})=1\}=\prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right)\cdot\prod_{p\le\sqrt{2m-n}}\left(1-\frac{1}{p}\right)$$

Proof.

Independence of terms in the sequence $\left(\xi(\nu_{mk})\right)_{3\le k\le 2m-k}$ implies

$$\begin{aligned}
&P\{\gamma(\nu_n,\nu'_{2m-n})=1\}=P\{\nu_k(\omega)=n\in\mathbb{P} \text{ and } \nu_{k'}(\omega)=2m-n\in\mathbb{P}\}\\
&=P\{\nu_k(\omega)=n\in\mathbb{P}\}\cdot P\{\nu'_{2m-n}(\omega)=2m-n\in\mathbb{P}\}\\
&=P\{\xi(n)=1\}\cdot P\{\xi(2m-n)=1\}
\end{aligned}$$

Due to (2.11) and (4.5), $P\left\{\xi(n)=1\right\}=\prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right)$, and $P\left\{\xi(2m-n)=1\right\}=\prod_{p\le\sqrt{2m-n}}\left(1-\frac{1}{p}\right)$.

Thus, we have $P\left\{\xi(n)=1\right\}\cdot P\left\{\xi(2m-n)=1\right\}=\prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right)\cdot\prod_{p\le\sqrt{2m-n}}\left(1-\frac{1}{p}\right)$

Q.E.D.

Denote $g(n)=\prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right)$ and $\beta(m,n)=\prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right)\cdot\prod_{p\le\sqrt{2m-n}}\left(1-\frac{1}{p}\right)=g(n)\cdot g(2m-n)$

Notice that the defined above function $g(n)$ is a monotonically decreasing function $g:\mathbb{N}\to(0,1)$.

The following Lemma adresses the behavior of $\beta(m,n)$ for $3\le n\le 2m$.

Defining $\gamma\left(n,n'\right)$ as above in (4.4), we write

$$G(2m,\vec{v}_m)=\sum_{k=3}^{2m-3}\gamma(k,2m-k) \tag{4.4}$$

Lemma 4.2

Let $g:[a,2m]\to(0,1)$ be a monotonically decreasing function of natural n where $a\le n\le 2m$. Then $\beta(m,n)=g(n)\cdot g(2m-n)$ gets its minimum value on $[a,2m]$ at $n=m$,

that is $\min_{a\le n\le 2m}\left[g(n)\cdot g(2m-n)\right]=g^2(m)$.

Proof.

For $n\le m$ we have $g(n)\cdot g(2m-n)\ge g(m)\cdot g(m)$ since $2m-n\ge m$ for $n\le m$.

Similarly, for $n\ge m$ have $g(n)\cdot g(2m-n)\ge g(m)\cdot g(m)$ since $2m-n\le m$ for $n\ge m$.

The statement folows from the decreasing behavior of function $g(n)$.

Q.E.D.

Lemma 4.2 implies the inequality

$$\beta(m,n)\ge\beta(m,m)\ \text{ for } m,n \text{ such that } 3\le n\le 2m-n \tag{4.5}$$

for functions $g(n)=\prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right)$ and $\beta(m,n)=g(n)\cdot g(2m-n)$

Lemma 4.3.

Both models A and B are asymptotically equivalent, that is $P_A\{\xi(n)\} \sim P_B\{\xi(n)\}$ as $n \to \infty$,

where $P_A\{\xi(n)=1\} = \dfrac{1}{\ln n}$, $P_B\{\xi(n)=1\} = \prod\limits_{p\le\sqrt{n}}\left(1-\dfrac{1}{p}\right)$.

Proof.

Validity of the choice of probabilities in the Cramer's model (A) and in Zeta probability model (B) is supported by formula the (2.11) in Theorem 2.3, and by Merten's 2nd theorem ('Merten's Formula') [9, p.21-22]. Indeed, by using (2.11) and the Merten's 2-nd theorem (30), we have:

$$P_B\{\xi(n)=1\} = \prod_{p\le\sqrt{n}}\left(1-\frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\frac{1}{2}\ln(n)}\left[1+O\left(\frac{1}{\ln(n)}\right)\right] = \frac{c}{\ln(n)}\left[1+O\left(\frac{1}{\ln(n)}\right)\right] \tag{4.6}$$

where $c = \dfrac{2}{e^{\gamma}} \approx 1.122918968$ and $\gamma = \lim\limits_{n\to n}\left(\sum\limits_{k=1}^{n}\dfrac{1}{k} - \ln n\right) \approx 0.577215664$ is Euler's constant.

Then, we have $P_A\{\xi(n)=1\} = E\{\xi(n)\} \sim \dfrac{c}{\ln(n)}$. This implies $P_A\{\xi(n)\} \sim P_B\{\xi(n)\}$ as $n \to \infty$.

Q.E.D.

Due to (4.5, 4.6), we can evaluate

$$\beta(m,n) = c^2 \cdot \frac{1}{\ln(n)} \cdot \frac{1}{\ln(2m-n)} \cdot \left[1+\frac{C}{\ln(n)}\right] \cdot \left[1+\frac{C}{\ln(2m-n)}\right] \tag{4.7}$$

for $3 \le n \le 2m-3$ with an certain choice of a constant $C > 0$. Then, (4.7) implies

$\beta(m,m) = \dfrac{C'}{\ln^2(m)}\left[1+\dfrac{C'}{\ln(m)}\right]^2$ for an appropriate choice of constant $C' > 0$.

Some authors assume the constant $c = \dfrac{2}{e^{\gamma}}$ in (4.6) appears as a correcting coefficient for the Cramér's model as a compensation for possible pairwise dependence of prime occurences ignored in the model.

Then, since $E\{\gamma(v_n, v'_{2m-n})\} = \beta(m,n)$ and $E\{G(2m,\vec{v}_m)\} = \sum\limits_{n=3}^{2m-3} E\{\gamma(v_n, v'_{2m-n})\}$, we have the following expressions for expectation and a variance, respectively:

$$E\left\{G(2m,\vec{v}_m)\right\} = \sum_{n=3}^{2\cdot m-3} \beta(m,n) \sim \int_3^{2m-3} \beta(m,t)dt, \tag{4.8}$$

Due to independence of v_{mn} in $\vec{v}_m = \left(v_{mn}\right)_{3\le n\le m-3}$, we have

$$\begin{aligned} Var\left\{G(2m,\vec{v}_m)\right\} &= \sum_{n=3}^{2m-3} Var\left\{\gamma_m(v_n, v'_{2m-n})\right\} \\ &= \sum_{n=3}^{2\cdot m-3} \left[\beta(m,n)\cdot(1-\beta(m,n)\right] \sim \int_3^{2m-3} \beta(m,t)\cdot\left(1-\beta(m,t)\right)dt \end{aligned} \tag{4.9}$$

Notice that we use the approximations (4.6), (4.7) to prove the following Theorem 4.1 related to the Goldbach Conjecture.

Figure 4.1 below illustrates growth of functions $E\left\{G(2m,\vec{v}_m)\right\}$ for $m = 10, 10^2, 10^3, 10^4, 10^5, 10^6, 10^7$.

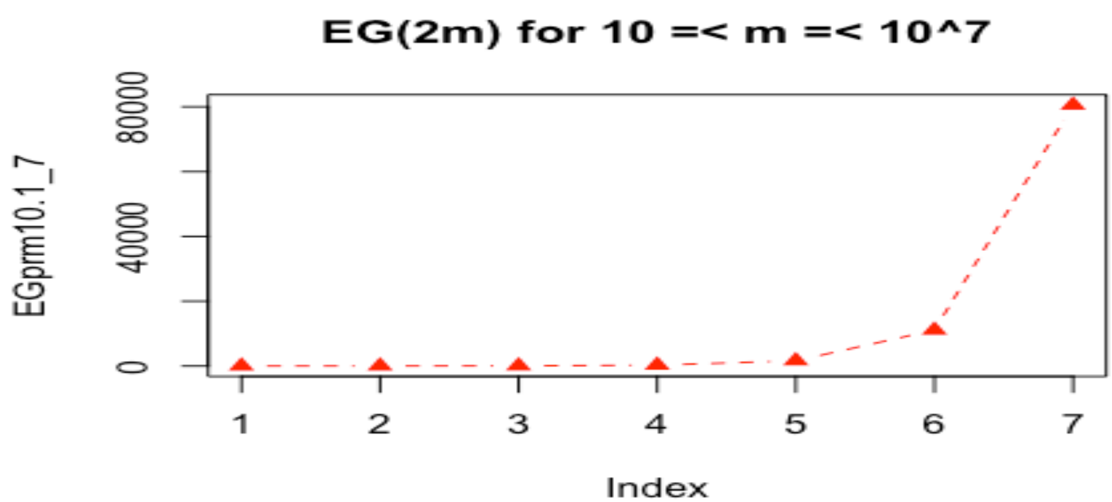


Figure 4.1

Remark 4.3

Notice meanwhile that the Goldbach function $G(2m)$ is not monotonic as ilustrated below in the domain $3 \le m \le 5000$ in Figure 4.2.

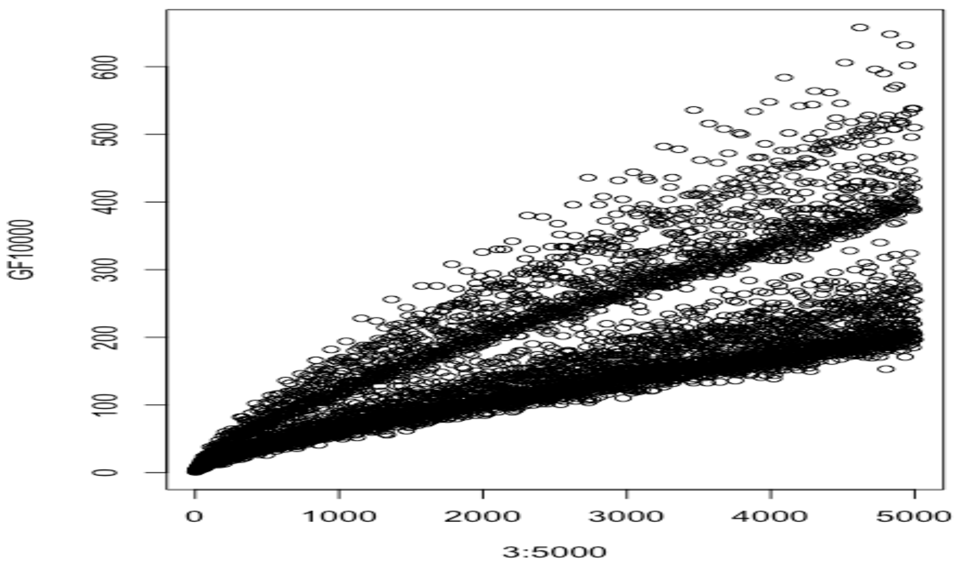


Figure 4.2

The Goldbach Conjecture for large values of can be stated in the form: probability $P\left\{G(2m,\vec{v}_{mi}) = \sum_{i=3}^{2m-3} \gamma_m(v_n, v'_{2m-n}) > 0\right\} \to 1$ for all $m \ge M$ as $M \to \infty$. Assumption that $G(2m,\vec{v}_m) = 0$ for some arbitrary large value of m contradicts to stochastic behavior of $G(2m,\vec{v}_m)$ when m increases, as demonstrated in the following theorem.

Theorem 4.1

Let $G_m\mathbb{P}$ for $m \ge 3$ be a set of all G-primes, that is prime numbers $p, p' \in \mathbb{P}$ such that $p + p' = 2m$. Let each random variable ν_k in the sequence of independent random variables $(\nu_k)_{k\in\mathbb{P}}$ follow Zeta probability distribution: $P\{\nu_k = n\} = \dfrac{n^{-s}}{\zeta(s)}$ $(s > 1)$ and $\vec{\nu}_m = (\nu_k)_{3\le k\le 2m-3}$ is a subsequence of the sequence of primes $(\nu_k)_{k\in\mathbb{P}}$ such that

$$\gamma(\nu_n, \nu'_{2m-n}) = \begin{cases} 1 \text{ if } \nu_n = n \in \mathbb{P} \text{ and } \nu_{n'} = n' = (2m-n) \in \mathbb{P} \\ 0 \text{ otherwise} \end{cases}, \text{ where } \; n, n' \in [3, 2m-3].$$

Then $\left(\gamma(\nu_n, \nu'_{2m-n})\right)_{3\le n\le 2m-3}$ is a sequence of independent Bernoulli variables and the randomized Goldbach function $G(2m, \vec{\nu}_m) = \sum_{n=3}^{2m-3} \gamma(\nu_n, \nu'_{2m-n})$ has the following properties:

(1) $$P\{G(2m, \vec{\nu}_m) = 0\} = P\left\{\bigcap_{n=3}^{2m-3} \{\gamma(\nu_n, \nu'_{2m-n}) = 0\}\right\} \to 0 \text{ as } m \to \infty.$$

(2) $$\sum_{m=3}^{\infty} P\{G(2m, \vec{\nu}_m) = 0\} < \sum_{m=3}^{\infty} e^{-C\cdot\frac{2m-5}{\ln^2(m)}} < \infty \quad (C>0).$$

(3) $$\lim_{M\to\infty} P\left\{\bigcap_{m=M}^{\infty} \{G(2m, \vec{\nu}_m) = |G_m\mathbb{P}| \; > 0\}\right\} \to 1$$

Proof.

Independence of the Bernoulli variables in the set $\{\gamma_m(\nu_{mi}) \,|\, 3 \le i \le 2m-3\}$ follows from the assumed independence of ν_k in the sequence $\vec{\nu}_m = (\nu_k)_{3\le k\le 2m-3}$. This implies:

$$P\{G(2m, \vec{\nu}_m) = 0\} = P\left\{\bigcap_{n=3}^{2m-3} \{\gamma_m(\nu_n, \nu'_{2m-n}) = 0\}\right\} = \prod_{i=3}^{2m-3} P\{\gamma_m(\nu_n, \nu'_{2m-n}) = 0\} = \prod_{n=3}^{2m-3} [1 - \beta(m,n)].$$

Due to Lemma 4.1 and (4.5), $\beta(m,n) \ge \beta(m,m) \sim \dfrac{C}{(\ln m)^2}\left[1 + \dfrac{C}{\ln m}\right]^2$ for an appropriate choice of constant $C > 0$. From this follows $1 - \beta(m,n) \le 1 - \beta(m,m)$ and

$$P\left\{G(2m,\vec{v}_m)=0\right\} \le \prod_{n=3}^{2m-3}\left[1-\frac{C}{(\ln m)^2}\cdot\left(1+\frac{C}{\ln m}\right)^2\right] = \left[1-\frac{C}{(\ln m)^2}\cdot\left(1+\frac{C}{\ln m}\right)^2\right]^{2m-5} \sim e^{-\frac{2m-5}{D(m)}}$$

where $D(m)=\frac{C}{(\ln m)^2}\cdot\left(1+\frac{C}{\ln m}\right)^2$ and $e^{-\frac{2m-5}{D(m)}}\to 0$ as $m\to\infty$.

This proves that $P\left\{G(2m,\vec{v}_m)>0\right\}\to 1$ as $m\to\infty$.

A critical question for the Goldbach Conjecture can be stated as follows:

is this true that for 'sufficiently large' values of m such that $m > M \ge 3$ the probability that all sets $G_m\mathbb{P}$ are not empty is equal to 1:

$$P\left\{\bigcap_{m=M}^{\infty}\left\{G(2m,\vec{v}_m)=|G_m\mathbb{P}|\ >0\right\}\right\}\to 1 \text{ as } M\to\infty.$$

Consider the probability of the opposite event: $P\left\{\bigcup_{m=M}^{\infty}\left\{G(2m,\vec{v}_m)\right\}=0\right\}$ and prove that

$$P\left\{\bigcup_{m=M}^{\infty}\left\{G(2m,\vec{v}_m)\right\}=0\right\}\to 0 \text{ as } M\to\infty.$$

This is a probability that for sufficiently large value of M there exists at least one value of $m\ge M$ such that the set $G_m\mathbb{P}$ is empty.

We have: $P\left\{\bigcup_{m=3}^{\infty}\left\{G(2m,\vec{v}_m)=0\right\}\right\} \le \sum_{m=3}^{\infty}P\left\{G(2m,\vec{v}_m)=0\right\} < \sum_{m=3}^{\infty}e^{-\frac{2m-5}{D(m)}} < \infty$ (by the limit test).

Then, $P\left\{\bigcup_{m=M}^{\infty}\left\{G(2m,\vec{v}_m)\right\}=0\right\} \le \sum_{m=M}^{\infty}P\left\{G(2m,\vec{v}_m)=0\right\}\to 0$ as $M\to\infty$,

due to convergence of the series $\sum_{m=3}^{\infty}P\left\{G(2m,\vec{v}_m)=0\right\}$.

Q.E.D.

There is another way to evaluate the probability $P\left\{\left|G_m\mathbb{P}\right|<1\right\}$.

Denote $Y_m=\sum_{n=3}^{2m-3}Y_{n,2m-n}$ where $Y_{nn'}=\gamma_{nn'}-E\left\{\gamma_{nn'}\right\}$ and $\gamma_{n,n'}=\gamma_m(\nu_n,\nu'_{n'})$ for each n $(3\le n\le 2m-3)$.

Theorem 4.2

Let $G_m\mathbb{P}$ for $m \ge 3$ be a set of all G_m-primes, that is prime numbers $p, p' \in \mathbb{P}$ such that $p + p' = 2m$.

Let each random variable ν_k in the sequence $\left(\nu_k\right)_{k\in\mathbb{N}}$ follows Zeta probability distribution:

$P\left\{\nu_k = n\right\} = \dfrac{n^{-s}}{\zeta(s)}$ $(s > 1)$. Consider $\vec{\nu}_m = \left(\nu_k\right)_{3\le k\le 2m-3}$ as a subsequence of $\left(\nu_k\right)_{k\in\mathbb{N}}$ such that

$$\gamma_m(\nu_n, \nu'_{n'}) = \begin{cases} 1 \text{ if } \nu_n = n \in \mathbb{P} \text{ and } \nu'_n = n' = 2m - n \in \mathbb{P} \\ 0 \text{ otherwise} \end{cases}.$$

We have $Y_{nn'} = \gamma_{nn'} - E\left\{\gamma_{nn'}\right\} = \gamma_{nn'} - \beta(m,n)$, where $E\left\{\gamma_{nn'}\right\} = \beta(m,n)$.

Then, for $X_m = \dfrac{Y_m - E\left\{Y_m\right\}}{\sqrt{Var\left\{Y_m\right\}}} = \dfrac{G(2m,\vec{\nu}_m) - E\left\{G(2m,\vec{\nu}_m)\right\}}{\sqrt{Var\left\{G(2m,\vec{\nu}_m)\right\}}}$ we have for $m \to \infty$

$$P\left\{G(2m,\vec{\nu}_m) < 1\right\} = P\left\{X_m < x_{cr}(m)\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_{cr}(m)} e^{-\frac{1}{2}t^2}\, dt,$$

where $x_{cr}(m) = \dfrac{1 - E\left\{G(2m,\vec{\nu}_m)\right\}}{\sqrt{Var\left\{G(2m,\vec{\nu}_m)\right\}}} \to -\infty$ and $\lim\limits_{m\to\infty} P\left\{G(2m,\vec{\nu}_m) < 1\right\} = 0$ or, equivalently,

$$\lim_{m\to\infty} P\left\{\left|G_m\mathbb{P}\right| \ge 1\right\} = \lim_{m\to\infty} P\left\{G(2m,\vec{\nu}_m) \ge 1\right\} = 1$$

Proof.

We have:

$Var\left\{Y_{n,2m-n}\right\} = Var\left\{\gamma_{n,2m-n}\right\} = \beta(m,n)\cdot\left(1-\beta(m,n)\right) \ge \beta(m,m)\cdot\left(1-\beta(m,n)\right) \ge \beta(m,m)\cdot\left(1-\beta(m,3)\right).$

since $\beta(m,n) \ge \beta(m,m)$ for all n: $3 \le n \le 2m-3$ due to Lemma 4.2.

Denoting $Y_m = \sum\limits_{n=3}^{2m-3} Y_{n,2m-n}$, we write $Var\left\{Y_m\right\} = Var\left\{G(2m,\vec{\nu}_m)\right\} = \sum\limits_{n=3}^{2m-3} Var\left\{Y_{n,m-n}\right\}$.

Then, $E\left\{\left|Y_{n,2m-n}\right|\right\} = E\left\{\left|\gamma_{n,2m-n} - \beta(m,n)\right|\right\} \le \left(1+\beta(m,n)\right) \le 2$,

$$\beta_{3,mn} = E\left\{\left|Y_{n,2m-n}\right|^3\right\} = E\left\{\left|\gamma_{n,2m-n} - \beta(m,n)\right|^3\right\} = p_{mn}\cdot q_{mn}\cdot(p_{mn}^2 + q_{mn}^2) \le p_{mn}\cdot q_{mn} = \sigma_{mn}^2,$$

where $p_{mn} = E\left\{\gamma_{n,2m-n}\right\} = \beta(m,n),\ q_{mn} = 1 - p_{mn}, \sigma_{mn}^2 = Var\left\{\gamma_{n,2m-n}\right\} = p_{mn}\cdot q_{mn}.$

Then, $E\{Y_m\}=0$, $Var\{Y_m\}=\sum_{n=3}^{2m-3} Var\{Y_{n,2m-n}\}=\sum_{n=3}^{2m-3}\sigma^2_{n,2m-n}=\sum_{n=3}^{2m-3}\left[\beta(m,n)\cdot\left(1-\beta(n,m)\right)\right]=\sigma^2_m$.

Due to (4.7), we have $\beta(m,m)=\frac{C'}{\ln^2(m)}\left[1+\frac{C'}{\ln(m)}\right]^2$.

This implies: $\sigma^2_m = Var\{Y_m\}=\sum_{k=3}^{2m-3}\sigma^2_{k,2m-k}\ge\left(2m-5\right)\cdot\beta(m,m)\cdot\left(1-\frac{1}{\ln^2 3}\right)\to\infty$ as $m\to\infty$

so that $\sigma^2_m = Var\{Y_m\}\to\infty$ as $m\to\infty$.

All terms $Y_{n,2m-n}=\gamma_{n,2m-n}-p_{mn}$ in the sum $Y_m=\sum_{n=1}^{2m-5}Y_{n,2m-n}$ are uniformly bounded

($\left|Y_{n,2m-n}\right|\le 1$ for all m and n) and centered, because $Y_{n,2m-n}=\gamma_{n,2m-n}-E\{\gamma_{n,2m-n}\}=\gamma_{n,2m-n}-p_{mn}$,

so that $E\{Y_m\}=0$. We have also $\beta_{3,mn}=E\left\{\left|Y_{mn}\right|^3\right\}\le\sigma^2_{mn}$. Since $\sigma^2_m=Var\{Y_m\}\to\infty$ as $m\to\infty$,

this implies the sufficient Liapunov condition

$$\frac{1}{\sigma^3_m}\sum_{k=3}^{2m-3}E\left\{\left|Y_{mk}\right|^3\right\}=\frac{1}{\sigma^3_m}\sum_{k=3}^{2m-3}\beta_{3,mk}\le\frac{1}{\sigma^3_m}\sum_{k=3}^{2m-3}\sigma^2_{mk}=\frac{\sigma^2_m}{\sigma^3_m}=\frac{1}{\sigma_m}\to 0 \text{ as } m\to\infty$$

for *Central Limit Theorem* (called in this situation *Bounded Liapunov Theorem* [18], [23])

for the sequence of normed and centered variables $X_m=\frac{Y_m}{\sigma_m}$, such that $E\{X_m\}=0, Var\{X_m\}=1$.

This guarantees the uniform convergence of probability distribution function $F_{X_m}(x)$ of X_m

to the standard normal probability distribution $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}t^2}\,dt$.

Recall that $X_m=\frac{Y_m-E\{Y_m\}}{\sqrt{Var\{Y_m\}}}=\frac{G(2m,\vec{v}_m)-E\{G(2m,\vec{v}_m)\}}{\sqrt{Var\{G(2m,\vec{v}_m)\}}}$

where $E\{G(2m,\vec{v}_m)\}=\sum_{n=3}^{m-3}E\{\gamma_m(v_{m,n})\}$, $G(2m,\vec{v}_m)=\sum_{i=3}^{2m-3}\gamma_m(v_{mi})$.

Then, we have

$$P\left\{G(2m,\vec{v}_m)<1\right\}=P\left\{X_m<x_{cr}(m)\right\}\approx\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_{cr}(m)}e^{-\frac{1}{2}t^2}\,dt,$$

where $x_{cr}(m)=\dfrac{1-E\left\{G(2m,\vec{v}_m)\right\}}{\sqrt{Var\left\{G(2m,\vec{v}_m)\right\}}}=\dfrac{1-\sum\limits_{k=3}^{3m-3}p_{mk}}{\sigma_m}$.

Since $\dfrac{1}{\sigma_m}\sum\limits_{k=3}^{2m-3}p_{mk}\geq\dfrac{1}{\sigma_m}\sum\limits_{k=3}^{2m-3}p_{mk}q_{mk}=\dfrac{\sigma_m^2}{\sigma_m}=\sigma_m\to\infty$ as $m\to\infty$, we have $x_{cr}(m)\to-\infty$ as $m\to\infty$.

This implies $\lim\limits_{m\to\infty}P\left\{G(2m,\vec{v}_m)<1\right\}=0$, which means that

$$\lim_{m\to\infty}P\left\{\left|G_m\mathbb{P}\right|\geq 1\right\}=\lim_{m\to\infty}P\left\{G(2m,\vec{v}_m)\geq 1\right\}=1.$$

Q.E.D.

The values of $P\left\{\left|G_m\mathbb{P}\right|<1\right\}$ and $x_{cr}(m)$ for $m=10^3,10^4,\ldots,10^8$ are given in the following table.

m	10^3	10^4	10^5	10^6	10^7	10^8
$x_{cr}(m)$	-6.866973	-16.130926	-40.343498	-105.469447	-284.348502	-783.836910
$P\left\{\left|G(2m)\right|<1\right\}$	3.278916×10^{-12}	7.734173×10^{-59}	0.0000000	0.0000000	0.0000000	0.0000000

6. A recursive algorithm generating the sequence of consecutive Goldbach sets

Abstract

A Recursive Algorithm described here generates consecutive sequences of Goldbach sets

$$\left\{G_k\mathbb{P} \mid 3 \le k \le m\right\}, \text{ where } G_k\mathbb{P} = \left\{n, n' \mid n \in \mathbb{P}, n' = 2\cdot k - n \in \mathbb{P}\right\}$$

toward the proof of the Strong Goldbach Conjecture. This approach is grounded in the fundamental principle of mathematical induction and uses rather elementary set-theoretical technique.

I tried to follow the idea of Martin Aigner and Günter M. Ziegler [10] to make the content accessible to the readers with their background only in the basics of discrete mathematics.

The main idea is to develop a recursive algorithm toward building the sequence of consecutive Goldbach sets $\left\{G_k\mathbb{P} \mid 3 \le k \le m\right\}$ representing solutions to the system of Goldbach equations

$$\left\{x + y = 2\cdot k \mid 3 \le k \le m\right\} \text{ in the intervals } I_k = [3, 2\cdot k - 3],\ 3 \le k \le m.$$

Validity of the algorithm follows from the proved here recursive formula

$$\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2\cdot(m-k)\right) \cap S_m\right] = G_m\mathbb{P} \ne \varnothing\ ,$$

given the inductive assumption that $G_k\mathbb{P} \ne \varnothing$ for all $k : 3 \le k < m$, where $S_m = I_m \cap \mathbb{P}$, and $\mathbb{P}$ is a set of all odd prime numbers. We establish a definite connection between the Goldbach function $G(2m)$ and some invariant properties for Diophantine variety of Goldbach sets.

"The most interesting facts are those which can be used several times, those which have a chance of recurring ..."

(Henry Poincaré, The Value of Science)

1. Shift invariance of Goldbach Set.

We approach here one of old classical problems in Number Theory known as the strong form of Goldbach Conjecture (SGC) [1, 5]. According to the conjecture stated by Goldbach in his letter to Euler in 1742, "every even number $2m \geq 6$ is the sum of two odd primes" [1]. Regardless numerous attempts to prove the statement, supported in our days by computer calculations up to 4×10^{18}, it remains unproven till now.

Let $\mathbb{N}$ be a set of natural numbers, and $\mathbb{P}$ a set of odd primes (all prime numbers excluding 2).

The Goldbach's Conjecture (GC), as one of the oldest and notoriously known unsolved problems in Number theory, raises a question why it seems so difficult to decide whether the equation

$$p + p' = 2m, \qquad (*)$$

where p and p' are prime numbers, has at least one solution for each even number $2m \geq 6$. Indeed, occurrences of primes look very sporadic, so that it is hard to predict, that there exists a pair of primes (p, p') related by the equation (*), especially for 'big' values of m.

Notice that every solution $(n,n') = (p,p')$ in primes to the equation $p + p' = 2m$, must satisfy condition: $(n,n') \in [3,2m-3]^2$.

We call a prime number p a G_m- prime (*Goldbach prime*) if $p' = 2m - p$ is also a prime number.

Then, denote $G_m\mathbb{P}$ as set of all G_m*- primes*, and call $G_m\mathbb{P}$ *Goldbach set.*

The number of elements in set $G_m\mathbb{P}$, denoted $G(2m)$, is called Goldbach function.

Obviously, for all $m \geq 3$ we have $G_m\mathbb{P} \subset I_m = [3, 2\cdot m - 3]$. A set $G_m\mathbb{P}$ is empty if for some $m \geq 3$ G_m- primes do not exist. Goldbach function $G(2m)$ counts the number of solutions to the equation

$$n + n' = 2m, \ (n,n') \in \mathbb{P}^2 \qquad (1)$$

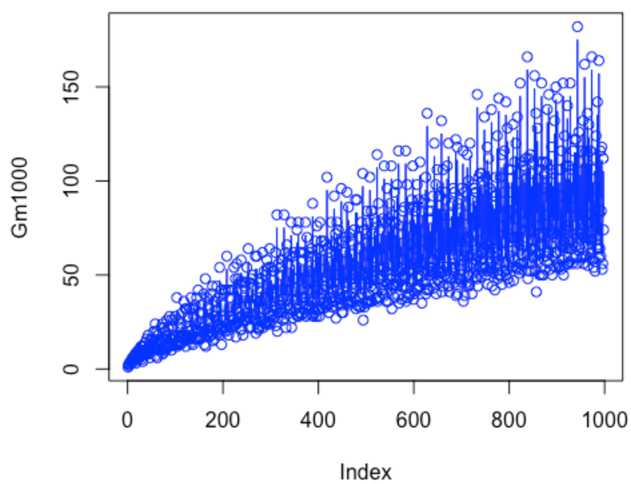
where n and n' are prime numbers, m is any integer $m \geq 3$.

Obviously, any pair (p, p') of primes greater then 2 solves (1) for $2m = p + p'$.

Due to infinity of $\mathbb{P}$, this implies that the Goldbach function has $\limsup G(2m) = \infty$.

The Strong Goldbach Conjecture states that $\min G(2m) = 1$, so that every set $G_m\mathbb{P}$ is nonempty set for all $m \geq 3$. Calculations show that $\max\limits_{m \leq M} G(2m)$ increases with M, though $G(2m)$ is not a monotonically increasing function (Fig.1)

Goldbach function $G(2 \cdot m)$ for $m = 3, 4, \ldots, 1000$ (Fig.1)



We observe that each pair (n, n') which solves (1) must belong to a set $[3, 2m-3]^2 = I_m^2$, where $I_m = [3, 2m-3] = \{3, 4, \ldots, 2m-3\}$. Since 3 is prime, if $n' = (2 \cdot m - 3) \in \mathbb{P}$, the pair $(3, 2 \cdot n - 3)$ solves (1), so that the prime $(2 \cdot m - 3) \in G_m\mathbb{P}$, and we need to consider the case $(2 \cdot m - 3) \notin \mathbb{P}$. In general, if p is a prime number such that $(2 \cdot m - p) \in \mathbb{P}$, then $(2 \cdot m - p) \in G_m\mathbb{P}$.

Consider a *shift mapping* of an interval of integers $\theta_m : I_m \to I_m$ given by the formula

$$\theta_m(n) = 2m - n. \tag{2}$$

Denote $\mathcal{F}_m$ an algebra of all subsets of the interval $I_m = [3, 2m-3]$.

Obviously, θ_m is one-to-one and has an inverse θ_m^{-1}, so that for all $A \in \mathcal{F}_m$

we have $\theta_m(A) \in \mathcal{F}_m, \theta_m^{-1}(A) \in \mathcal{F}_m$. Denote $I_m^- = [3, m-1], I_m^0 = \{m\}, I_m^+ = [m+1, 2m-3]$.

Obviously, θ_m is idempotent: $\theta_m^2 = id$ (an identical map), that is $\theta_m^{-1} = \theta_m$.

Indeed, $\theta_m^2(n) = \theta_m(\theta_m(n)) = \theta_m(2 \cdot m - n) = 2m - (2m - n) = n$.

Let S_m denote a set of prime numbers in the interval of integers $I_m = [3, 2m-3]$,

that is $S_m = I_m \cap \mathbb{P}$, and $S_m^c = I_m \setminus S_m$ its complement in I_m so that

$I_m = S_m \cup S_m^c$, $S_m \cap S_m^c = \varnothing$. While S_m stands for the set of primes in I_m,

S_m^c is the set of composite numbers in I_m.

We denote $\theta_m(S_m) = 2 \cdot m - S_m = \{n' \mid n' = 2 \cdot m - n, n \in S_m\}$.

The Strong Goldbach Conjecture asserts that for any $m \geq 3$ the set $G_m\mathbb{P}$ is not empty:

$$G_m\mathbb{P} = \{n, n' \mid n \in \mathbb{P}, n' = (2m - n) \in \mathbb{P}\} = (2m - S_m) \cap S_m = \theta_m(S_m) \cap S_m \neq \varnothing.$$

Lemma 1.

Golbach sets $G_m\mathbb{P}$ on intervals I_m are θ_m- shift invariant: $\theta_m(G_m\mathbb{P}) = G_m\mathbb{P}$.

Proof.

Notice that the sets $I_m, \{m\}, \{3, 2 \cdot m - 3\}$ and $\theta_m(S_m) \cap S_m$ are invariant sets of the map

$\theta_m : I_m \to I_m$ since for all $n \in I_m$ we have $\theta_m(\{n, \theta_m(n)\}) = \{\theta_m(n), \theta_n^2(n)\} = \{\theta_m(n), n\}$,

due to $\theta_m^2 = id$. θ_m-invariance of $G_m\mathbb{P}$ also follows directly from the equalities

$$\theta_m(G_m\mathbb{P}) = \theta_m(S_m \cap \theta_m(S_m)) = \theta_m(S_m) \cap \theta_m^2(S_m) = \theta_m(S_m) \cap S_m = G_m\mathbb{P}.$$

Q.E.D.

In what follows we need several recursively derived formulas.

Lemma 2.

1) $I_m = I_{m-1} \cup \{2m-4, 2m-3\}$, where $I_m = [3, 2m-3]$ (3)

2) $S_m = S_{m-1} \cup \left(\mathbb{P} \cap \{2m-3\}\right) = \begin{cases} S_{m-1} \cup \{2m-3\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \end{cases}$ (4)

3) $\theta_m(S_m) = \begin{cases} \theta_m\left(S_{m-1}\right) \cup \{3\} \text{ if } \{2m-3\} \in \mathbb{P} \\ \theta_m\left(S_{m-1}\right) \text{ if } \{2m-3\} \notin \mathbb{P} \end{cases}$ (5)

4) $G_m\mathbb{P} = \begin{cases} \theta_m(S_m) \cap S_m = \theta_m\left(S_{m-1}\right) \cap S_{m-1} \cup \{3\} \text{ if } (2m\text{-}3) \in \mathbb{P} \\ \theta_m(S_m) \cap S_m = \theta_m\left(S_{m-1}\right) \cap S_{m-1} \text{ if } (2m-1) \notin \mathbb{P} \end{cases}$ (6)

Proof.

1) We observe that $I_{m-1} = [3, 2\cdot(m-1)-3] = [3, 2\cdot m-5]$

so that $I_m = [3, 2m-3] = I_{m-1} \cup \{2m-4, 2m-3\}$.

2) $\left(\{2m-4, 2m-3\} \cap \mathbb{P}\right) = \left(\{2m-3\} \cap \mathbb{P}\right)$ implies

$$S_m = I_m \cap \mathbb{P} = \left(I_{m-1} \cap \mathbb{P}\right) \cup \left(\{2m-4, 2m-3\} \cap \mathbb{P}\right) = S_{m-1} \cup \left(\mathbb{P} \cap \{2m-3\}\right),$$

Thus, $S_m = I_m \cap \mathbb{P} = S_{m-1} \cup \{2m-3\}$ if $(2m-3) \in \mathbb{P}$ and $S_m = S_{m-1}$ otherwise.

3) $\theta_m(S_m) = \theta_m\left(S_{m-1}\right) \cup \theta_m\left(\{2m-3\} \cap \mathbb{P}\right) = \begin{cases} \theta_m\left(S_{m-1}\right) \cup \{3\} \text{ if } \{2m-3\} \in \mathbb{P} \\ \theta_m\left(S_{m-1}\right) \text{ if } \{2m-3\} \notin \mathbb{P} \end{cases}$,

since $\theta_m(2m-3) = 2\cdot m - (2m-3) = 3$.

4) $G_m\mathbb{P} = \theta_m(S_m) \cap S_m = \left(\theta_m(S_{m-1}) \cup \{3\}\right) \cap S_{m-1} = \left(\theta_m(S_{m-1}) \cap S_{m-1}\right) \cup \{3\}$

if $(2m\text{-}3) \in \mathbb{P}$, and $G_m\mathbb{P} = \theta_m(S_{m-1}) \cap S_{m-1}$ if $(2m-3) \notin \mathbb{P}$, that is

$$G_m\mathbb{P} = \begin{cases} \theta_m\left(S_{m-1}\right) \cap S_{m-1} \text{ if } (2m\text{-}3) \notin \mathbb{P} \\ \left(\theta_m\left(S_{m-1}\right) \cap S_{m-1}\right) \cup \{3\} \text{ if } (2m-3) \in \mathbb{P} \end{cases}$$

Q.E.D.

Notice that (5) implies that if $(2m-3) \notin \mathbb{P}$, then $S_m = S_{m-1}$ and $G_m\mathbb{P} = \theta_m(S_{m-1}) \cap S_{m-1}$.

In case when $(2\cdot m-3) \in \mathbb{P}$, we have $G_m\mathbb{P} \neq \varnothing$ since $(2m-3) \in \mathbb{P}$, so that $3 + (2m-3) = 2m$.

Thus, we need to consider the case $(2m-3) \notin \mathbb{P}$. Observe that $\theta_m(S_{m-1}) = \theta_{m-1}(S_{m-1}) + 2$, due to (7) in Lemma 3 below. This implies $G_m\mathbb{P}=\left(\theta_{m-1}(S_{m-1})+2\right)\cap S_{m-1}$. If $p\in G_{m-1}\mathbb{P}\neq\varnothing$,

then $p\in S_{m-1}$ and $p\in\theta_{m-1}(S_{m-1})$. Assuming now that p is a twin prime, that is $(p+2)\in S_m=S_{m-1}$, we have that $G_{m-1}\mathbb{P}\neq\varnothing$ implies $G_m\mathbb{P}=\left(\theta_{m-1}(S_{m-1})+2\right)\cap S_{m-1}\neq\varnothing$.

See in what follows the more detailed discussion and definitions of sets of twin primes $T_1\mathbb{P}$ and t-primes $T_t\mathbb{P}$ related to the Goldbach Conjecture.

The next Lemma concerns some properties of the shift transformation θ_m $(m\geq 3)$

Lemma 3.

Consider a shift transformation $\theta_m:\ \mathbb{Z}\to\mathbb{Z}$ such that $\theta_m(n)=2\cdot m-n,$ where $n\in\mathbb{N},\ m\in\mathbb{N}\ (m\geq 3)$.

Then, for any subset $A\subseteq\mathbb{Z}$ and integer $t\in\mathbb{N}$ the following properties of θ_m hold true:

$$\begin{aligned}\theta_{m+t}(A)&=\theta_m(A)+2\cdot t\\ \theta_m(A)&=\theta_{m-t}(A)+2\cdot t\end{aligned}\tag{7}$$

Proof.

$$\begin{aligned}\theta_{m+t}(A)&=2\cdot(m+t)-A=2\cdot m-A+2\cdot t=\theta_m(A)+2\cdot t\\ \theta_m(A)&=2\cdot m-A=2\cdot(m-t)-A+2\cdot t=\theta_{m-t}(A)+2\cdot t\end{aligned}$$

Q.E.D.

2. Twin primes, t-primes and Goldbach sets.

Let $T_1\mathbb{P}=\left\{p\mid p\in\mathbb{P}\text{ and }(p+2)\in\mathbb{P}\right\}$ stand for a set of all twin primes and consider $G_k\mathbb{P}\cap T_1\mathbb{P}$ for each $k\ (3\leq k\leq m)$. Thus, if for some $k\ (3\leq k\leq m)$ $G_k\mathbb{P}\cap T_1\mathbb{P}\neq\varnothing,$ then $G_{k+1}\mathbb{P}\neq\varnothing$.

Lemma 3 implies that if for a prime $p\in G_k\mathbb{P}$ there exists a twin prime $(p+2)\in\mathbb{P}$, then $\left(p+2\right)\in G_{k+1}\mathbb{P}$. This shows some connection between the Twin Prime Conjecture and the Strong Goldbach Conjecture (SGC), and, moreover, between the t-Prime Conjecture (de Polignac Conjecture (1849)) and SGC, as we observe below. This also shows how nonempty Goldbach sets $G_k\mathbb{P}$ can propagate further with increasing values of k.

Definition.

Denote, in general, by $T_t\mathbb{P}$ $(t \in \mathbb{N})$ a set *of t-primes* for some $t \in \mathbb{N}$, that is

$T_t\mathbb{P} = \{p \mid p \in \mathbb{P} \text{ and } (p + 2 \cdot t) \in \mathbb{P}\}$. Notice that $\bigcup_{t=0}^{\infty} T_t\mathbb{P} = T_0\mathbb{P} = \mathbb{P}$,

where $\mathbb{P}$ stands for the set of all odd prime numbers. Consider examples below:

$\{3,5,11,17,29,41\} \subset T_1\mathbb{P}$, $\{3,7,13,19,37,43\} \subset T_2\mathbb{P}$, $\{5,7,11,17,23,31,37,41\} \subset T_3\mathbb{P}$, and so on.

Propagation of nonempty $G_k\mathbb{P}$ for all $k \geq 3$ is based on the following observations.

Lemma 4.

Let $p \in G_k\mathbb{P} \neq \varnothing$ and $p \leq k$. There exist $q \in \mathbb{P}$ $(q > p)$ and $t \in \mathbb{N}$ ($1 \leq t < k-1$) such that $q = p + 2 \cdot t \in \mathbb{P}$, and $p \in T_t\mathbb{P}$. This implies that there exists $t \in \mathbb{N}$ ($1 \leq t < k-1$) such that $p \in G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ and $q = (p + 2 \cdot t) \in G_{k+t}\mathbb{P} \neq \varnothing$.

Proof.

Let $p \in G_k\mathbb{P}$, $p \leq k$. Thanks to the Bertrand's postulate [4], there exists a prime q between integers k and $2 \cdot k$ $(k > 3)$. This implies that there exists $t \in \mathbb{N}$ such that this prime q can be expressed in the form $q = (p + 2 \cdot t) \in \mathbb{P}$, where $1 \leq t < k-1$.

Indeed, we can take $t = \dfrac{q-p}{2}$, so that $q = p + 2 \cdot t \in \mathbb{P}$. Then, $p \in G_k\mathbb{P}$ implies that $p + \theta_k(p) = 2 \cdot k$ and $(p + 2 \cdot t) + \theta_k(p) = 2 \cdot k + 2 \cdot t = 2 \cdot (k + t)$.

Since $q = p + 2 \cdot t \in \mathbb{P}$ and $\theta_k(p) \in G_k\mathbb{P}$, we have $p \in G_k\mathbb{P} \cap T_t\mathbb{P} \neq \varnothing$ and $q = (p + 2 \cdot t) \in G_{k+t}\mathbb{P} \neq \varnothing$.

Q.E.D.

This shows how nonempty Goldbach sets $G_3\mathbb{P}, G_4\mathbb{P}, G_5\mathbb{P}, \ldots\ G_{12}\mathbb{P}, \ldots$ have been generated:

$$G_3\mathbb{P} = \{3\}, G_4\mathbb{P} = \{3, 3+2\}, G_5\mathbb{P} = \{3,5,5+2\}, G_6\mathbb{P} = \{5,5+2\},$$

$$G_7\mathbb{P} = \{3,7,7+4\},\ G_8\mathbb{P} = \{3,5,11,11+2\},\ G_9\mathbb{P} = \{5,7,11,11+2\},$$

$$G_{10}\mathbb{P}=\{3,7,13,13+4\},G_{11}\mathbb{P}=\{3,11,17,17+2\},G_{12}\mathbb{P}=\{5,7,11,13,17+2\}\ldots$$

Let $G_k\mathbb{P}\neq\varnothing$, so that there exist $p\in G_k\mathbb{P}$ and $p'=\theta_k(p)\in G_k\mathbb{P}$, where $p'=\theta_k(p)=2\cdot k-p$.

Assume that q is prime and $q>p$ such that $q=p+2\cdot t$, where $t=\dfrac{q-p}{2}$.

This implies that $q+\theta_k(p)=(p+2\cdot t)+\theta_k(p)=(p+\theta_k(p))+2\cdot t=2\cdot(k+t)$.

Since both q and $\theta_k(p)$ are primes and $q+\theta_k(p)=2\cdot(k+t)$, we have q and $\theta_k(p)$ belong to $G_{k+t}\mathbb{P}\neq\varnothing$. For instance, if for some k $(3\le k\le m)$ we have $G_{k-1}\mathbb{P}\cap T_2\mathbb{P}\neq\varnothing$, then, due to Lemma 4, $G_{k+1}\mathbb{P}\neq\varnothing$. Consider $p=3$ and $q=5$, that is we start from $G_3\mathbb{P}=\{3\}$.

Then, $t=\dfrac{5-3}{2}=1$ and $\theta_3(3)=3$. Thus, we have $q+\theta_k(p)=5+3=2\cdot(3+1)=8$, where 3 and 5 both belong to $G_{3+1}\mathbb{P}=G_4=\{3,5\}$.

Then, due to Lemma 4, for each $G_k\mathbb{P}\neq\varnothing$ there exists $t\in\mathbb{N}$ such that $G_k\mathbb{P}\cap T_t\mathbb{P}\neq\varnothing$, which implies $G_{k+t}\mathbb{P}\neq\varnothing$. This means that the occurrence of a t-prime in a non-empty set $G_k\mathbb{P}$ implies that $G_{k+t}\mathbb{P}$ is necessarily non-empty. This provides proliferation of non-empty sets $G_k\mathbb{P}$ t steps forward, so that $G_{k+t}\mathbb{P}$ is not empty for any k.

Starting from $k=3$ and $t=1$ the 'wave' of G_k-primes propagates forward recursively as $k\to\infty$ without gaps, supported by the existence of such t-primes.

Observe that each pair of primes (p,q) such that $p\in G_k\mathbb{P}$, $q>p$ and $q\in S_m=I_m\cap\mathbb{P}$, generates a nonempty set $G_{k+t}\mathbb{P}$, where $t=\dfrac{q-p}{2}$ and p is a t-prime in $G_k\mathbb{P}$.

Notice that each prime number in $G_m\mathbb{P}$ for $m\ge 3$ is a t-prime for an appropriate value of t. Our goal is to demonstrate that we can build a nonempty Goldbach set $G_m\mathbb{P}$ for every $m>3$, given a sequence of nonempty Goldbach sets $\{G_k\mathbb{P}\}_{3\le k\le m-1}$, by using assumption of mathematical induction. We need the following simple Lemmas.

Lemma 5.

Let $S_m = I_m \cap \mathbb{P}$, where $I_m = [3, 2\cdot m - 3]$.

For any primes $p \in S_m$ and $q \in S_m$ there exists $k \le m$ such that both p and q belong to $G_k\mathbb{P}$.

Proof.

Indeed, take $k = \dfrac{p+q}{2}$. Then, $p \in G_k\mathbb{P}$ and $q \in G_k\mathbb{P}$, since $p + q = 2\cdot k \le 2\cdot m$.

Notice that we can choose $p = 3$ and any prime $q \ge 3$ in S_m. Obviously, $G_k\mathbb{P} \subseteq S_m$.

Q.E.D.

Lemma 6.

For all $m \ge 3$ we have

$$S_m = \bigcup_{k=3}^{m} G_k\mathbb{P} = G^{(m)}\mathbb{P}. \tag{8}$$

Proof.

For any $p \in S_m$, due to Lemma 4, there exists $k \le m$ such that $p \in G_k\mathbb{P}$, so that $p \in G^{(m)}\mathbb{P}$.

And vice versa, if $p \in G^{(m)}\mathbb{P}$, then $p \in G_k\mathbb{P}$ for some $k \le m$, so that $p \in S_m$.

Q.E.D.

The following statement concerns a recurrent formula that generates an infinite sequence of nonempty Goldbach sets $G_m\mathbb{P}$ for all $m \ge 3$.

Theorem 1.

Let $G_k\mathbb{P} \ne \varnothing$ for all $k : 3 \le k \le m-1$. If $2\cdot m - 3 \in \mathbb{P}$, then $2\cdot m - 3 \in G_m\mathbb{P} \ne \varnothing$.

Otherwise, if $2\cdot m - 3 \ne G_m\mathbb{P}$, we have $S_m = S_{m-1}$, due to Lemma 2.

Then, for any $m \ge 3$ the following equality holds true:

$$\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2\cdot(m-k)\right) \cap S_m\right] = G_m\mathbb{P} = \varnothing \tag{9}$$

Proof.

Denote $A_{k,m} = \left(G_k\mathbb{P} + 2\cdot(m-k)\right)$.

Then, $\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P}+2\cdot(m-k)\right)\cap S_m\right] = \bigcup_{k=3}^{m-1}\left[A_{k,m}\cap S_m\right] = \left(\bigcup_{k=3}^{m-1}A_{k,m}\right)\cap S_m$.

Consider $\theta_m(A_{k,m}) = 2\cdot m - A_{k,m} = 2\cdot m - G_k\mathbb{P} - 2\cdot(m-k) = 2\cdot k - G_k\mathbb{P} = \theta_k(G_k\mathbb{P}) = G_k\mathbb{P}$.

Indeed, due to Lemma 1 about θ_m- shift invariance of $G_k\mathbb{P}$, we have $\theta_k(G_k\mathbb{P}) = G_k\mathbb{P}$.

Equality $\theta_m(A_{k,m}) = G_k\mathbb{P}$ implies that

$$\theta_m\left(\left(\bigcup_{k=3}^{m-1}A_{k,m}\right)\cap S_m\right) = \left(\bigcup_{k=3}^{m-1}\theta_m(A_{k,m})\right)\cap\theta_m(S_m) = \left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right)\cap\theta_m(S_m).$$

According to equality $S_m = \bigcup_{k=3}^{m}G_k\mathbb{P} = G^{(m)}\mathbb{P}$ (see formula (8) in Lemma 6) we find

$S_{m-1} = \left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right) = G^{(m-1)}\mathbb{P}$. Then, $2\cdot m-3 \neq G_m\mathbb{P}$ implies $S_m = S_{m-1}$. Hence

$$\theta_m\left(\left(\bigcup_{k=3}^{m-1}A_{k,m}\right)\cap S_m\right) = \left(\bigcup_{k=3}^{m-1}G_k\mathbb{P}\right)\cap\theta_m(S_m) = S_m\cap\theta_m(S_m) = G_m\mathbb{P}.$$

We have $\left(\bigcup_{k=3}^{m-1}A_{k,m}\right)\cap S_{m-1} = G^{(m-1)}\mathbb{P}\cap S_{m-1} = G^{(m-1)}\mathbb{P}\neq\varnothing$, due to the assumption of mathematical

induction. Then, from $\left(\bigcup_{k=3}^{m-1}A_{k,m}\right)\cap S_{m-1}\neq\varnothing$ it follows $\theta_m\left(\left(\bigcup_{k=3}^{m-1}A_{k,m}\right)\cap S_m\right) = G_m\mathbb{P}\neq\varnothing$.

Q.E.D.

The recursive formula (9) in Theorem 1 proves the Strong Goldbach Conjecture.
Lemma 4 and Theorem 1 show a definite connecion between the number of solutions to the Goldbach equation $p + p' = 2\cdot m$ in the intervals $I_m = [3, 2\cdot m-3]$ and the number of t-primes in sets $G_k\mathbb{P}$ for $k: 3\leq k<m$. We discuss this in what follows.

3. Diophantine variety of Goldbach sets.

A seqience of Goldbach sets $\{G_k\mathbb{P} \mid 3 \le k \le m\}$ represents solutions to the system of Goldbach equations $\{x + y = 2 \cdot k \mid 3 \le k \le m\}$ in the intervals $I_k = [3, 2 \cdot k - 3]$.

This system is an algebraic variety given by linear equations $x + y = 2 \cdot k$ $(3 \le k \le m)$, which solutions (if exist) are pairs of prime numbers $(p, p') \in \mathbb{P}^2$.

Geometrically, each Goldbach set $G_k\mathbb{P}$ is a sequence of points with coordinates $(p, p') \in \mathbb{P}^2$ on the segment of a straight line given by $x + y = 2 \cdot k$, $(x, y) \in [0, 2 \cdot k]$, symmetrically located on the line with respect to a point (k, k), due to invariance $\theta_k(G_k\mathbb{P}) = G_k\mathbb{P}$, where $\theta_k(x) = 2k - x = y$. See below Fig. 1 and 2 representing Diophantine geometry of Goldbach sets, where dots are points with coordinates $(p, p') \in \mathbb{P}^2$ on the corresponding lines. These dots are solutions to the Goldbach equations $x + y = 2 \cdot k$ $(3 \le k \le m)$. The theorem below answers the question how many solutions are in each Golbach set.

Theorem 2.

The number of solutions to the Goldbach equation $p + p' = 2 \cdot m$ in primes $(p, p') \in \mathbb{P}^2$, where $p < m$ and m is not prime, in each inerval $I_m = [3, 2 \cdot m - 3]$ is equal to the number of t-primes in the set $G_m\mathbb{P}$ such that $t = \dfrac{p' - p}{2}$. We have then, $p = m - t$, $p' = m + t$.

Proof.

Consider a quadratic polynomial $P_m(x) = x^2 + 2 \cdot m \cdot x + c$ for m, c and $x \in \mathbb{Z}$.

Let a pair of primes (p, p') be a solution to the Goldbach equation $p + p' = 2 \cdot m$ in the interval $I_m = [3, 2 \cdot m - 3]$. Obviously, for $c = p \cdot p'$, the pair of prime numbers

$(p, p') \in \mathbb{P}^2$ are roots of the polynomial $P_m(x) = x^2 - 2\cdot m\cdot x + p\cdot p' = (x-p)\cdot(x-p')$.

Discriminant of $P_m(x)$ is $D = 4\cdot(m^2 - p\cdot p') = 4\cdot t^2$, where t is a nonnegative integer.

Observe that $(m^2 - p\cdot p') = t^2$ implies $(m-t)\cdot(m+t) = p\cdot p'$.

Since p and p' are not equal prime numbers, the equation $(m-t)\cdot(m+t) = p\cdot p'$

for an integer t and $p \le p'$ implies: $m - t = p$ and $m + t = p'$, so that

$t = \dfrac{p'-p}{2}$ and $p' = p + 2\cdot t$. This means that $p \in G_m\mathbb{P}$ is a t-prime in $G_m\mathbb{P}$, where $t = \dfrac{p'-p}{2}$.

Therefore, we have as many solutions $(p, p') \in \mathbb{P}^2$ to the equation $P_m(x) = x^2 + 2\cdot m\cdot x + p\cdot p'$

as there are t-primes, $t = \dfrac{p'-p}{2}$, in the set $G_m\mathbb{P}$. Assume now that $2\cdot m = p + q$, $c = p\cdot q$,

where $p \in \mathbb{P}$ and $q = 2\cdot m - p$ is an unknown integer. **Q.E.D.**

For $p \in G_m\mathbb{P}$ the polynomial $P_m(x)$ takes a form: $P_m(x) = x^2 + 2\cdot m\cdot x + p\cdot(2\cdot m - p)$.

Its discrimimant is $D_m = 4\cdot\left(m^2 - p\cdot(2\cdot m - p)\right) = 4\cdot\left(m^2 - 2\cdot m\cdot p + p^2\right) = 4\cdot(m-p)^2$.

The solutions to the equation $P_m(x) = 0$ are $x_{1,2} = m \pm (m-p)$, where $x_1 = p, x_2 = 2\cdot m - p$.

For instance, let $m = 9$, $c = 45$. Then, $P_9(x) = x^2 - 18x + 45$ has 2 roots:

$x_1 = 3 \in \mathbb{P}, x_2 = 2\cdot 9 - 3 = 15 \notin \mathbb{P}$, but they do not belong to $G_9\mathbb{P}$. Meanwhile, for $m = 9$

and $c = 65$ we have $P_9(x) = x^2 - 18x + 65$ with roots $x_1 = 5 \in \mathbb{P}, x_2 = 2\cdot 9 - 5 = 13 \in \mathbb{P}$,

so that $5 \in G_9\mathbb{P}$ and $13 \in G_9\mathbb{P}$. Notice that $3, \theta_9(3) = 15$ and $2\cdot 9 = 18$ are not coprime numbers,

while $5, \theta_9(5) = 13$ and $2\cdot 9 = 18$ are all coprime.

Denote $[x]_p = \mathrm{mod}(x, p)$. Then, $[P_m(x)]_p = [x]_p^2 - [2\cdot m]_p\cdot[x]_p$. The equation.

$$[P_m(x)]_p = [x]_p^2 - [2\cdot m]_p\cdot[x]_p = [x]_p\cdot\left([x]_p - [2\cdot m]_p\right) = 0$$

has the following different sets of solutions: $[x]_p = 0$ and $[x]_p = [2\cdot m]_p$.

Since we are solving equation $P_m(x) = 0$ in primes within interval $I_m = [3, 2\cdot m - 3]$,

the solutions are restricted to the set $S_m = I_m \cap \mathbb{P}$. Therefore, in interval I_m equations

$[x]_p = 0$ and $[x]_p = [2 \cdot m]_p$ have solutions: $x_1 = p < m$, and $x_2 = 2 \cdot m - p > m$.

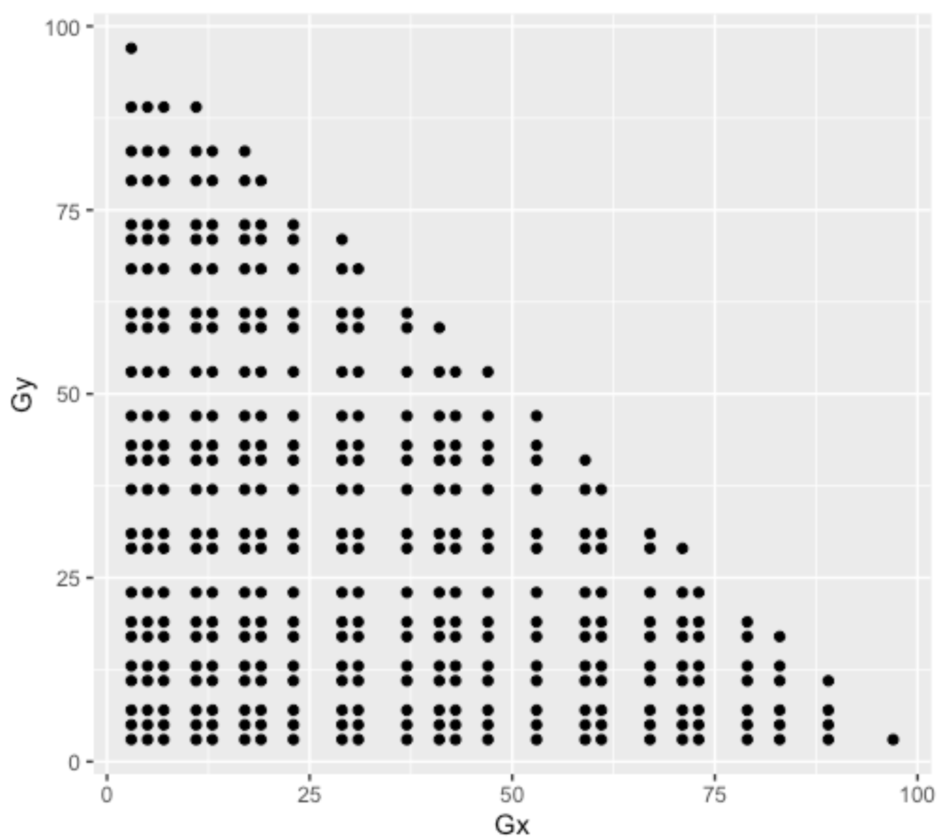
Example 1.

$$G_{48}\mathbb{P} = \{7, 13, 17, 23, 29, 37, 43, 53, 59, 67, 73, 79, 83, 89\}, \quad m = 48$$

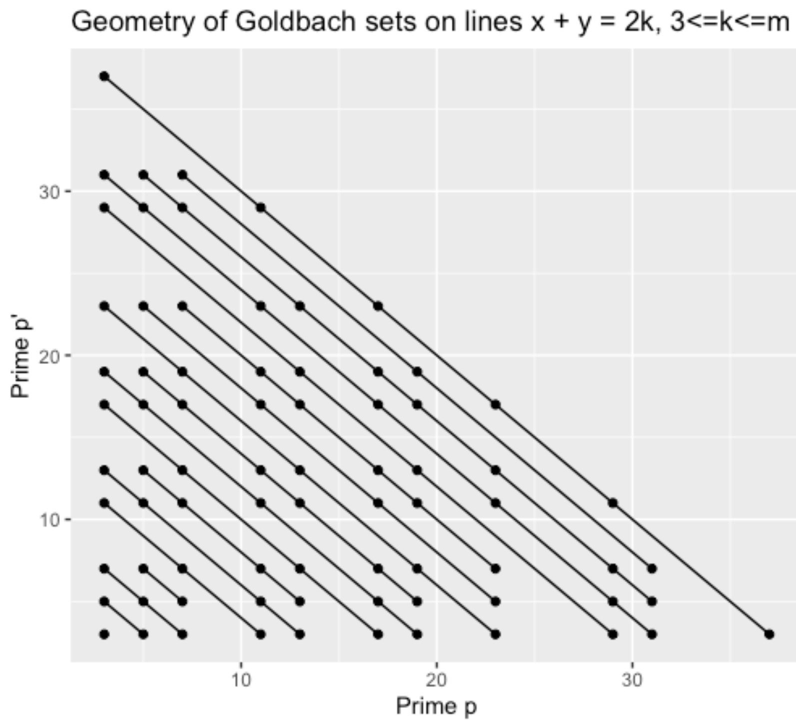
$p = m - t$	7	13	17	23	29	37	43
$p' = m + t$	89	83	79	73	67	59	53
t	41	35	31	25	19	11	5

Diophantine Geometry of Goldbach Sets (Fig.2)

$G_k\mathbb{P} \quad (k = 3,4,\ldots,50)$ (Fig.1)



$G_k\mathbb{P}$ $(k = 3,4,\dots,40)$ (Fig.3)



Every dot in the above figure denotes a point with coordinates (p, p') such that $p + p' = 2 \cdot k$ on the line $x + y = 2 \cdot k$, where $3 \le k \le m$.

4. Recursive Algorithm generating the infinite sequence of nonempty Goldbach sets $G_m\mathbb{P} \ne \varnothing$ for all natural $m \ge 3$.

We apply now one of the most fundamental and simple proof techniques in mathematics known as *mathematical induction* [3]. Let $\text{Prop}(m)$ denote a statement about a natural number m, and let m_0 be a fixed number. A proof that $\text{Prop}(m)$ is true for all $m \ge m_0$ by induction requires two steps:

Basis step: Verify that $\text{Prop}(m_0)$ is true.

Induction step: Assuming that $\text{Prop}(k)$ is true for all k such that $k : m_0 < k \le m$, verify that $\text{Prop}(m+1)$ is true.

Theorem 3.

$\text{Prop}(m)$: For all integer $m \ge 3$, the set $G_m\mathbb{P}$ of solutions to the equation $n + n' = 2m$, $\left(n, n'\right) \in \mathbb{P}^2$, in prime numbers is not empty: $G_m\mathbb{P} \ne \varnothing$.

The $\text{Prop}(m)$ can be equivalently stated as: $G_m\mathbb{P} = \theta_m(S_m) \cap S_m \ne \varnothing$ for all integers $m \ge 3$.

Proof.

(1) Basic step.

As we know [2], $\text{Prop}(m)$ is true for all m up to $M = 4 \cdot 10^{18}$.

Let $m_0 = 3$. Then $2 \cdot 3 = 6 = 3 + 3$.

(2) Induction step.

Assume that $G_k\mathbb{P} = \theta_k(S_k) \cap S_k \ne \varnothing$ for all integer $k : m_0 \le k \le m\text{-}1$.

Let $k = m$. We denote: $I_m = [3, 2 \cdot m - 3]$ and $S_m = I_m \cap \mathbb{P}$.

In Lemma 2 we proved (3) that $S_m = \begin{cases} S_{m-1} \cup \left\{2m-3\right\} & \text{if } (2m\text{-}3) \in \mathbb{P} \\ S_{m-1} & \text{if } (2m\text{-}3) \notin \mathbb{P} \end{cases}$.

In the case $(2 \cdot m - 3) \in \mathbb{P}$, the formula (5) implies $G_m\mathbb{P} = \theta_m(S_{m-1}) \cap S_{m-1} \ne \varnothing$.

We can also confirm in this case directly that $G_m\mathbb{P} \ne \varnothing$, because $3 + (2 \cdot m - 3) = 2 \cdot m$

Consider now a general situation, which includes the case $(2 \cdot m - 3) \notin \mathbb{P}$.

This part of the proof consists of two steps.

On the first step we prove that $G_m\mathbb{P} = \bigcup_{k=3}^{m-1} \left[\left(G_k\mathbb{P} + 2 \cdot (m-k) \right) \cap S_m \right]$ by applying shift transformation θ_m to both sides of the above equation and by using the property of θ_m- invariance of Goldbach sets $G_m\mathbb{P}$ for all $m \ge 3$.

Next, we observe that $S_m = S_{m-1}$ if $(2\cdot m-3)\neq \mathbb{P}$. Then, due to Lemma 6, we obtain

$$G_m\mathbb{P} \subseteq S_m = S_{m-1} = \bigcup_{k=3}^{m-1} G_k\mathbb{P} = G^{(m-1)}\mathbb{P}.$$

On the second step we show that the set $\bigcup_{k=3}^{m-1}\left[G_k\mathbb{P} + 2\cdot(m-k)\cap S_m\right]$ is not empty.

This follows from the induction assumption $G_k\mathbb{P}\neq\varnothing$ for all $k\ (3\leq k<m)$.

Finally, Theorem 1 states the recursive formula for nonempty Goldbach sets:

$$\bigcup_{k=3}^{m-1}\left[\left(G_k\mathbb{P} + 2\cdot(m-k)\right)\cap S_m\right] = G_m\mathbb{P}\neq\varnothing\ (m=3,4,5,\ldots). \tag{10}$$

This formula allows to write a computer program to generate recursively potentially infinite sequence of Goldbach sets $G_m\mathbb{P}$ for all $m\geq 3$.

Q.E.D.

See in APPENDIX the text of R-script `GenGS.R` and data lists of the calculated $G_k\mathbb{P}$ for $k: 3\leq k\leq m$. An example below illustrates the above statement with some computer calculations.

In this example we consider sets $G_m\mathbb{P} = \theta_m(S_m)\cap S_m$ for m from 105 to 110.

Notice that many of those sets can be calculated based on the rule that if a prime $p\in G_k\mathbb{P}$ has a twin prime $(p+2)\in\mathbb{P}$, that is $t=1$ and $p\in T_1\mathbb{P}$, then $\left(p+2\right)\in G_{k+1}\mathbb{P}$.

For example, terms in $G_{106}\mathbb{P}$ are calculated with this rule by using terms in $G_{105}\mathbb{P}$.

Meanwhile, terms in $G_{110}\mathbb{P}$ are calculated by using terms in $G_{108}\mathbb{P}$ for $t=2$ based on the general rule: if $p<k$ and $p\in G_k\mathbb{P}\cap T_t\mathbb{P}$, then $p\in\mathbb{P}$ and $p+2\cdot t\in\mathbb{P}$ implies $(p+2\cdot t)\in G_{k+t}\mathbb{P}$

(Lemma 4): $23+197=(19+2\cdot 2)+197=220=2\cdot 110,$ since $19+197=216=2\cdot 108$.

The calculations below illustrate the conclusion of the Theorem 1 (see the data referred in Example 2).

We would like to verify that $G_{110}\mathbb{P}\neq\varnothing$, by using that $G_k\mathbb{P}\neq\varnothing$ for all $k\leq 110$. Consider $G_{110}\mathbb{P}$ $(m=110, 2\cdot m=220)$. If we choose $t=1$ it would not work with $G_{109}\mathbb{P}$, because $G_{109}\mathbb{P}\cap T_1S_{109}=\varnothing$. We try then $G_{108}\mathbb{P}$ and $t=2$.

We have $G_{108}\mathbb{P}\cap T_2S_{108}\neq\varnothing$ and $p=19\in G_{108}\mathbb{P}\cap T_2S_{108}$. Then, $p+2\cdot t=19+2\cdot 2=23$ should belong (due to Lemma 3) to $G_{110}\mathbb{P}$. Therefore, $2\cdot 110-23=197\in G_{110}\mathbb{P}$.

Thus, we have $23+197=2\cdot 110$, which means that $G_{110}\mathbb{P}\neq\varnothing$. Notice that in this instance $k=109$, $k+1-t=109+1-2=108$ and $(k+1-t)+t=108+2=110$, which allows us to establish that $G_{(k+1-t)+t}=G_{110}\mathbb{P}\neq\varnothing$, by using the fact that $G_{108}\mathbb{P}\cap T_2S_{108}\neq\varnothing$.

Example 2.

Sets $G_m\mathbb{P}=\theta_m(S_m)\cap S_m$ for m from 105 to 110

$$G_{105}\mathbb{P}=\begin{Bmatrix} 11\ 13\ 17\ 19\ 29\ 31\ 37\ 43\ 47\ 53\ 59\ 61\ 71\ 73 \\ 79\ 83\ 97\ 101 103 107\ 109\ 113\ 127\ 131\ 137\ 139 149 \\ 151\ 157\ 163\ 167\ 173\ 179\ 181\ 191\ 193\ 197\ 199 \end{Bmatrix}$$

$$G_{106}\mathbb{P}=\{13\ 19\ 31\ 61\ 73\ 103\ 109\ 139\ 151\ 181\ 193\ 199\}$$

$$G_{107}\mathbb{P}=\{3\ 17\ 23\ 41\ 47\ 83\ 101\ 107\ 113\ 131\ 167\ 173\ 191\ 197\ 211\}$$

$$G_{108}\mathbb{P}=\begin{Bmatrix} 5\ 17\ 19\ 23\ 37\ 43\ 53\ 59\ 67\ 79\ 89\ 103\ 107\ 109 \\ 113\ 127\ 137\ 149\ 157\ 163\ 173\ 179\ 193\ 197\ 199\ 211 \end{Bmatrix}$$

$$G_{109}\mathbb{P}=\{7\ 19\ 37\ 61\ 67\ 79\ 109\ 139\ 151\ 157\ 181\ 199\ 211\}$$

$$G_{110}\mathbb{P}=\begin{Bmatrix} 23\ 29\ 41\ 47\ 53\ 71\ 83\ 89\ 107\ 113 \\ 131\ 137\ 149\ 167\ 173\ 179\ 191\ 197 \end{Bmatrix}$$

Thus, we can predict that $G_{110}\mathbb{P}\neq\varnothing$without explicit calculation of this set, just by using the previously calculated sets $G_{109}\mathbb{P},G_{108}\mathbb{P},G_{107}\mathbb{P},\ldots$ By using the algorithm described in Lemma 5, we find that $G_{109}\mathbb{P}\cap T_1\mathbb{P}=\varnothing$, but $G_{108}\mathbb{P}\cap T_2\mathbb{P}\neq\varnothing$, since, for instance, $19\in G_{108}\mathbb{P}\cap T_2\mathbb{P}$,
and $19+2\cdot 2=23\in G_{110}\mathbb{P}$.

Conclusion

By using probabilitic approach I have proved that $P\left\{\bigcup_{m=M}^{\infty}\left\{G(2m,\vec{v}_m)\right\}=0\right\}\to 0$ as $M\to\infty$.

This means that probability that for sufficiently large value of M there exists at least one value of $m\geq M$ such that the set $G_m\mathbb{P}$ is empty, tends to zero. Equivalently, it is proved that

$$\lim_{m\to\infty}P\left\{\left|G_m\mathbb{P}\right|>0\right\}=\lim_{m\to\infty}P\left\{G(2m,\vec{v}_m)>0\right\}=1.$$

To support the probabilistic conclusions, I developed a recursive deterministic algorithm, which gives constructive means to build infinite sequence of consecutive nonempty Goldbach sets $G_m\mathbb{P}$ for all $m\geq 3$.

I tried to follow the 'natural logic' of the problem, by being more exploratory rather than artificially creative and used a computer as my permanent companion and advisor.
As to simplicity of the used methods, I recall to the point the well-known Poincaré Recurrence Theorem [7], which proof takes only a few lines of the text and uses mainly elementary set-theoretical operations. Meanwhile the significance of the Poincaré Recurrence Theorem can be hardly overestimated. Notice, by the way, that the proof of the famous Poincaré recurrence theorem is not constructive, since it does not provide an algorithm to establish a number n of iterations, after which the recurrence occurs. The Poincaré theorem states only that such number n exists. Meanwhile the statement of Theorem 1 above is quite constructive since it leads to the recursive formula (9) (see the computed examples of Goldbach set sequences and the text of R script in Appendix), which allows potentially unlimited computation of consecutive nonempty Goldbach sets $\left\{G_k\mathbb{P}\mid 3\leq k\leq m\right\}$ for any $m\geq 3$. This means that Strong Goldbach Conjecture holds true.

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The spirit of friendly interaction in our numerous discussions was very crucial for me.

APPENDIX

The text of R-script for computer realization of Recursive Algorithm generating sequences of Goldbach sets $G_k\mathbb{P}$ for $k = 3,4,5,\ldots,m$

```
# Function GenGS(M1,M2) recursively generates a sequence of Goldbach sets
# G(m) where M1 <= m <= M2, for an natural M1, M2 such that 5 < M1 < M2.
# Here each G(m) is calculated by using the function
# GenG(m) and the function supply:  GenGS(M1,M2) = sapply(M1:M2,GenG)
# Function GenG(m) generates sets G(m) of Goldbach primes such that
# p + p' = 2m (3 <= m <= 2m-3) for each natural m (3 <= m <= 2m-3).
# This function is based on formula (9) from Theorem 1:
# G(m) includes each p + 2t if p is a t-prime in the Goldbach set G(k)
# (3 <= k <= m-1) for t = m-k.
# Thus, G(m) is a inion of subsets tG(k) of t-primes in G(K) such that
# tG(k) = {p + 2t| p is in G(k), p + 2t is prime for each t = m - k}.
# Notice that G(m) is recursively generated from the Goldbach sets G(k),
# where 3 < k <= m-1, starting from G(3) = {3} (3+3=6).
# This is confirming non-emptiness of Goldbach sets G(m) for all natural
# m = 3,4, 5... (the Goldbach Conjucture)
# by the principle of mathematical induction.
# Needed packages: 'numbers' and 'sets'.
# Created by GMS
# Date: 06.30.21.
#
GenGS <- function (M1,M2) {
  Gen_GS <- sapply(M1:M2, GenG)
  return(Gen_GS)
}
#source('~/Documents/R/Number Theory/GenGS.R')

GenG <- function(m) {
 if (isPrime(2*m-3)){Gm <- 3 }
  else {  Gm <- NULL
  }
  for (k in (3: m-1)) {
    Gk <- Gm(k)
    t <- m - k
    tGk <- Gk + 2*t
    pr_tGk <- tGk[isPrime(tGk)]
    Gm <- union(Gm, pr_tGk)
}
  return(sort(Gm))
}

#source('~/Documents/R/Number Theory/GenG.R')
```

Data lists of calculated $G_k\mathbb{P}$ for $k = 3,4,5,\ldots,m$

m	Goldbach sets $G_m\mathbb{P}$ $(m = 3,4,5,\ldots,43)$
3	3
4	3 5
5	3 5 7
6	5 7
7	3 7 11
8	3 5 11 13
9	5 7 11 13
10	3 7 13 17
11	3 5 11 17 19
12	5 7 11 13 17 19
13	3 7 13 19 23
14	5 11 17 23
15	7 11 13 17 19 23
16	3 13 19 29
17	3 5 11 17 23 29 31
18	5 7 13 19 23 29 31
19	7 19 31
20	3 11 17 23 29 37
21	5 11 13 19 23 29 31 37
22	3 7 13 31 37 41
23	3 5 17 23 29 41 43
24	5 7 11 17 19 29 31 37 41 43
25	3 7 13 19 31 37 43 47
26	5 11 23 29 41 47
27	7 11 13 17 23 31 37 41 43 47
28	3 13 19 37 43 53
29	5 11 17 29 41 47 53

30	7 13 17 19 23 29 31 37 41 43 47 53
31	3 19 31 43 59
32	3 5 11 17 23 41 47 53 59 61
33	5 7 13 19 23 29 37 43 47 53 59 61
34	7 31 37 61
35	3 11 17 23 29 41 47 53 59 67
36	5 11 13 19 29 31 41 43 53 59 61 67
37	3 7 13 31 37 43 61 67 71
38	3 5 17 23 29 47 53 59 71 73
39	5 7 11 17 19 31 37 41 47 59 61 71 73
40	7 13 19 37 43 61 67 73
41	3 11 23 29 41 53 59 71 79
42	5 11 13 17 23 31 37 41 43 47 53 61 67 71 73 79
43	3 7 13 19 43 67 73 79 83

m	Goldbach sets $G_m\mathbb{P}$ ($m = 100,101,\ldots,128$)
100	3 7 19 37 43 61 73 97 103 127 139 157 163 181 193 197
101	3 5 11 23 29 53 71 87 101 113 149 173 179 191 197 199
102	5 7 11 13 23 31 37 41 47 53 67 73 97 101 103 107 131 137 151 157 163 167 173 191 193 197 199
103	7 13 43 67 79 97 103 109 127 139 163 193 199
104	11 17 29 41 59 71 101 107 137 149 167 179 191 197
105	11 13 17 19 29 31 37 43 47 53 59 61 71 73 79 83 97 101 103 107 109 113 127 131 137 139 149 151 157 163 167 173 179 181 191 193 197 199
106	13 19 31 61 73 103 109 139 151 181 193 199
107	3 17 23 41 47 83 101 107 113 131 167 173 191 197 211
108	5 17 19 23 37 43 53 59 67 79 103 107 109 113 127 137 149 157 163 173 179 193 197 199 211
109	7 19 37 61 67 79 109 139 151 157 181 199 211
110	23 29 41 53 71 83 89 107 113 131 137 149 167 173 179 191 197

111	11 23 29 31 41 43 59 71 73 83 109 113 139 149 151 163 179 181 191 193 199 211
112	13 31 43 61 67 73 97 127 151 157 163 181 193 211
113	3 29 47 53 59 89 113 137 167 173 179 197 223
114	5 17 29 31 37 47 61 71 79 89 97 101 127 131 139 149 157 167 181 191 197 199 211 223
115	3 7 19 31 37 67 73 79 103 127 151 157 163 193 199 211 223 227
116	3 5 41 53 59 83 101 131 149 173 179 191 227 229
117	5 7 11 23 37 41 43 53 61 67 71 83 97 103 107 127 131 137 151 163 167 173 181 191193 197 211 223 227 229
118	3 7 13 37 43 73 79 97 109 127 139 157 163 193 199 223 229 233
119	5 11 41 47 59 71 89 101 107 131 137 149 167 179 191 197 227 233
120	7 11 13 17 29 41 43 47 59 61 67 73 83 89 101 103 109 113 127 131 137 139 151 157 167 173 179 181 193 197 199 211 223 227 229 233
121	3 13 19 31 43 61 79 103 139 163 181 199 211 223 229 239
122	3 5 11 17 47 53 71 107 113 131 137 173 191 197 227 233 239 241
123	5 7 13 17 19 23 47 53 67 73 79 83 89 97 107 109 137 139 149 157 163 167 173 179193 199 223 227 229 233 239 241
124	7 19 37 67 97 109 139 151 181 211 229 241
125	11 17 23 53 59 71 83 101 113 137 149 167 179 191 197 227 233 239
126	11 13 19 23 29 41 53 59 61 71 73 79 89 101 103 113 139 149 151 163 173 179 181 191193 199 211 223 229 233 239 241
127	3 13 31 43 61 73 97 103 127 151 157 181 193 211 223 241 251
128	5 17 23 29 59 83 89 107 149 167 173 197 227 233 239 251

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