Tutorial: Electromagnetic Violations of Newtonian Precepts in Basic Experiments

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Abstract Since acceleration is invariant under constant-velocity Galilean transformations, a system moving at constant velocity cannot, in Newtonian physics, exert new forces it doesn't already exert when it is at rest. But a bar magnet moving at nonzero constant velocity exerts a force on electric charges that it doesn't exert when it is at rest (Faraday's Law), and a charge moving at nonzero constant velocity exerts a torque on the needle of a magnetic compass that it doesn't exert when it is at rest (Biot-Savart Law). Thus basic electromagnetic experiments which are feasible in undergraduate or secondary-school physics labs illustrate the need to replace the Galilean transformations. That seems pedagogically much more compelling than the standard practice of merely discussing experiments which use extremely high-precision equipment such as Michelson interferometers. What should replace the Galilean transformations? A key qualification obviously is compatibility with the electromagnetic Laws. Those Laws can be presented as wave equations with source terms, and wherever the source terms are zero, the free waves travel exclusively at the fixed constant speed c. Thus to be compatible with the electromagnetic Laws, the replacements of the Galilean transformations must preserve the speed c, but they of course must in addition become Galilean when the ratio of the untransformed speed to c goes to zero.

1. Basic electromagnetic experiments versus Galilean-transformation force invariance

The Galilean transformation of time t and displacement $\mathbf{r}$ due to travel *at constant velocity* $\mathbf{v}$ is,

$$t' = t \quad \text{and} \quad \mathbf{r}' = \mathbf{r} - \mathbf{v}t. \tag{1.1a}$$

Therefore *the effect on velocity* $d\mathbf{r}/dt$ of the constant-velocity-$\mathbf{v}$ Galilean transformation *is to subtract* $\mathbf{v}$,

$$d\mathbf{r}'/dt' = d(\mathbf{r} - \mathbf{v}t)/dt = d\mathbf{r}/dt - \mathbf{v}, \tag{1.1b}$$

and the constant-velocity-$\mathbf{v}$ Galilean transformation *has no effect on acceleration* $d^2\mathbf{r}/dt^2$—it is *invariant,*

$$d^2\mathbf{r}'/d(t')^2 = d(d\mathbf{r}'/dt')/dt' = d(d\mathbf{r}/dt - \mathbf{v})/dt = d^2\mathbf{r}/dt^2. \tag{1.1c}$$

Since forces *produce accelerations* in Newtonian physics, a constant-velocity-$\mathbf{v}$ Galilean transformation *is incapable of introducing new forces* which were *absent* before that transformation was made.

This Newtonian/Galilean precept notwithstanding, it was observed hundreds of years ago that the magnetic-dipole needle of a compass which is lying sufficiently close to a metal wire is deflected away from its equilibrium position of pointing toward magnetic north upon that wire being connected to a battery. It is surmised that the battery sets the invisible microscopic free electrons in the metal wire into motion with, at least on average, a nonzero constant speed that causes them to produce a magnetic field which is *absent* when the battery *isn't connected* and those free electrons are, at least on average, *at rest.*

Of course surmises about the state of motion of the completely invisible microscopic free electrons in a metal wire are hardly immediately persuasive. Such an experiment would be more compelling if the wire and its invisible microscopic free electrons *were replaced by a macroscopic object which has been statically charged.* Issues regarding such an approach include getting enough charge on a macroscopic object and/or getting its speed high enough to produce a strong enough magnetic field to visibly deflect a magnetic compass needle. A major concern is too-rapid dissipation of the object's charge into the air around it, which might be ameliorated by artificial cooling and dehumidification of that air. The object's charge may also need to be shielded from air currents associated with its speed, or which occur spontaneously in its surroundings.

In 1831 Michael Faraday showed that thrusting a bar magnet lengthwise through the center of a metal wire coil produces a transient current in the coil that is detected by a galvanometer connected to the coil. Thus a bar magnet moving at a nonzero constant velocity in the direction of its magnetic moment apparently produces *a moving azimuthal electric field* that is *absent* when the magnet *is at rest,* but transiently drives the free electrons in the wire coil *around that coil* when the magnet *is moving.* Here one moving object, the bar magnet, is indeed *macroscopic,* so its motion, or the lack thereof, *is plain to see.* The moving azimuthal electric field that its motion supposedly produces is inferred, however, from supposed transient azimuthal motion of invisible microscopic free electrons in the metal wire coil.

It would be more compelling *to instead verify that transient electric field by the visible deflection of a macroscopic entity which has been statically charged.* A low-mass charged object which hangs downward

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by a thread directly above the bar magnet's horizontal-line trajectory should be deflected horizontally perpendicular to that trajectory (i.e., azimuthally), first toward one side and then toward the opposite side, as the magnet passes beneath it at constant velocity. As might be expected, this concept comes with its list of caveats and pitfalls. The object's charge must be great enough and its mass low enough to produce a visible deflection. (Of course the stronger the moving bar magnet's electric field is, the greater is the deflection of the charged object; that electric field strength increases with the magnet's velocity and the strength of its dipole moment.) Too-rapid dissipation of the object's charge into the air around it needs to be ameliorated, possibly by cooling and dehumidifying that air. The low-mass hanging charged object would be ultra-sensitive to deflection by stray air currents, and so would need to hang inside an airtight transparent case.

The deflection of the needle of a magnetic compass by nearby *moving* charges (but *not* by such charges *at rest)* became in due course the essence of the Biot-Savart Law of electromagnetic theory, and James Clerk Maxwell distilled Faraday's demonstration that a *moving* magnetic field (but *not* a *stationary* magnetic field) produces a moving electric field into Faraday's Law of electromagnetic theory.

These two Laws of electromagnetic theory obviously *flatly contradict* the Newtonian/Galilean precept pointed out below Eq. (1.1c) that a constant-velocity-$\mathbf{v}$ Galilean transformation *is incapable of introducing new forces* which were *absent* before that transformation was made.

But the *existence* of this *blatant contradiction of a basic Newtonian precept* by the Laws of electromagnetism *completely eluded the awareness of the physics community* until *the null result* of the Michelson-Morley experiment *challenged another basic Galilean precept,* namely the Eq. (1.1b) *Galilean subtraction of the constant transformation velocity* $\mathbf{v}$ *from the untransformed velocity* $d\mathbf{r}/dt$—which *also* clashes with the electromagnetic Laws. Those Laws can be presented *as wave equations with source terms,* where *the fixed constant* c *is invariably the speed of the free waves* which can exist wherever the source terms are zero. Since *the completely fixed constant* c *is the only free wave speed which the electromagnetic Laws permit,* physically-correct transformations $d\mathbf{r}'/dt'$ of untransformed velocity $d\mathbf{r}/dt$ *must preserve speed* c. But the Eq. (1.1b) constant-velocity-$\mathbf{v}$ Galilean transformation of velocity, i.e., $d\mathbf{r}'/dt' = d\mathbf{r}/dt - \mathbf{v}$, *clearly doesn't preserve speed* c. That the electromagnetic free wave speed *invariably is* c is supported by the Michelson-Morley *null result,* which, *contrary to the* Eq. (1.1b) *subtraction of the transformation velocity* $\mathbf{v}$ *from the untransformed velocity* $d\mathbf{r}/dt$, *found no variation in the speed* c *of light from sources traveling at various different velocities.*

We next develop the constant-velocity-$\mathbf{v}$ *Lorentz transformation* which *preserves speed* c, but *of course in addition becomes constant-velocity-*$\mathbf{v}$ *Galilean when the ratio of the untransformed speed to* c *goes to zero.*

2. Developing the Lorentz transformations and exploring some of their consequences

The constant-velocity-$\mathbf{v}$ Lorentz transformation *is required to preserve speed* c, i.e., *if the untransformed speed* $|d\mathbf{r}/dt|$ *is* c, *the constant-velocity-*$\mathbf{v}$ *Lorentz-transformed speed* $|d\mathbf{r}'/dt'|$ *is required to be* c *as well.* In addition, the constant-velocity-$\mathbf{v}$ Lorentz transformation *is required to become constant-velocity-*$\mathbf{v}$ *Galilean when the ratio* $(|d\mathbf{r}/dt|/c)$ *of the untransformed speed to* c *goes to zero;* specifically, the constant-velocity-$\mathbf{v}$ Lorentz-transformed velocity $d\mathbf{r}'/dt'$ is required to become the Eq. (1.1b) constant-velocity-$\mathbf{v}$ Galilean-transformed velocity $d\mathbf{r}/dt - \mathbf{v}$ when the ratio $(|d\mathbf{r}/dt|/c) \to 0$.

It turns out that the above two requirements for the constant-velocity-$\mathbf{v}$ Lorentz transformation can be satisfied *by an appropriately-modified form of the* Eq. (1.1a) *constant-velocity-*$\mathbf{v}$ *Galilean transformation.*

A *salient property* of the Eq. (1.1a) constant-velocity-$\mathbf{v}$ Galilean transformation, $t' = t$ and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, of $(t,\, \mathbf{r})$ into $(t',\, \mathbf{r}')$ is its trivial *identity nature for the part of* $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$; *only the component of* $\mathbf{r}$ *in the direction of* $\mathbf{v}$ *is,* in conjunction with time t, *transformed in a nontrivial manner.*

The *direction* of the constant velocity $\mathbf{v}$ is given by the constant unit vector,

$$\hat{\mathbf{u}} \stackrel{\text{def}}{=} (\mathbf{v}/|\mathbf{v}|), \tag{2.1a}$$

which has the following readily-verified properties,

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1, \quad \mathbf{v} \cdot \hat{\mathbf{u}} = |\mathbf{v}| \quad \text{and} \quad \mathbf{v} = |\mathbf{v}|\hat{\mathbf{u}} = (\mathbf{v} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}. \tag{2.1b}$$

The component of $\mathbf{r}$ *in the direction of* $\mathbf{v}$ is $(\mathbf{r} \cdot \hat{\mathbf{u}})$, and *the part of* $\mathbf{r}$ *that is parallel to* $\mathbf{v}$ is,

$$\mathbf{r}_{\parallel} = (\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}}. \tag{2.1c}$$

Subtraction of $\mathbf{r}_{\|}$ from $\mathbf{r}$ produces,

$$\mathbf{r}_{\perp} \stackrel{\text{def}}{=} \mathbf{r} - \mathbf{r}_{\|} = \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}, \tag{2.1d}$$

which is *the part of* $\mathbf{r}$ *that is perpendicular to* $\mathbf{v}$ because,

$$\mathbf{r}_{\perp}\cdot\mathbf{v} = (\mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}})\cdot\mathbf{v} = \big((\mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}})\cdot\hat{\mathbf{u}}\big)|\mathbf{v}| = \big(((\mathbf{r}\cdot\hat{\mathbf{u}}) - (\mathbf{r}\cdot\hat{\mathbf{u}}))\big)|\mathbf{v}| = 0 \quad \text{and} \quad \mathbf{r} = \mathbf{r}_{\|} + \mathbf{r}_{\perp}. \tag{2.1e}$$

From the Eq. (1.1a) constant-velocity-$\mathbf{v}$ Galilean transformation, $t' = t$ and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, we obtain,

$$\mathbf{r}'_{\perp} = \mathbf{r}' - (\mathbf{r}'\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = (\mathbf{r} - \mathbf{v}t) - ((\mathbf{r} - \mathbf{v}t)\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = (\mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}) - (\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}})t = \mathbf{r}_{\perp} - (\mathbf{v} - \mathbf{v})t = \mathbf{r}_{\perp}, \tag{2.2a}$$

where we have applied the relation $\mathbf{v} = (\mathbf{v}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}$ of Eq. (2.1b) *to reveal the trivial* $\mathbf{r}_{\perp}$ *identity transformation,*

$$\mathbf{r}'_{\perp} = \mathbf{r}_{\perp}, \;\; \text{which in greater detail reads,} \;\; \mathbf{r}' - \mathbf{r}'_{\|} = \mathbf{r} - \mathbf{r}_{\|} \;\; \text{or} \;\; \mathbf{r}' - (\mathbf{r}'\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} = \mathbf{r} - (\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}}. \tag{2.2b}$$

But when the dot product of *the displacement part* $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ of the Eq. (1.1a) constant-velocity-$\mathbf{v}$ Galilean transformation with the unit vector $\hat{\mathbf{u}}$ is taken, it is seen that that transformation *also yields a nontrivial homogeneous linear transformation of the two-dimensional pair* $(t,\, (\mathbf{r}\cdot\hat{\mathbf{u}}))$ *into* $(t',\, (\mathbf{r}'\cdot\hat{\mathbf{u}}))$, namely,

$$t' = t \;\; \text{and} \;\; (\mathbf{r}'\cdot\hat{\mathbf{u}}) = (\mathbf{r}\cdot\hat{\mathbf{u}}) - |\mathbf{v}|t. \tag{2.2c}$$

Furthermore, multiplying *the second equation of* Eq. (2.2c) by the unit vector $\hat{\mathbf{u}}$ yields $\mathbf{r}'_{\|} = \mathbf{r}_{\|} - \mathbf{v}t$, which added to *the trivial* Eq. (2.2b) *identity transformation* $\mathbf{r}'_{\perp} = \mathbf{r}_{\perp}$ produces *the displacement part* $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ of the Eq. (1.1a) constant-velocity-$\mathbf{v}$ Galilean transformation. Thus we see that the Eq. (2.2c) homogeneous linear transformation of $(t,\, (\mathbf{r}\cdot\hat{\mathbf{u}}))$ into $(t',\, (\mathbf{r}'\cdot\hat{\mathbf{u}}))$ *is the only nontrivial part of the* Eq. (1.1a) *constant-velocity-*$\mathbf{v}$ *Galilean transformation of* $(t,\, \mathbf{r})$ *into* $(t',\, \mathbf{r}')$. We *likewise* expect that *the only nontrivial part of the constant-velocity-*$\mathbf{v}$ *Lorentz transformation of* $(t,\, \mathbf{r})$ *into* $(t',\, \mathbf{r}')$ *will be a homogeneous linear transformation of* $(t,\, (\mathbf{r}\cdot\hat{\mathbf{u}}))$ *into* $(t',\, (\mathbf{r}'\cdot\hat{\mathbf{u}}))$. We therefore *begin with the most general possible form for a homogeneous linear transformation of* $(t,\, (\mathbf{r}\cdot\hat{\mathbf{u}}))$ *into* $(t',\, (\mathbf{r}'\cdot\hat{\mathbf{u}}))$, which we conveniently express as,

$$t' = \lambda(t - (\sigma/|\mathbf{v}|)(\mathbf{r}\cdot\hat{\mathbf{u}})) \;\; \text{and} \;\; (\mathbf{r}'\cdot\hat{\mathbf{u}}) = \gamma((\mathbf{r}\cdot\hat{\mathbf{u}}) - \kappa|\mathbf{v}|t), \tag{2.2d}$$

where λ, σ, γ and κ *are dimensionless,* and *the Galilean* Eq. (2.2c) *corresponds to* $\lambda = 1$, $\sigma = 0$, $\gamma = 1$ *and* $\kappa = 1$. Multiplying the second equation of Eq. (2.2d) by the unit vector $\hat{\mathbf{u}}$ yields $\mathbf{r}'_{\|} = \gamma\mathbf{r}_{\|} - \gamma\kappa\mathbf{v}t$, which added to *the trivial* Eq. (2.2b) *identity transformation* $\mathbf{r}'_{\perp} = \mathbf{r}_{\perp} = \mathbf{r} - \mathbf{r}_{\|}$ produces *the displacement part* of the following transformation of $(t,\, \mathbf{r})$ into $(t',\, \mathbf{r}')$,

$$t' = \lambda(t - (\sigma/|\mathbf{v}|)(\mathbf{r}\cdot\hat{\mathbf{u}})) \;\; \text{and} \;\; \mathbf{r}' = \mathbf{r} + (\gamma - 1)\mathbf{r}_{\|} - \gamma\kappa\mathbf{v}t = \mathbf{r} + (\gamma - 1)(\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\kappa\mathbf{v}t, \tag{2.2e}$$

where the constant-velocity-$\mathbf{v}$ *Galilean* Eq. (1.1a), $t' = t$ and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, *corresponds to* $\lambda = 1$, $\sigma = 0$, $\gamma = 1$ *and* $\kappa = 1$. We next obtain from Eq. (2.2e) *its transformation of velocity* $d\mathbf{r}'/dt' = [d\mathbf{r}'/dt]/[dt'/dt]$. Since,

$d\mathbf{r}'/dt = d\mathbf{r}/dt + (\gamma - 1)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\kappa\mathbf{v}$ and $dt'/dt = \lambda(1 - (\sigma/|\mathbf{v}|)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}}))$, the result is,

$$d\mathbf{r}'/dt' = \big[d\mathbf{r}/dt + (\gamma - 1)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\kappa\mathbf{v}\big]/\big[\lambda(1 - (\sigma/|\mathbf{v}|)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}}))\big], \tag{2.2f}$$

where the *Galilean* Eq. (1.1b), namely $d\mathbf{r}'/dt' = d\mathbf{r}/dt - \mathbf{v}$, *corresponds to* $\lambda = 1$, $\sigma = 0$, $\gamma = 1$ *and* $\kappa = 1$. One of *the two properties required of the constant-velocity-*$\mathbf{v}$ *Lorentz transformation* is that *when the ratio* $(|d\mathbf{r}/dt|/c) \to 0$, $d\mathbf{r}'/dt'$ *becomes the Galilean* $d\mathbf{r}/dt - \mathbf{v}$ *of* Eq. (1.1b). Therefore, *for the constant-velocity-*$\mathbf{v}$ *Lorentz transformation, when* $d\mathbf{r}/dt = \mathbf{0}$, $d\mathbf{r}'/dt' = -\mathbf{v}$, which inserted into Eq. (2.2f) yields,

$$\gamma\kappa = \lambda. \tag{2.2g}$$

Consequently, *for the constant-velocity-*$\mathbf{v}$ *Lorentz transformation,* Eq. (2.2f) can be rewritten without κ,

$$d\mathbf{r}'/dt' = \big[d\mathbf{r}/dt + (\gamma - 1)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \lambda\mathbf{v}\big]/\big[\lambda(1 - (\sigma/|\mathbf{v}|)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}}))\big], \tag{2.3a}$$

which of course implies that,

$$|d\mathbf{r}'/dt'|^2\lambda^2\big(1 - (\sigma/|\mathbf{v}|)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})\big)^2 = \big|d\mathbf{r}/dt + (\gamma - 1)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \lambda\mathbf{v}\big|^2. \tag{2.3b}$$

The *other property required of Lorentz transformations* is that *when* $|d\mathbf{r}/dt| = c$, $|d\mathbf{r}'/dt'| = c$ *as well.* Inserting these equalities respectively into the expanded right side and left side of Eq. (2.3b) produces,

$$\lambda^2c^2 - 2\lambda^2c^2(\sigma/|\mathbf{v}|)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}}) + \lambda^2c^2(\sigma/|\mathbf{v}|)^2((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})^2$$
$$= (c^2 + \lambda^2|\mathbf{v}|^2) - 2\lambda\gamma|\mathbf{v}|((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}}) + (\gamma^2 - 1)((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})^2. \tag{2.3c}$$

Because of *the linear independence of distinct powers of* $((d\mathbf{r}/dt)\cdot\hat{\mathbf{u}})$, Eq. (2.3c) implies *the three equations,*

$$\lambda^2 = 1/\big(1 - |\mathbf{v}/c|^2\big),\quad \sigma = (\gamma/\lambda)|\mathbf{v}/c|^2 \quad\text{and}\quad \lambda^2\sigma^2 = (\gamma^2 - 1)|\mathbf{v}/c|^2. \tag{2.3d}$$

The insertion of the second equation of Eq. (2.3d) into its third equation allows us to conclude that,

$$\gamma^2 = 1/\big(1 - |\mathbf{v}/c|^2\big) = \lambda^2 \quad\text{and}\quad \sigma = (\gamma/\lambda)|\mathbf{v}/c|^2. \tag{2.3e}$$

which leaves the *signs* of γ, λ and σ *undetermined.* However, *for the constant-velocity-*$\mathbf{v}$ *Lorentz transformation, when the ratio* $(|d\mathbf{r}/dt|/c) \to 0$, $d\mathbf{r}'/dt'$ *becomes the Galilean* $d\mathbf{r}/dt - \mathbf{v}$ *of* Eq. (1.1b). Requiring that of the Eq. (2.3a) expression for $d\mathbf{r}'/dt'$ has the consequence that when $c \to \infty$, $\gamma \to 1$, $\lambda \to 1$ and $\sigma \to 0$. Comparison of these $c \to \infty$ *limits of* γ, λ *and* σ with the Eq. (2.3e) *results for* γ^2, λ^2 *and* σ yields,

$$\gamma = 1/\sqrt{1 - |\mathbf{v}/c|^2} = \lambda \quad\text{and}\quad \sigma = |\mathbf{v}/c|^2. \tag{2.3f}$$

Insertion of the results given by Eq. (2.3f) into Eq. (2.2g) yields that,

$$\kappa = 1. \tag{2.4a}$$

Insertion into Eq. (2.2e) of the results for γ, λ, σ and κ given by Eqs. (2.3f) and (2.4a) yields that *the constant-velocity-*$\mathbf{v}$ *Lorentz transformation of time* t *and displacement* $\mathbf{r}$ *is,*

$$t' = \gamma(t - (|\mathbf{v}/c|/c)(\mathbf{r}\cdot\hat{\mathbf{u}})) \;\text{ and }\; \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\mathbf{v}t, \;\text{ where }\; \gamma \stackrel{\text{def}}{=} 1/\sqrt{1 - |\mathbf{v}/c|^2}, \tag{2.4b}$$

which, *when* $c \to \infty$, *becomes the* Eq. (1.1a) *constant-velocity-*$\mathbf{v}$ *Galilean transformation,* as it *must.*

It will later on be easier to extract the widely applicable dimensionless matrix form of the Lorentz transformation if the two equations of Eq. (2.4b) *are made to have the same dimension.* We multiply the first equation of Eq. (2.4b) by c, and then conveniently define $(x^0)' \stackrel{\text{def}}{=} ct'$, $x^0 \stackrel{\text{def}}{=} ct$ and $\beta \stackrel{\text{def}}{=} |\mathbf{v}/c|$ to produce,

$$(x^0)' = \gamma(x^0 - \beta(\mathbf{r}\cdot\hat{\mathbf{u}})) \;\text{ and }\; \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\beta x^0\hat{\mathbf{u}};\; \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|,\; \gamma \stackrel{\text{def}}{=} 1/\sqrt{1 - \beta^2}. \tag{2.4c}$$

We next briefly return to the Galilean transformation, $t' = t$ and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, *to study its inverse,*

the Galilean transformation, $t' = t$ and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, is easily inverted to yield, $t = t'$ and $\mathbf{r} = \mathbf{r}' + \mathbf{v}t'$. (2.5)

Thus the *inverse* of the Galilean transformation has *almost* the same form as the Galilean transformation itself *except that* $\mathbf{v} \to -\mathbf{v}$. This property of the Galilean transformation, sometimes called *relativistic reciprocity,* is *a fundamental relationship of the two coordinate systems:* they are equivalent *except that an observer at rest in the "moving" system attributes velocity* $-\mathbf{v}$ *to the "stationary" system.* Since relativistic reciprocity is a fundamental relationship of the two coordinate systems, *it ought to hold as well for the Lorentz transformation.* We next undertake *the cumbersome task* of *inverting* the Lorentz transformation of Eq. (2.4c), which is also displayed in Eq. (2.6a) below, *to check whether relativistic reciprocity holds.*

$$(x^0)' = \gamma(x^0 - \beta(\mathbf{r}\cdot\hat{\mathbf{u}})) \;\text{ and }\; \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\beta x^0\hat{\mathbf{u}};\; \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|,\; \gamma \stackrel{\text{def}}{=} 1/\sqrt{1 - \beta^2}. \tag{2.6a}$$

The first equation of Eq. (2.6a), taken together with the the result of taking the dot product of the second equation of Eq. (2.6a) with $\hat{\mathbf{u}}$, produces the following $x^0 \leftrightarrow (\mathbf{r}\cdot\hat{\mathbf{u}})$ "mirror pair" of scalar equations,

$$(x^0)' = \gamma(x^0 - \beta(\mathbf{r}\cdot\hat{\mathbf{u}})) \;\text{ and }\; (\mathbf{r}'\cdot\hat{\mathbf{u}}) = \gamma((\mathbf{r}\cdot\hat{\mathbf{u}}) - \beta x^0);\; \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|,\; \gamma \stackrel{\text{def}}{=} 1/\sqrt{1 - \beta^2}, \tag{2.6b}$$

whose *solution* for x^0 and $(\mathbf{r}\cdot\hat{\mathbf{u}})$ in terms of $(x^0)'$ and $(\mathbf{r}'\cdot\hat{\mathbf{u}})$ is the following $x^0 \leftrightarrow (\mathbf{r}\cdot\hat{\mathbf{u}})$ "mirror pair",

$$x^0 = \gamma((x^0)' + \beta(\mathbf{r}'\cdot\hat{\mathbf{u}})) \;\text{ and }\; (\mathbf{r}\cdot\hat{\mathbf{u}}) = \gamma((\mathbf{r}'\cdot\hat{\mathbf{u}}) + \beta(x^0)');\; \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|,\; \gamma \stackrel{\text{def}}{=} 1/\sqrt{1 - \beta^2}. \tag{2.6c}$$

The Eq. (2.6c) solution of Eq. (2.6b) is easily verified upon noting that $\gamma^2(1 - \beta^2) = 1$. Also, *as is generally the case for Galilean and Lorentz transformations,* $\mathbf{r}_\perp = \mathbf{r}'_\perp$, which is apparent from the second equation of Eq. (2.6a). Multiplying the second Eq. (2.6c) equation for $(\mathbf{r}\cdot\hat{\mathbf{u}})$ in terms of $(\mathbf{r}'\cdot\hat{\mathbf{u}})$ and $(x^0)'$ by $\hat{\mathbf{u}}$ yields,

$$\mathbf{r}_\parallel = \gamma\mathbf{r}'_\parallel + \gamma\beta(x^0)'\hat{\mathbf{u}}, \tag{2.6d}$$

which when added to $\mathbf{r}_\perp = \mathbf{r}'_\perp = \mathbf{r}' - \mathbf{r}'_\parallel$ produces,

$$\mathbf{r} = \mathbf{r}' + (\gamma - 1)\mathbf{r}'_{\|} + \gamma\beta(x^0)'\hat{\mathbf{u}} = \mathbf{r}' + (\gamma - 1)(\mathbf{r}' \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \gamma\beta(x^0)'\hat{\mathbf{u}}. \tag{2.6e}$$

Combining the first Eq. (2.6c) equation for x^0 in terms of $(x^0)'$ and $\mathbf{r}'$ with the Eq. (2.6e) result for $\mathbf{r}$ in terms of $\mathbf{r}'$ and $(x^0)'$ produces *the inverse of the* Eq. (2.6a) *Lorentz transformation,*

$$x^0 = \gamma((x^0)' + \beta(\mathbf{r}' \cdot \hat{\mathbf{u}})) \text{ and } \mathbf{r} = \mathbf{r}' + (\gamma - 1)(\mathbf{r}' \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \gamma\beta(x^0)'\hat{\mathbf{u}};\ \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|,\ \gamma \stackrel{\text{def}}{=} 1\big/\sqrt{1-\beta^2}. \tag{2.6f}$$

We note that *reversing the sign of* $\hat{\mathbf{u}}$ *reverses of the sign of* $\mathbf{v} = |\mathbf{v}|\hat{\mathbf{u}}$. The Eq. (2.6f) *inverse* of the Eq. (2.6a) Lorentz transformation has *almost* the same form as the Eq. (2.6a) Lorentz transformation itself *except that* $\hat{\mathbf{u}} \to -\hat{\mathbf{u}}$. Thus the Eq. (2.6a) Lorentz transformation *indeed manifests relativistic reciprocity.*

We next investigate time/spatial-coordinate *forms* which *are left invariant* by the Eq. (2.4b) time t and displacement $\mathbf{r}$ Lorentz transformation. Since the Lorentz transformation *preserves speed* c, it *necessarily also preserves the space-time locus of the expanding spherical light surface;* i.e., if $|\mathbf{r}|^2 - c^2t^2 = 0$, then $|\mathbf{r}'|^2 - c^2(t')^2 = 0$, or in terms of the variables x^0 and $\mathbf{r}$ used in the Eq. (2.4c) Lorentz transformation, if $|\mathbf{r}|^2 - (x^0)^2 = 0$ then $|\mathbf{r}'|^2 - ((x^0)')^2 = 0$. But the Eq. (2.4c) Lorentz transformation *goes well beyond merely preserving the space-time locus of the expanding spherical light surface* $|\mathbf{r}|^2 - (x^0)^2 = 0$; it in fact *preserves the indefinite quadratic form* $|\mathbf{r}|^2 - (x^0)^2$ *regardless of that form's value,*

$$|\mathbf{r}'|^2 - ((x^0)')^2 = |\mathbf{r} + (\gamma - 1)(\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\beta x^0\hat{\mathbf{u}}|^2 - \gamma^2(x^0 - \beta(\mathbf{r} \cdot \hat{\mathbf{u}}))^2 =$$
$$|\mathbf{r}|^2 - (x^0)^2\gamma^2(1-\beta^2) - 2\gamma(1 + (\gamma - 1) - \gamma)\beta x^0(\mathbf{r} \cdot \hat{\mathbf{u}}) + ((\gamma-1)^2 + 2(\gamma - 1) - \gamma^2\beta^2)(\mathbf{r} \cdot \hat{\mathbf{u}})^2 =$$
$$|\mathbf{r}|^2 - (x^0)^2 + (\gamma^2 - 1 - \gamma^2\beta^2)(\mathbf{r} \cdot \hat{\mathbf{u}})^2 = |\mathbf{r}|^2 - (x^0)^2 + (\gamma^2(1-\beta^2) - 1)(\mathbf{r} \cdot \hat{\mathbf{u}})^2 = |\mathbf{r}|^2 - (x^0)^2. \tag{2.7}$$

Constant-velocity-$\mathbf{v}$ Lorentz transformation applies *not only* to x^0 and $\mathbf{r}$, as in Eq. (2.4c), but *as well* to energy and momentum, electromagnetic field components *and many other physical entities.* That *widely applicable form of the constant-velocity-*$\mathbf{v}$ *Lorentz transformation exists within* Eq. (2.4c) *as a dimensionless* 4×4 *matrix.* To *extract those sixteen dimensionless matrix elements* from the Eq. (2.4c) x^0 and $\mathbf{r}$ form of the constant-velocity-$\mathbf{v}$ Lorentz transformation, *we explicitly replace* $\mathbf{r}'$ *by* $((x^1)', (x^2)', (x^3)')$, $\mathbf{r}$ *by* (x^1, x^2, x^3) *and* $\hat{\mathbf{u}}$ by (u^1, u^2, u^3). The sixteen dimensionless matrix elements then *are the coefficients of* x^0, x^1, x^2 *and* x^3 *in the* Eq. (2.4c) x^0 *and* $\mathbf{r}$ *Lorentz transformation form's expressions for* $(x^0)'$, $(x^1)'$, $(x^2)'$ *and* $(x^3)'$. For example, the $(x^0)' = \gamma(x^0 - \beta(\mathbf{r} \cdot \hat{\mathbf{u}}))$*scalar part* of the Eq. (2.4c) x^0 and $\mathbf{r}$ Lorentz transformation form *is its expression for* $(x^0)'$, which is parsed as follows to extract the coefficients of x^0, x^1, x^2 and x^3,

$$(x^0)' = \gamma(x^0 - \beta(\mathbf{r} \cdot \hat{\mathbf{u}})) = (\gamma)x^0 + \textstyle\sum_{i=1}^3(-\gamma\beta u^i)x^i = \Lambda^{00}(\beta, \hat{\mathbf{u}})x^0 + \sum_{i=1}^3 \Lambda^{0i}(\beta, \hat{\mathbf{u}})x^i, \tag{2.8a}$$

from which we read off the following *four* of the constant-velocity-$\mathbf{v}$ Lorentz-transformation's *sixteen* dimensionless matrix elements $\Lambda^{\mu\nu}(\beta, \hat{\mathbf{u}})$, $\mu, \nu = 0, 1, 2, 3$,

$$\Lambda^{00}(\beta, \hat{\mathbf{u}}) = \gamma, \quad \Lambda^{0i}(\beta, \hat{\mathbf{u}}) = -\gamma\beta u^i,\ i = 1, 2, 3. \tag{2.8b}$$

The *remaining* $\mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\beta x^0\hat{\mathbf{u}}$ *vector part* of the Eq. (2.4c) x^0 and $\mathbf{r}$ Lorentz transformation form *encompasses in three-vector-shorthand notation its following expressions for* $(x^1)'$, $(x^2)'$ *and* $(x^3)'$,

$$(x^i)' = x^i + (\gamma - 1)(\mathbf{r} \cdot \hat{\mathbf{u}})u^i - \gamma\beta x^0 u^i,\ i = 1, 2, 3, \tag{2.8c}$$

which are parsed as follows to extract their coefficients of x^0, x^1, x^2 and x^3,

$$(x^i)' = \textstyle\sum_{j=1}^3\big(\delta^{ij} + (\gamma - 1)u^iu^j\big)x^j + (-\gamma\beta u^i)x^0 = \sum_{j=1}^3 \Lambda^{ij}(\beta, \hat{\mathbf{u}})x^j + \Lambda^{i0}(\beta, \hat{\mathbf{u}})x^0,\ i = 1, 2, 3, \tag{2.8d}$$

from which we read off the following *twelve* of the constant-velocity-$\mathbf{v}$ Lorentz-transformation's *sixteen* dimensionless matrix elements $\Lambda^{\mu\nu}(\beta, \hat{\mathbf{u}})$, $\mu, \nu = 0, 1, 2, 3$,

$$\Lambda^{ij}(\beta, \hat{\mathbf{u}}) = \big(\delta^{ij} + (\gamma - 1)(u^iu^j)\big),\ i, j = 1, 2, 3, \quad \Lambda^{i0}(\beta, \hat{\mathbf{u}}) = -\gamma\beta u^i,\ i = 1, 2, 3. \tag{2.8e}$$

Combining Eq. (2.8b) with Eq. (2.8e) provides the constant-velocity-$\mathbf{v}$ Lorentz-transformation's *sixteen* dimensionless matrix elements $\Lambda^{\mu\nu}(\beta, \hat{\mathbf{u}})$, $\mu, \nu = 0, 1, 2, 3$,

$$\Lambda^{00}(\beta, \hat{\mathbf{u}}) = \gamma, \quad \Lambda^{ij}(\beta, \hat{\mathbf{u}}) = \big(\delta^{ij} + (\gamma - 1)(u^iu^j)\big),\ i, j = 1, 2, 3,$$
$$\Lambda^{0i}(\beta, \hat{\mathbf{u}}) = -\gamma\beta u^i \ \&\ \Lambda^{i0}(\beta, \hat{\mathbf{u}}) = -\gamma\beta u^i,\ i = 1, 2, 3. \tag{2.9a}$$

It is apparent by inspection of Eq. (2.9a) that the Lorentz-transformation matrix $\Lambda^{\mu\nu}(\beta, \hat{\mathbf{u}})$ is *symmetric,*

$$\Lambda^{\mu\nu}(\beta, \hat{\mathbf{u}}) = \Lambda^{\nu\mu}(\beta, \hat{\mathbf{u}}),\ \mu, \nu = 0, 1, 2, 3. \tag{2.9b}$$

It is also apparent by inspection of Eq. (2.9a) that the effect *of reversing the sign of* $\hat{\mathbf{u}}$ is,

$$\Lambda^{00}(\beta,-\hat{\mathbf{u}}) = \gamma = \Lambda^{00}(\beta,\hat{\mathbf{u}}), \quad \Lambda^{ij}(\beta,-\hat{\mathbf{u}}) = \big(\delta^{ij} + (\gamma-1)(u^i u^j)\big) = \Lambda^{ij}(\beta,\hat{\mathbf{u}}),\ i,j=1,2,3,$$
$$\Lambda^{0i}(\beta,-\hat{\mathbf{u}}) = \gamma\beta u^i = -\Lambda^{0i}(\beta,\hat{\mathbf{u}}) \ \&\ \Lambda^{i0}(\beta,-\hat{\mathbf{u}}) = \gamma\beta u^i = -\Lambda^{i0}(\beta,\hat{\mathbf{u}}),\ i=1,2,3. \tag{2.9c}$$

The principle of relativistic reciprocity implies that *reversing the sign of* $\hat{\mathbf{u}}$ *inverts the Lorentz transformation,* so $\Lambda(\beta,-\hat{\mathbf{u}}) = (\Lambda(\beta,\hat{\mathbf{u}}))^{-1}$. That fact can, of course, be confirmed by (rather tediously) verifying that,

$$\textstyle\sum_{\sigma=0}^{3} \Lambda^{\mu\sigma}(\beta,-\hat{\mathbf{u}})\,\Lambda^{\sigma\nu}(\beta,\hat{\mathbf{u}}) = \delta^{\mu\nu} \text{ for } \mu,\nu=0,1,2,3. \tag{2.9d}$$

It *also* follows, however, from the Lorentz transformation's preservation of the indefinite quadratic form $|\mathbf{r}|^2-(x^0)^2$, which was demonstrated in Eq. (2.7). To show that, we begin by extracting *a matrix relation satisfied by* $\Lambda(\beta,\hat{\mathbf{u}})$ from the Eq. (2.7) result $|\mathbf{r}'|^2-((x^0)')^2 = |\mathbf{r}|^2-(x^0)^2$ *written in vector/symmetric-matrix format.* The two quadratic forms $|\mathbf{r}|^2-(x^0)^2$ and $|\mathbf{r}'|^2-((x^0)')^2$ *are written as follows in that format,*

$$|\mathbf{r}|^2-(x^0)^2 = -(x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2 = \textstyle\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} x^\mu G^{\mu\nu} x^\nu = x^T G x,$$
$$\text{where } G^{00}=-1,\ G^{11}=G^{22}=G^{33}=+1 \text{ and } G^{\mu\nu}=0 \text{ when } \mu\neq\nu, \ \text{ and}$$
$$|\mathbf{r}'|^2-((x^0)')^2 = (x')^T G x' = (\Lambda(\beta,\hat{\mathbf{u}})x)^T G(\Lambda(\beta,\hat{\mathbf{u}})x) = x^T\big(\Lambda^T(\beta,\hat{\mathbf{u}})G\Lambda(\beta,\hat{\mathbf{u}})\big)x = x^T\big(\Lambda(\beta,\hat{\mathbf{u}})G\Lambda(\beta,\hat{\mathbf{u}})\big)x, \tag{2.9e}$$

where the last equality reflects $\Lambda^T(\beta,\hat{\mathbf{u}}) = \Lambda(\beta,\hat{\mathbf{u}})$, since $\Lambda(\beta,\hat{\mathbf{u}})$ is a *symmetric* matrix (see Eq. (2.9b)). The Eq. (2.7) equality $|\mathbf{r}'|^2-((x^0)')^2 = |\mathbf{r}|^2-(x^0)^2$ in conjunction with Eq. (2.9e) yields *the matrix relation,*

$$\Lambda(\beta,\hat{\mathbf{u}})G\Lambda(\beta,\hat{\mathbf{u}}) = G. \tag{2.9f}$$

Since it is apparent from Eq. (2.9e) that $G^2 = \mathrm{I}$, multiplying both sides of Eq. (2.9f) by G yields,

$$G\Lambda(\beta,\hat{\mathbf{u}})G\Lambda(\beta,\hat{\mathbf{u}}) = \mathrm{I}, \tag{2.9g}$$

which implies that the matrix $G\Lambda(\beta,\hat{\mathbf{u}})G$ *is the inverse of* $\Lambda(\beta,\hat{\mathbf{u}})$,

$$(\Lambda(\beta,\hat{\mathbf{u}}))^{-1} = G\Lambda(\beta,\hat{\mathbf{u}})G. \tag{2.9h}$$

Because $G^{\mu\nu}=0$ when $\mu\neq\nu$, the matrix elements of $G\Lambda(\beta,\hat{\mathbf{u}})G$ have the relatively simple form,

$$(G\Lambda(\beta,\hat{\mathbf{u}})G)^{\mu\nu} = G^{\mu\mu}\big(\Lambda^{\mu\nu}(\beta,\hat{\mathbf{u}})\big)G^{\nu\nu}. \tag{2.9i}$$

Since $G^{00}=-1$ and $G^{ii}=+1$ for $i=1,2,3$, Eqs. (2.9h), (2.9i) and (2.9c) yield that,

$$\big((\Lambda(\beta,\hat{\mathbf{u}}))^{-1}\big)^{00} = \Lambda^{00}(\beta,\hat{\mathbf{u}}) = \Lambda^{00}(\beta,-\hat{\mathbf{u}}), \ \big((\Lambda(\beta,\hat{\mathbf{u}}))^{-1}\big)^{ij} = \Lambda^{ij}(\beta,\hat{\mathbf{u}}) = \Lambda^{ij}(\beta,-\hat{\mathbf{u}}),\ i,j=1,2,3,$$
$$\big((\Lambda(\beta,\hat{\mathbf{u}}))^{-1}\big)^{0i} = -\Lambda^{0i}(\beta,\hat{\mathbf{u}}) = \Lambda^{0i}(\beta,-\hat{\mathbf{u}}) \ \&\ \big((\Lambda(\beta,\hat{\mathbf{u}}))^{-1}\big)^{i0} = -\Lambda^{i0}(\beta,\hat{\mathbf{u}}) = \Lambda^{i0}(\beta,-\hat{\mathbf{u}}),\ i=1,2,3, \tag{2.9j}$$

from which it follows by inspection that,

$$(\Lambda(\beta,\hat{\mathbf{u}}))^{-1} = \Lambda(\beta,-\hat{\mathbf{u}}), \text{ the principle of relativistic reciprocity.} \tag{2.9k}$$

In this tutorial we will need to Lorentz transform *two key differential operators of the electromagnetic field equations.* The *most fundamental differential operator* of those equations is *the space-time gradient,*

$$((\partial/\partial x^0),\nabla_{\mathbf{r}}) = (\partial/\partial x^\mu), \tag{2.10a}$$

which is *vital* for expressing *local charge conservation,* and *underlies the d'Alembertian differential operator,*

$$\big(\partial^2/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2\big) = -\textstyle\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}(\partial/\partial x^\mu)G^{\mu\nu}(\partial/\partial x^\nu), \text{ \textit{the differential core} of basic wave dynamics.} \tag{2.10b}$$

To make our notation more compact, *we henceforth assume repeated Greek indices are summed over.* We will also prefer using $\eta^{\mu\nu} \stackrel{\text{def}}{=} -G^{\mu\nu}$ to using $G^{\mu\nu}$. The first part of Eq. (2.10b) is therefore modified to read,

$$\big(\partial^2/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2\big) = -(\partial/\partial x^\mu)G^{\mu\nu}(\partial/\partial x^\nu) = (\partial/\partial x^\mu)\eta^{\mu\nu}(\partial/\partial x^\nu). \tag{2.10c}$$

Lorentz transformations of the space-time gradient $(\partial/\partial x^\mu)$ are $(\partial/\partial(x^\mu)')$, where $(x^\mu)' = \Lambda^{\mu\nu}(\beta,\hat{\mathbf{u}})x^\nu$. We wish to express them as *linear combinations of the untransformed partials,* i.e., $(\partial/\partial(x^\mu)') = C^{\mu\nu}(\partial/\partial x^\nu)$. Since $(\partial/\partial(x^\mu)')$ is a simple first-order differential operator, the $C^{\mu\nu}$ *follow from the appropriate application of the chain rule of the calculus,* which is,

$$(\partial/\partial(x^\mu)') = (\partial x^\nu/\partial(x^\mu)')(\partial/\partial x^\nu). \tag{2.11a}$$

To *evaluate the* $C^{\mu\nu} = (\partial x^\nu/\partial(x^\mu)')$ *of* Eq. (2.11a), *we need the untransformed* x^ν *in terms of the transformed* $(x^\mu)'$, That needed relation is formally given by, $x^\nu = ((\Lambda(\beta,\hat{\mathbf{u}}))^{-1})^{\nu\mu}(x^\mu)'$, and relativistic reciprocity implies that $(\Lambda(\beta,\hat{\mathbf{u}}))^{-1} = \Lambda(\beta,-\hat{\mathbf{u}})$ (see Eq. (2.9k)), so $x^\nu = \Lambda^{\nu\mu}(\beta,-\hat{\mathbf{u}})(x^\mu)'$, which implies that,

$$(\partial x^\nu/\partial(x^\mu)') = \Lambda^{\nu\mu}(\beta,-\hat{\mathbf{u}}) = \Lambda^{\mu\nu}(\beta,-\hat{\mathbf{u}}), \tag{2.11b}$$

where the last equality in Eq. (2.11b) reflects the *symmetry* of $\Lambda^{\nu\mu}(\beta,-\hat{\mathbf{u}})$ in its two indices (see Eq. (2.9c)). Inserting the Eq. (2.11b) result $(\partial x^\nu/\partial(x^\mu)') = \Lambda^{\mu\nu}(\beta,-\hat{\mathbf{u}})$ into Eq. (2.11a) yields,

$$(\partial/\partial(x^\mu)') = \Lambda^{\mu\nu}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\nu), \tag{2.11c}$$

which when *formally compared to the familiar Lorentz transformation of* x^μ, $(x^\mu)' = \Lambda^{\mu\nu}(\beta,\hat{\mathbf{u}})x^\nu$, *exhibits only one notable departure,* namely the replacement of $\Lambda(\beta,\hat{\mathbf{u}})$ by its inverse $(\Lambda(\beta,\hat{\mathbf{u}}))^{-1} = \Lambda(\beta,-\hat{\mathbf{u}})$.

To evaluate *the Lorentz transformation of the d'Alembertian operator* $(\partial/\partial x^\mu)\eta^{\mu\nu}(\partial/\partial x^\nu)$, *which is* $(\partial/\partial(x^\mu)')\eta^{\mu\nu}(\partial/\partial(x^\nu)')$, *where* $(x^\mu)' = \Lambda^{\mu\kappa}(\beta,\hat{\mathbf{u}})x^\kappa$ *and* $(x^\nu)' = \Lambda^{\nu\lambda}(\beta,\hat{\mathbf{u}})x^\lambda$, we use Eq. (2.11c) *both* to *replace* $(\partial/\partial(x^\mu)')$ *by* $\Lambda^{\mu\alpha}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\alpha)$ *and* to *replace* $(\partial/\partial(x^\nu)')$ *by* $\Lambda^{\nu\chi}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\chi)$, with the result,

$$(\partial/\partial(x^\mu)')\eta^{\mu\nu}(\partial/\partial(x^\nu)') = \Lambda^{\mu\alpha}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\alpha)\eta^{\mu\nu}\Lambda^{\nu\chi}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\chi) =$$
$$(\partial/\partial x^\alpha)\Lambda^{\alpha\mu}(\beta,-\hat{\mathbf{u}})\eta^{\mu\nu}\Lambda^{\nu\chi}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\chi), \tag{2.12a}$$

where the last expression follows from reversing the order of the commuting factors $\Lambda^{\mu\alpha}(\beta,-\hat{\mathbf{u}})$ and $(\partial/\partial x^\alpha)$, and then applying the Eq. (2.9c) index symmetry of $\Lambda^{\mu\alpha}(\beta,-\hat{\mathbf{u}})$. Eq. (2.9f) implies that, since $G = -\eta$, $\Lambda(\beta,-\hat{\mathbf{u}})\,\eta\,\Lambda(\beta,-\hat{\mathbf{u}}) = \eta$, which, *when its indices are displayed,* reads, $\Lambda^{\alpha\mu}(\beta,-\hat{\mathbf{u}})\eta^{\mu\nu}\Lambda^{\nu\chi}(\beta,-\hat{\mathbf{u}}) = \eta^{\alpha\chi}$, which in turn implies that $(\partial/\partial x^\alpha)\Lambda^{\alpha\mu}(\beta,-\hat{\mathbf{u}})\eta^{\mu\nu}\Lambda^{\nu\chi}(\beta,-\hat{\mathbf{u}})(\partial/\partial x^\chi) = (\partial/\partial x^\alpha)\eta^{\alpha\chi}(\partial/\partial x^\chi)$. Replacing the last expression of Eq. (2.12a) by the right side of this just-given equality yields,

$$(\partial/\partial(x^\mu)')\eta^{\mu\nu}(\partial/\partial(x^\nu)') = (\partial/\partial x^\alpha)\eta^{\alpha\chi}(\partial/\partial x^\chi) = (\partial/\partial x^\mu)\eta^{\mu\nu}(\partial/\partial x^\nu) = \left(\partial^2/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2\right), \tag{2.12b}$$

so the d'Alembertian operator $(\partial/\partial x^\mu)\eta^{\mu\nu}(\partial/\partial x^\nu) = \left(\partial^2/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2\right)$ *is Lorentz-transformation invariant,* just as the Eq. (2.7) indefinite quadratic form $-x^\mu\eta^{\mu\nu}x^\nu = \left(|\mathbf{r}|^2 - (x^0)^2\right)$ *is Lorentz-transformation invariant.*

3. Setting static electromagnetic fields into motion at low-speed constant velocity

In this tutorial we are interested in conceivable educational-physics electromagnetic lab experiments *which violate Newtonian precepts.* Galilean constant-velocity transformations leave accelerations *invariant,* so constant-velocity transformations *can't produce additional forces* in Newtonian physics. Nevertheless, a *moving,* but *not a stationary,* charge is accompanied *by a moving magnetic field,* and a *moving,* but *not a stationary,* dipole magnet is accompanied *by a moving electric field.*

We shall obtain the electromagnetic fields of point charges and point magnetic dipoles *which are moving at low-speed constant velocity* by applying low-speed *Lorentz transformation* to their *at-rest static electromagnetic fields.* We *haven't* so far discussed *any of the details* of Lorentz transformation of electromagnetic fields or potentials; those details of course *must be compatible with the electromagnetic equations which the electromagnetic fields or potentials satisfy.*

We *begin* with the four Laws which govern the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$,

$$\text{Coulomb's Law: } \nabla_{\mathbf{r}}\cdot\mathbf{E} = 4\pi d^0, \quad \text{Faraday's Law: } \nabla_{\mathbf{r}}\times\mathbf{E} + \partial\mathbf{B}/\partial x^0 = 0,$$
$$\text{Gauss' Law: } \nabla_{\mathbf{r}}\cdot\mathbf{B} = 0 \text{ and the Biot-Savart/Maxwell Law: } \nabla_{\mathbf{r}}\times\mathbf{B} - \partial\mathbf{E}/\partial x^0 = 4\pi\mathbf{d}, \tag{3.1}$$

where $d^0 \stackrel{\text{def}}{=} \rho$, the charge density, $\mathbf{d} \stackrel{\text{def}}{=} (\mathbf{j}/c)$, the current density divided by c, and, of course, $x^0 \stackrel{\text{def}}{=} ct$.

When the magnetic field $\mathbf{B}$ and electric field $\mathbf{E}$ *are attributed as follows to a four-potential* $A^\mu = (A^0,\mathbf{A})$,

$$\mathbf{B} = \nabla_{\mathbf{r}}\times\mathbf{A} \text{ and } \mathbf{E} = -\nabla_{\mathbf{r}}A^0 - \partial\mathbf{A}/\partial x^0, \tag{3.2}$$

then Gauss' Law and Faraday's Law are satisfied. Coulomb's Law and the Biot-Savart/Maxwell Law become,

$$-\nabla_{\mathbf{r}}^2 A^0 - \partial(\nabla_{\mathbf{r}}\cdot\mathbf{A})/\partial x^0 = 4\pi d^0 \quad\text{and}\quad \nabla_{\mathbf{r}}(\nabla_{\mathbf{r}}\cdot\mathbf{A}) - \nabla_{\mathbf{r}}^2\mathbf{A} + \nabla_{\mathbf{r}}(\partial A^0/\partial x^0) + \partial^2\mathbf{A}/\partial(x^0)^2 = 4\pi\mathbf{d}. \tag{3.3a}$$

At this point it is important to take note of the fact that Eq. (3.2) *doesn't determine* $A^\mu = (A^0,\mathbf{A})$ *uniquely.* It is readily verified that, given *an arbitrary scalar function* $S(x^0,\mathbf{r})$ (which has the appropriate dimension), then if $(A^0,\mathbf{A})$ satisfies Eq. (3.2), *so does* $(A^0 - \partial S/\partial x^0,\ \mathbf{A} + \nabla_{\mathbf{r}}S)$, the "gauge freedom" in $(A^0,\mathbf{A})$. We *take advantage of this scalar-function freedom in* $(A^0,\mathbf{A})$ *to simplify* Eq. (3.3a) *by requiring* $(A^0,\mathbf{A})$ *to satisfy the scalar-equation "Lorentz condition"* $\nabla_{\mathbf{r}}\cdot\mathbf{A} = -\partial A^0/\partial x^0$. Eq. (3.3a) then reads,

$$\partial^2 A^0/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2 A^0 = 4\pi d^0 \quad \text{and} \quad \partial^2 \mathbf{A}/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2 \mathbf{A} = 4\pi \mathbf{d}. \tag{3.3b}$$

The d'Alembertian $\left(\partial^2/\partial(x^0)^2 - \nabla_{\mathbf{r}}^2\right) = \left((\partial/\partial x^\alpha)\eta^{\alpha\chi}(\partial/\partial x^\chi)\right)$ *acts on both parts of the four-potential* $A^\mu = (A^0, \mathbf{A})$ *in these two equations,* so we *also* combine $d^0 \stackrel{\text{def}}{=} \rho$ *and* $\mathbf{d} \stackrel{\text{def}}{=} (\mathbf{j}/c)$ *into* $d^\mu \stackrel{\text{def}}{=} (d^0, \mathbf{d}) = (\rho, (\mathbf{j}/c))$, *the charge density/current.* The Lorentz condition $\nabla_{\mathbf{r}} \cdot \mathbf{A} = -\partial A^0/\partial x^0$ reads $(\partial/\partial x^\mu)A^\mu(x^\sigma) = 0$ in terms of the space-time gradient $(\partial/\partial x^\mu)$. Therefore *the four-potential* A^μ *equations of electromagnetic theory are,*

$$\left((\partial/\partial x^\alpha)\eta^{\alpha\chi}(\partial/\partial x^\chi)\right)A^\mu(x^\sigma) = 4\pi d^\mu(x^\sigma) \quad \text{and} \quad (\partial/\partial x^\mu)A^\mu(x^\sigma) = 0. \tag{3.4}$$

If Eq. (3.4) has been solved for $A^\mu = (A^0, \mathbf{A})$, *then the* $\mathbf{E}$ *and* $\mathbf{B}$ *fields can be obtained from* Eq. (3.2).

Applying the space-time gradient $(\partial/\partial x^\mu)$ to both sides of the first equation in Eq. (3.4) and summing over the index μ produces zero on the left side *because of the second equation in* Eq. (3.4), i.e., because of the Lorentz condition. Therefore this procedure *must produce zero on the right side as well,* namely,

$$(\partial/\partial x^\mu)d^\mu(x^\sigma) = 0. \tag{3.5a}$$

Eq. (3.5a) is called the charge density/current "equation of continuity"; *it enforces local charge conservation.*

It is *physically apparent* that any constant-velocity Lorentz transformation $(d^\mu)'(x^\sigma)$ of a charge density/current $d^\mu(x^\sigma)$ *is itself a charge density/current.* As such, any constant-velocity Lorentz transformation $(d^\mu)'(x^\sigma)$ of a charge density/current $d^\mu(x^\sigma)$ *must also satisfy the equation of continuity in order to enforce local charge conservation,* i.e.,

$$(\partial/\partial x^\mu)\,(d^\mu)'(x^\sigma) = 0. \tag{3.5b}$$

It would furthermore be expected of a Lorentz transformation $(d^\mu)'$ of a charge density/current d^μ that $(d^\mu)'$ is a homogeneous linear transformation of the components of d^μ *evaluated at the Lorentz-transformed coordinates,* i.e.,

$$(d^\mu)'(x^\sigma) = \Omega^{\mu\nu} d^\nu((x^\sigma)'), \text{ where } (x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau. \tag{3.5c}$$

Putting Eq. (3.5c) into Eq. (3.5b) produces,

$$(\partial/\partial x^\mu)(\Omega^{\mu\nu} d^\nu((x^\sigma)')) = 0, \text{ where } (x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau. \tag{3.5d}$$

The question now is, *which matrix* $\Omega^{\mu\nu}$ *ensures that* Eq. (3.5d) *holds,* given that the charge density/current $d^\mu(x^\sigma)$ is such that Eq. (3.5a) holds? A radical shortcut to answering this question in fact exists, namely the systematic replacement of all occurrences of the independent variable x^σ in Eq. (3.5a) by its Lorentz-transformed counterpart $(x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau$, which changes Eq. (3.5a) to,

$$(\partial/\partial(x^\nu)')d^\nu((x^\sigma)') = 0, \text{ where } (x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau. \tag{3.5e}$$

Applying Eq. (2.11c), we replace the differential operator $(\partial/\partial(x^\nu)')$ in Eq. (3.5e) by $\Lambda^{\nu\mu}(\beta, -\hat{\mathbf{u}})(\partial/\partial x^\mu)$ and then reverse the order of the first two factors to obtain,

$$(\partial/\partial x^\mu)(\Lambda^{\nu\mu}(\beta, -\hat{\mathbf{u}})\, d^\nu((x^\sigma)')) = 0, \text{ where } (x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau. \tag{3.5f}$$

Comparison of Eq. (3.5d) to the result obtained in Eq. (3.5f) shows that,

$$\Omega^{\mu\nu} = \Lambda^{\nu\mu}(\beta, -\hat{\mathbf{u}}) = \Lambda^{\mu\nu}(\beta, -\hat{\mathbf{u}}), \tag{3.5g}$$

where $\Lambda^{\nu\mu}(\beta, -\hat{\mathbf{u}})$ is, of course, *symmetric* in its two indices (see Eq. (2.9c)).

We now insert the Eq. (3.5g) result for $\Omega^{\mu\nu}$ into Eq. (3.5c) to obtain the Lorentz transformation $(d^\mu)'(x^\sigma)$ of the charge density/current $d^\mu(x^\sigma)$,

$$(d^\mu)'(x^\sigma) = \Lambda^{\mu\nu}(\beta, -\hat{\mathbf{u}})\, d^\nu((x^\sigma)'), \text{ where } (x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau. \tag{3.6}$$

In Eq. (3.4) the four-potential $A^\mu(x^\sigma)$ is linked to its charge density/current source $d^\mu(x^\sigma)$ *by only the d'Alembertian operator* $((\partial/\partial x^\alpha)\eta^{\alpha\chi}(\partial/\partial x^\chi))$, which is *Lorentz-transformation invariant* (see Eq. (2.12b)). Therefore the Lorentz-transformation characteristics of $A^\mu(x^\sigma)$ *are identical to those of* $d^\mu(x^\sigma)$,

$$(A^\mu)'(x^\sigma) = \Lambda^{\mu\nu}(\beta, -\hat{\mathbf{u}})A^\nu((x^\sigma)'), \text{ where } (x^\sigma)' = \Lambda^{\sigma\tau}(\beta, \hat{\mathbf{u}})x^\tau. \tag{3.7}$$

A *consequence* of Eq. (3.7) is that $(A^\mu)'$ *satisfies the Lorentz condition, i.e.,* $(\partial/\partial x^\mu)\,(A^\mu)'(x^\sigma) = 0$, just as $(d^\mu)'$ *satisfies the equation of continuity,* $(\partial/\partial x^\mu)\,(d^\mu)'(x^\sigma) = 0$. Therefore the Eq. (3.4) electromagnetic four-potential equations *are form-invariant when* A^μ *and* d^μ *are Lorentz-transformed to* $(A^\mu)'$ *and* $(d^\mu)'$.

We next write the Eq. (3.7) Lorentz transformation of $A^\mu(x^\sigma) = (A^0(x^0, \mathbf{r}), \mathbf{A}(x^0, \mathbf{r}))$ in the A^0 and $\mathbf{A}$ form *that is analogous to the* x^0 *and* $\mathbf{r}$ *form of the Lorentz transformation of* x^μ *presented in* Eq. (2.4c),

$$(A^0)'(x^0, \mathbf{r}) = \gamma(A^0((x^0)', \mathbf{r}') + \beta(\mathbf{A}((x^0)', \mathbf{r}') \cdot \hat{\mathbf{u}})) \ \text{ and}$$
$$\mathbf{A}'(x^0, \mathbf{r}) = \mathbf{A}((x^0)', \mathbf{r}') + (\gamma - 1)(\mathbf{A}((x^0)', \mathbf{r}') \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} + \gamma\beta A^0((x^0)', \mathbf{r}')\hat{\mathbf{u}}, \text{ where}$$
$$(x^0)' = \gamma(x^0 - \beta(\mathbf{r} \cdot \hat{\mathbf{u}})) \ \text{ and } \ \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r} \cdot \hat{\mathbf{u}})\hat{\mathbf{u}} - \gamma\beta x^0\hat{\mathbf{u}}; \quad \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|, \gamma \stackrel{\text{def}}{=} 1\big/\sqrt{1 - \beta^2}. \tag{3.8}$$

In this tutorial we will Lorentz-transform *only static scalar or vector potentials produced by at-rest time-independent point sources,* so there will be *no dependence on the variable* $(x^0)'$, which we therefore drop. We will also Lorentz-transform these static potentials *to only low speeds, i.e.,* $\beta \ll 1$, so we drop effects of order β^2 or higher, which entails setting $\gamma = 1\big/\sqrt{1 - \beta^2}$ to unity. For *these special cases,* Eq. (3.8) becomes,

$$(A^0)'(x^0, \mathbf{r}) = A^0(\mathbf{r}') + \beta(\mathbf{A}(\mathbf{r}') \cdot \hat{\mathbf{u}}) + \mathrm{O}(\beta^2) \ \text{ and } \ \mathbf{A}'(x^0, \mathbf{r}) = \mathbf{A}(\mathbf{r}') + \beta A^0(\mathbf{r}')\hat{\mathbf{u}} + \mathrm{O}(\beta^2), \text{ where}$$
$$\mathbf{r}' = \mathbf{r} - \beta x^0\hat{\mathbf{u}} + \mathrm{O}(\beta^2) = \mathbf{r} - \mathbf{v}t + \mathrm{O}(\beta^2); \quad \beta \stackrel{\text{def}}{=} |\mathbf{v}/c|. \tag{3.9}$$

Rewritten in terms of the standard variables $(\mathbf{v}/c) = \beta\hat{\mathbf{u}}$ and $t = (x^0/c)$, Eq. (3.9) becomes,

$$(A^0)'(\mathbf{r}, t) = A^0(\mathbf{r} - \mathbf{v}t) + ((\mathbf{v}/c) \cdot \mathbf{A}(\mathbf{r} - \mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) \ \text{ and}$$
$$\mathbf{A}'(\mathbf{r}, t) = \mathbf{A}(\mathbf{r} - \mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r} - \mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.10}$$

The Eq. (3.10) $|\mathbf{v}/c| \ll 1$ Lorentz-transformed static vector and scalar potentials *also yield the corresponding* $|\mathbf{v}/c| \ll 1$ *Lorentz-transformed static magnetic and electric fields by applying* Eq. (3.2). Thus *the* $|\mathbf{v}/c| \ll 1$ *Lorentz-transformed static magnetic field* $\mathbf{B}'(\mathbf{r}, t) = \nabla_\mathbf{r} \times \mathbf{A}'(\mathbf{r}, t)$, where $\mathbf{A}'(\mathbf{r}, t)$ is given by Eq. (3.10),

$$\mathbf{B}'(\mathbf{r}, t) = \nabla_\mathbf{r} \times \mathbf{A}'(\mathbf{r}, t) = \nabla_\mathbf{r} \times \big(\mathbf{A}(\mathbf{r} - \mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r} - \mathbf{v}t)\big) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{B}(\mathbf{r} - \mathbf{v}t) - (\mathbf{v}/c) \times (\nabla_\mathbf{r} A^0(\mathbf{r} - \mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) = \mathbf{B}(\mathbf{r} - \mathbf{v}t) + (\mathbf{v}/c) \times \mathbf{E}(\mathbf{r} - \mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.11a}$$

Likewise, applying Eq. (3.2) yields that *the* $|\mathbf{v}/c| \ll 1$ *Lorentz-transformed static electric field* $\mathbf{E}'(\mathbf{r}, t) = -\nabla_\mathbf{r}(A^0)'(\mathbf{r}, t) - (1/c)(\partial/\partial t)\mathbf{A}'(\mathbf{r}, t)$, where $(A^0)'(\mathbf{r}, t)$ and $\mathbf{A}'(\mathbf{r}, t)$ are given by Eq. (3.10),

$$\mathbf{E}'(\mathbf{r}, t) = -\nabla_\mathbf{r}(A^0)'(\mathbf{r}, t) - (1/c)(\partial/\partial t)\mathbf{A}'(\mathbf{r}, t) =$$
$$-\nabla_\mathbf{r}\big(A^0(\mathbf{r} - \mathbf{v}t) + ((\mathbf{v}/c) \cdot \mathbf{A}(\mathbf{r} - \mathbf{v}t))\big) - (1/c)(\partial/\partial t)\big(\mathbf{A}(\mathbf{r} - \mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r} - \mathbf{v}t)\big) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{E}(\mathbf{r} - \mathbf{v}t) - \nabla_\mathbf{r}\big((\mathbf{v}/c) \cdot \mathbf{A}(\mathbf{r} - \mathbf{v}t)\big) + ((\mathbf{v}/c) \cdot \nabla_\mathbf{r})\big(\mathbf{A}(\mathbf{r} - \mathbf{v}t) + (\mathbf{v}/c)A^0(\mathbf{r} - \mathbf{v}t)\big) + \mathrm{O}(|\mathbf{v}/c|^2) =$$
$$\mathbf{E}(\mathbf{r} - \mathbf{v}t) - (\mathbf{v}/c) \times (\nabla_\mathbf{r} \times \mathbf{A}(\mathbf{r} - \mathbf{v}t)) + \mathrm{O}(|\mathbf{v}/c|^2) = \mathbf{E}(\mathbf{r} - \mathbf{v}t) - (\mathbf{v}/c) \times \mathbf{B}(\mathbf{r} - \mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2), \tag{3.11b}$$

where the term $((\mathbf{v}/c) \cdot \nabla_\mathbf{r})\big((\mathbf{v}/c)A^0(\mathbf{r} - \mathbf{v}t)\big)$, which is of order $|\mathbf{v}/c|^2$, was dropped in the next-to-last step. In précis, Eqs. (3.11) show that the $|\mathbf{v}/c| \ll 1$ Lorentz-transformed static magnetic and electric fields are,

$$\mathbf{B}'(\mathbf{r}, t) = \mathbf{B}(\mathbf{r} - \mathbf{v}t) + (\mathbf{v}/c) \times \mathbf{E}(\mathbf{r} - \mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2),$$
$$\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}(\mathbf{r} - \mathbf{v}t) - (\mathbf{v}/c) \times \mathbf{B}(\mathbf{r} - \mathbf{v}t) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.12}$$

For the at-rest point charge of charge q, $\mathbf{B}(\mathbf{r}) = \mathbf{0}$ and $\mathbf{E}(\mathbf{r}) = q(\mathbf{r}/|\mathbf{r}|^3)$. So when the point charge has low-speed constant velocity $\mathbf{v}$, $|\mathbf{v}/c| \ll 1$, Eq. (3.12) implies that,

$$\mathbf{B}'(\mathbf{r}, t) = q(((\mathbf{v}/c) \times \mathbf{r})/|\mathbf{r} - \mathbf{v}t|^3) + \mathrm{O}(|\mathbf{v}/c|^2),$$
$$\mathbf{E}'(\mathbf{r}, t) = q((\mathbf{r} - \mathbf{v}t)/|\mathbf{r} - \mathbf{v}t|^3) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.13}$$

The Eq. (3.13) magnetic field $\mathbf{B}'(\mathbf{r}, t)$, which vanishes when $\mathbf{v} = \mathbf{0}$ (in contradiction to Newtonian precepts), is azimuthal, so arranging the trajectory of the point charge to run along a magnetic north-south line should maximize the deflection of the needle of a magnetic compass placed immediately above that point charge's trajectory. The maximum magnitude of the Eq. (3.13) magnetic field at the location of the compass is $(|q||\mathbf{v}/c|/d_\perp^2)$, where $d_\perp$ is the perpendicular distance from the compass to the point charge's trajectory.

For the at-rest point magnetic dipole of dipole moment $\mathbf{m}$, $\mathbf{E}(\mathbf{r}) = \mathbf{0}$ and $\mathbf{B}(\mathbf{r}) = ((3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}|\mathbf{r}|^2)/|\mathbf{r}|^5)$. So when the point magnetic dipole has low-speed constant velocity $\mathbf{v}$, $|\mathbf{v}/c| \ll 1$, Eq. (3.12) implies that,

$$\mathbf{E}'(\mathbf{r}, t) = ((-3((\mathbf{v}/c) \times \mathbf{r})((\mathbf{r} - \mathbf{v}t) \cdot \mathbf{m}) + ((\mathbf{v}/c) \times \mathbf{m})|\mathbf{r} - \mathbf{v}t|^2)/|\mathbf{r} - \mathbf{v}t|^5) + \mathrm{O}(|\mathbf{v}/c|^2),$$
$$\mathbf{B}'(\mathbf{r}, t) = ((3(\mathbf{r} - \mathbf{v}t)((\mathbf{r} - \mathbf{v}t) \cdot \mathbf{m}) - \mathbf{m}|\mathbf{r} - \mathbf{v}t|^2)/|\mathbf{r} - \mathbf{v}t|^5) + \mathrm{O}(|\mathbf{v}/c|^2). \tag{3.14}$$

The Eq. (3.14) electric field $\mathbf{E}'(\mathbf{r}, t)$, which vanishes when $\mathbf{v} = \mathbf{0}$ (in contradiction to Newtonian precepts), is azimuthal *when* $\mathbf{v}$ *is parallel to* $\mathbf{m}$. *In that case the* Eq. (3.14) $\mathbf{E}'(\mathbf{r}, t)$ *is,*

$$\mathbf{E}'_{\mathbf{v}\|\mathbf{m}}(\mathbf{r}, t) = ((-3((\mathbf{v}/c) \times \mathbf{r})((\mathbf{r} - \mathbf{v}t) \cdot \mathbf{m}))/|\mathbf{r} - \mathbf{v}t|^5) + \mathrm{O}(|\mathbf{v}/c|^2), \text{ where } \mathbf{m} = (|\mathbf{m}|\mathbf{v}/|\mathbf{v}|). \tag{3.15}$$

The Eq. (3.15) azimuthal electric field $\mathbf{E}'_{\mathbf{v}\|\mathbf{m}}(\mathbf{r}, t)$ can transiently propel the invisible microscopic free electrons in a metal wire coil through whose center the dipole passes (Faraday), or it can transiently deflect a macroscopic charged object which hangs by a thread immediately above the dipole's horizontal trajectory. Because of the factor $((\mathbf{r}-\mathbf{v}t)\cdot\mathbf{m})$ in Eq. (3.15), where $\mathbf{m} = (|\mathbf{m}|\mathbf{v}/|\mathbf{v}|)$, the Eq. (3.15) azimuthal electric field at $\mathbf{r}$ *vanishes* at time $t_0(\mathbf{r}) = ((\mathbf{r}\cdot\mathbf{v})/|\mathbf{v}|^2)$, and its *direction* thereafter *is reversed.* The *maximum magnitude at* $\mathbf{r}$ of the Eq. (3.15) azimuthal electric field *occurs twice,* at *the two times* $t_\mp(\mathbf{r}) = t_0(\mathbf{r}) \mp (d_\perp(\mathbf{r})/(2|\mathbf{v}|))$, where $d_\perp(\mathbf{r}) \stackrel{\text{def}}{=} |\mathbf{r} - (\mathbf{v}(\mathbf{v}\cdot\mathbf{r})/|\mathbf{v}|^2)|$ is the perpendicular distance from $\mathbf{r}$ to the point magnetic dipole's trajectory. At its *second* maximum magnitude at $\mathbf{r}$, the Eq. (3.15) azimuthal electric field's *direction is reversed.* The *value* of the Eq. (3.15) azimuthal electric field's *two equal maximum magnitudes at* $\mathbf{r}$, which have *opposite field directions* and time separation $\Delta t(\mathbf{r}) = (d_\perp(\mathbf{r})/|\mathbf{v}|)$, is $(.85865|\mathbf{m}||\mathbf{v}/c|/(d_\perp(\mathbf{r}))^3)$.