Discrete Markov Random Field Relaxation

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Abstract

This paper gives a technique to approximate (relaxation) discrete Markov Random Field (MRF) using convex programming. This approximated MRF can be used to approximate NP problem. This also proves that NP is not equal P because the MRF convex programming and the approximate MRF convex programming are not the same with removal of some product terms.

1 Introduction

A NP problem can be represented by a discrete Markov Random Field (MRF) [Tan21]. A constraint satisfaction problem (a NP problem) can also be represented by Markov Random Field [Sad19].

A discrete MRF has solution if

$$\sum_{v_1,v_2,\ldots,v_n} \prod_{i,j} H(a_i = v_i, a_j = v_j) > 0 \tag{1}$$

and if

$$\prod_{i,j} H(a_i = v_i, a_j = v_j) > 0 \tag{2}$$

then $a_1 = v_1, a_2 = v_2, \ldots, a_n = v_n$ is the solution to the MRF.

Note that $H(a_i, a_j) = 0$ or 1, $H(a_i, a_j) = H(a_j, a_i)$ and $H(a_i, a_i) = 1$.

The MRF can also be represented by indexing notation.

$$\sum_{v_1,v_2,\ldots,v_n} \prod_{i,j} b^{(i,j)}_{v_i,v_j} > 0 \tag{3}$$

Note that these equations mean the same thing:

$$\sum_{v_1,v_2,\ldots,v_n} \prod_{i,j} H(a_i = v_i, a_j = v_j) > 0 \tag{4}$$

and

$$\sum_{a_1,a_2,\ldots,a_n} \prod_{i,j} H(a_i, a_j) > 0. \tag{5}$$

2 Equation of 3 values 4 variables MRF

The equation below shows the Discrete Markov Random Field with 4 variables.

$$\sum_{a_1=0}^{2}\sum_{a_2=0}^{2}\sum_{a_3=0}^{2}\sum_{a_4=0}^{2} H(a_1,a_2)H(a_1,a_3)H(a_1,a_4)H(a_2,a_3)H(a_2,a_4)H(a_3,a_4) > 0. \tag{6}$$

Each variable can take 3 values. If the variable can only take 2 values, it is polynomial time solvable, otherwise 3 values or more will take NP time. If there is a solution, the equation above compute to greater than zero.

The solution of each MRF can be calculated as follows

$$\begin{aligned}&\hat{H}(a_1 = v_1, a_2 = v_2)\\ &= H(a_1 = v_1, a_2 = v_2)\\ &\sum_{a_1=v_1,a_2=v_2,a_3=0,1,2,a_4=0,1,2} H(a_1,a_3)H(a_1,a_4)H(a_2,a_3)H(a_2,a_4)H(a_3,a_4).\end{aligned} \tag{7}$$

If $\hat{H}(a_1 = v_1, a_2 = v_2) > 0$ then $a_1 = v_1$ and $a_2 = v_2$ is the solution to this MRF.

The MRF can be solved by a convex programming below

$$\max \sum_{a_1=0}^{2}\sum_{a_2=0}^{2}\sum_{a_3=0}^{2}\sum_{a_4=0}^{2} \hat{H}(a_1,a_2)+\hat{H}(a_1,a_3)+\hat{H}(a_1,a_4)+\hat{H}(a_2,a_3)+\hat{H}(a_2,a_4)+\hat{H}(a_3,a_4)$$

such that

$$\begin{aligned}&\hat{H}(a_i,a_j) \le \sum_{a_k=0,1,2,a_l=0,1,2} \hat{H}(a_i,a_k)\hat{H}(a_i,a_l)\hat{H}(a_j,a_k)\hat{H}(a_j,a_l)\hat{H}(a_k,a_l)\\ &\hat{H}(a_i,a_j) \le H(a_i,a_j)\\ &\hat{H}(a_i,a_j) \in \{0,1\}\\ &\text{for all } i,j,k,l \in \{1,2,3,4\}\\ &\text{and } i \ne j \ne k \ne l\\ &a_i,a_j,a_k,a_l \in \{0,1,2\}.\end{aligned} \tag{8}$$

3 Approximate MRF

The convex programming can be approximately simplified to

$$\max \sum_{a_1=0}^{2} \sum_{a_2=0}^{2} \sum_{a_3=0}^{2} \sum_{a_4=0}^{2} \hat{H}(a_1, a_2) + \hat{H}(a_1, a_3) + \hat{H}(a_1, a_4) + \hat{H}(a_2, a_3) + \hat{H}(a_2, a_4) + \hat{H}(a_3, a_4)$$

such that

$$\begin{aligned}
&\hat{H}(a_i, a_j) \leq \sum_{a_k=0,1,2} \hat{H}(a_i, a_k)\hat{H}(a_j, a_k) \\
&\hat{H}(a_i, a_j) \leq H(a_i, a_j) \\
&\hat{H}(a_i, a_j) \in \{0, 1\} \\
&\text{for all } i, j, k \in \{1, 2, 3, 4\} \\
&\text{and } i \neq j \neq k \\
&a_i, a_j, a_k \in \{0, 1, 2\}.
\end{aligned} \tag{9}$$

This convex programming is an approximation of original programming.

Approximated MRF can be expanded into

$$\max \sum_{a_1=0}^{2} \sum_{a_2=0}^{2} \sum_{a_3=0}^{2} \sum_{a_4=0}^{2} \hat{H}(a_1, a_2) + \hat{H}(a_1, a_3) + \hat{H}(a_1, a_4) + \hat{H}(a_2, a_3) + \hat{H}(a_2, a_4) + \hat{H}(a_3, a_4)$$

such that

$$\begin{aligned}
&\hat{H}(a_i, a_j) \leq \sum_{k=0}^{2} \hat{H}(a_j, a_k) \sum_{l=0}^{2} \hat{H}(a_k, a_l)\hat{H}(a_l, a_i) \\
&\hat{H}(a_i, a_j) \leq H(a_i, a_j) \\
&\hat{H}(a_i, a_j) \in \{0, 1\} \\
&\text{for all } i, j, k, l \in \{1, 2, 3, 4\} \\
&\text{and } i \neq j \neq k \\
&a_i, a_j, a_k, a_l \in \{0, 1, 2\},
\end{aligned} \tag{10}$$

then we can derive the recursive solution (later in the paper) using this equation.

4 Approximate MRF equal to 2 values MRF

This 2 values approximate MRF(MRF approximated with convex programming)

$$\max \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \sum_{a_4=0}^{1} \hat{H}(a_1, a_2) + \hat{H}(a_1, a_3) + \hat{H}(a_1, a_4) + \hat{H}(a_2, a_3) + \hat{H}(a_2, a_4) + \hat{H}(a_3, a_4)$$

such that

$$\begin{aligned}
&\hat{H}(a_i, a_j) \leq \sum_{a_k=0,1} \hat{H}(a_i, a_k)\hat{H}(a_j, a_k) \\
&\hat{H}(a_i, a_j) \leq H(a_i, a_j) \\
&\hat{H}(a_i, a_j) \in \{0, 1\} \\
&\text{for all } i, j, k \in \{1, 2, 3, 4\} \\
&\text{and } i \neq j \neq k \\
&a_i, a_j, a_k \in \{0, 1\}
\end{aligned} \tag{11}$$

is the same as 2 values MRF shown below.

$$\sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \sum_{a_4=0}^{1} H(a_1, a_2)H(a_1, a_3)H(a_1, a_4)H(a_2, a_3)H(a_2, a_4)H(a_3, a_4) > 0. \tag{12}$$

2 values MRF is actually a 2sat problem. Note that the approximate MRF for 2sat can only take 2 values, $a_i \in \{0, 1\}$.

5 Approximate MRF tells us NP not equal P

Recursion solution to 2sat in equation 12 is shown below.

$$\begin{aligned}
&H^{(0)}(a_i, a_j) = H(a_i, a_j) \\
&H^{(l)}(a_i, a_j) = H^{(l-1)}(a_i, a_j) \prod_{k=1}^{4} \sum_{a_k} H^{(l-1)}(a_i, a_k)H^{(l-1)}(a_j, a_k) \\
&\text{where } 1 <= l <= 4.
\end{aligned} \tag{13}$$

All dynamic programming problems are of this form. Same as my NP vs P solution [Tan21], there is no ‘Not’ operations but only ‘And’ and ‘Or’ operations.

The approximate MRF using convex programming can be converted to a Boolean algebra with ‘and’ and ‘or’ operations with no ‘Not’ operation. The approximate MRF convex programming removes product terms or potential functions from the original convex programming. Removal of terms cause the convex programming to be polynomial time complexity else it is not polynomial time but NP time. This means than NP problem cannot be simplified to polynomial problem. This has the same result as my NP vs P paper [Tan21].

6 Approximate MRF convex programming can be converted to Boolean algebra and linear programming

The approximate MRF can also be converted to boolean algebra shown below.

$$
\begin{aligned}
&H^{(0)}(a_i, a_j) \iff H(a_i, a_j) \\
&H^{(l)}(a_i, a_j) \iff H^{(l-1)}(a_i, a_j) \wedge \bigwedge_{k=1}^{4} \bigvee_{a_k} H^{(l-1)}(a_i, a_k) \wedge H^{(l-1)}(a_j, a_k) \\
&\text{where } 1 <= l <= n.
\end{aligned} \tag{14}
$$

The linear programming formulation of 2sat is

$$
\max \sum_{a_1=0}^{1} \sum_{a_2=0}^{1} \sum_{a_3=0}^{1} \sum_{a_4=0}^{1} \hat{H}(a_1, a_2) + \hat{H}(a_1, a_3) + \hat{H}(a_1, a_4) + \hat{H}(a_2, a_3) + \hat{H}(a_2, a_4) + \hat{H}(a_3, a_4)
$$

such that

$$
\begin{aligned}
&\hat{H}(a_i, a_j) \leq \hat{H}(a_i, a_k = 0) + \hat{H}(a_j, a_k = 1) \\
&\hat{H}(a_i, a_j) \leq \hat{H}(a_i, a_k = 1) + \hat{H}(a_j, a_k = 0) \\
&\hat{H}(a_i, a_j) \leq \hat{H}(a_i, a_k = 0) + \hat{H}(a_i, a_k = 1) \\
&\hat{H}(a_i, a_j) \leq \hat{H}(a_j, a_k = 0) + \hat{H}(a_j, a_k = 1) \\
&\hat{H}(a_i, a_j) \leq H(a_i, a_j) \\
&\hat{H}(a_i, a_j) \in \{0, 1\} \\
&\text{for all } i, j, k \in \{1, 2, 3, 4\} \\
&\text{and } i \neq j \neq k \\
&a_i, a_j, a_k \in \{0, 1\}.
\end{aligned} \tag{15}
$$

Note that in this linear programming formulation, no matter whether the product term is continuous or Discrete, it leads to the same solution. Whether $\hat{h}(a_i, a_j) \in \{0, 1\}$ or $0 \leq \hat{h}(a_i, a_j) \leq 1$, it leads to the same solution.

7 Conclusion

MRF can easily relaxed (simplified and approximated) by convex programming. Since relaxed MRF lead to different convex programming with different product sum of smaller number of product terms, NP is not equal P.

Arbitrary number of variables MRF can also be relaxed in the same manner.

References

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