# Elementary identities and Pi

Edgar Valdebenito

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abstract

In this note we are going to look at quite a few integrals involving hyperbolic functions

The number Pi is defined by

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

The number $\pi^2$ is defined by

$$\pi^2 = 6\sum_{n=1}^{\infty}\frac{1}{n^2}$$

In this note we give some formulas related to $\pi^2 = 4\int_0^{\infty}\sinh^{-1}(\operatorname{csch} x)\,dx$ .

Keywords: Integrals,hyperbolic functions,series,number Pi.

## Identities

Entry 1.

$$\frac{\pi^2}{4} = \int_0^{\infty}\sinh^{-1}(\operatorname{csch} x)\,dx = \int_0^{\infty}\cosh^{-1}(\coth x)\,dx = \int_0^{\infty}\tanh^{-1}(\operatorname{sech} x)\,dx$$

Entry 2. For $u > 0$ , we have

$$\frac{\pi^2}{4} = u\sinh^{-1}(\operatorname{csch} u) + \int_{\sinh^{-1}(\operatorname{csch} u)}^{\infty}\sinh^{-1}(\operatorname{csch} x)\,dx + \int_u^{\infty}\sinh^{-1}(\operatorname{csch} x)\,dx$$

$$\frac{\pi^2}{4} + u\sinh^{-1}(\operatorname{csch} u) = \int_0^{u}\sinh^{-1}(\operatorname{csch} x)\,dx + \int_0^{\sinh^{-1}(\operatorname{csch} u)}\sinh^{-1}(\operatorname{csch} x)\,dx$$

Entry 3. For $u > 0$ , we have

$$\frac{\pi^2}{4} = u\sinh^{-1}(\operatorname{csch} u) + \int_0^{u}\frac{x}{\sinh x}\,dx + \int_u^{\infty}\sinh^{-1}(\operatorname{csch} x)\,dx$$

$$\frac{\pi^2}{4} = u\sinh^{-1}(\operatorname{csch} u) + \int_0^{u}\frac{x}{\sinh x}\,dx + \int_0^{\sinh^{-1}(\operatorname{csch} u)}\frac{x}{\sinh x}\,dx$$

Entry 4. For $u > 0$ , we have

$$\frac{\pi^2}{4} + u\sinh^{-1}(\operatorname{csch} u) = \int_0^u \sinh^{-1}(\operatorname{csch} x)\,dx + \int_u^\infty \frac{x}{\sinh x}\,dx$$

$$\frac{\pi^2}{4} + u\sinh^{-1}(\operatorname{csch} u) = \int_u^\infty \frac{x}{\sinh x}\,dx + \int_{\sinh^{-1}(\operatorname{csch} u)}^\infty \frac{x}{\sinh x}\,dx$$

Entry 5. For $u > 0$ , we have

$$\pi^2 = 4u + 4(1+u)\sinh^{-1}(\operatorname{csch} u) - 8\sum_{n=1}^{\infty}(-1)^{n-1}\Big(u + \sinh^{-1}(\operatorname{csch} u) - n\pi\tan^{-1}\left(\frac{u}{n\pi}\right) - n\pi\tan^{-1}\left(\frac{1}{n\pi}\sinh^{-1}(\operatorname{csch} u)\right)\Big)$$

Entry 6. For $\sinh^{-1}(\operatorname{csch}\pi) < u < \pi$ , we have

$$\pi^2 = 4u + 4(1+u)\sinh^{-1}(\operatorname{csch} u) + 8\sum_{n=1}^{\infty}\frac{(-1)^n(2^{2n-1}-1)B_n}{(2n+1)!}\left(u^{2n+1} + (\sinh^{-1}(\operatorname{csch} u))^{2n+1}\right)$$

where $B_n$ are the Bernoulli numbers:

$$B_n = \left\{\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \dots\right\}$$

Entry 7.

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 8\ln\left(1+\sqrt{2}\right) + 16\sum_{n=1}^{\infty}\frac{(-1)^n(2^{2n-1}-1)B_n}{(2n+1)!}\left(\ln\left(1+\sqrt{2}\right)\right)^{2n+1}$$

Remark: $B_n$ are the Bernoulli numbers.

Entry 8. For $u \geq 0$ , we have

$$\pi^2 = 4\int_0^u \sinh^{-1}(\operatorname{csch} x)\,dx + 8\sum_{n=0}^{\infty}\frac{e^{-(2n+1)u}}{(2n+1)^2}$$

Entry 9. For $0 < u < \pi$ , we have

$$\pi^2 = 4u + 4u\ln 2 - 4u\ln u + 8\sum_{n=0}^{\infty}\frac{e^{-(2n+1)u}}{(2n+1)^2} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{n(2n+1)!}u^{2n+1}$$

Remark: $B_n$ are the Bernoulli numbers.

Entry 10. For $u > 0$ , we have

$$\pi^2 = 8\sum_{n=0}^{\infty}\frac{e^{-(2n+1)u}}{(2n+1)^2} + 8\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(\frac{\sinh u}{1+\cosh u}\right)^{2n+1} + 8\ln\left(\frac{1+\cosh u}{\sinh u}\right)\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{\sinh u}{1+\cosh u}\right)^{2n+1}$$

Remark: $\frac{\sinh u}{1+\cosh u} = \tanh\frac{u}{2}$.

Entry 11.

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 8\ln\left(1+\sqrt{2}\right) + 4\sqrt{2}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-2n}}{(2n+1)^2}\ F\left(\frac{1}{2},1,n+\frac{3}{2},\frac{1}{2}\right)$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 12. For $u > 0$ , we have

$$\frac{\pi^2}{4} = u\cosh^{-1}(\coth u) + \int_u^{\infty}\cosh^{-1}(\coth x)\,dx + \int_{\cosh^{-1}(\coth u)}^{\infty}\tanh^{-1}(\operatorname{sech} x)\,dx$$

$$\frac{\pi^2}{4} + u\cosh^{-1}(\coth u) = \int_0^{u}\cosh^{-1}(\coth x)\,dx + \int_0^{\cosh^{-1}(\coth u)}\tanh^{-1}(\operatorname{sech} x)\,dx$$

Entry 13.

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 8\int_{\ln(1+\sqrt{2})}^{\infty}\cosh^{-1}(\coth x)\,dx$$

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 8\int_{\ln(1+\sqrt{2})}^{\infty}\tanh^{-1}(\operatorname{sech} x)\,dx$$

Entry 14. For $u > 0$ , we have

$$\frac{\pi^2}{4} = \int_0^{u}\tanh^{-1}(\operatorname{sech} x)\,dx + \sum_{n=0}^{\infty}\frac{(\operatorname{sech} u)^{2n+1}}{(2n+1)^2}\ F\left(\frac{1}{2},n+\frac{1}{2},n+\frac{3}{2},(\operatorname{sech} u)^2\right)$$

$$\frac{\pi^2}{4} = \int_0^{u}\tanh^{-1}(\operatorname{sech} x)\,dx + \sum_{n=0}^{\infty}\frac{(2e^{-u})^{2n+1}}{(2n+1)^2}\ F\left(2n+1,n+\frac{1}{2},n+\frac{3}{2},-e^{-2u}\right)$$

$$\frac{\pi^2}{4} = \int_0^{u}\tanh^{-1}(\operatorname{sech} x)\,dx + 2\sum_{n=0}^{\infty}\frac{e^{-(2n+1)u}}{(2n+1)^2}$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 15. For $u > \ln 3$ , we have

$$\frac{\pi^2}{4} = \int_0^u \sinh^{-1}(\operatorname{csch} x)\, dx + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2e^{-u})^n}{n^2}\, F(n,n,n+1,e^{-u})$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 16. For $u > 0$ , we have

$$\frac{\pi^2}{4} = \int_0^u \sinh^{-1}(\operatorname{csch} x)\, dx + \sum_{n=1}^{\infty} \frac{(2e^{-u})^n}{n^2}\, F(n,n,n+1,-e^{-u})$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 17.

$$\pi^2 = \int_0^{\infty} \sinh^{-1}\left(\frac{1}{2}\left(\left(\coth\frac{x}{2}\right)^4 - \left(\tanh\frac{x}{2}\right)^4\right)\right) dx$$

$$\pi^2 = \int_0^{\infty} \tanh^{-1}\left(\frac{4\operatorname{sech} x\,(1+(\operatorname{sech} x)^2)}{1+6(\operatorname{sech} x)^2+(\operatorname{sech} x)^4}\right) dx$$

Entry 18. For $0 < u < \sqrt{2}-1$ , we have

$$\frac{\pi^2}{4} = 2\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n u^{2n+1}}{(2n+1)^2}\, F\left(2n+1, n+\frac{1}{2}, n+\frac{3}{2}, u^2\right) + \int_u^1 \frac{1}{x}\sinh^{-1}\left(\frac{2x}{1-x^2}\right) dx$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 19. For $u > 0$ , we have

$$\frac{\pi^2}{4} = u\sinh^{-1}(\operatorname{csch} u) + 2\int_{\sinh^{-1}(\operatorname{csch} u)}^{\infty} \sinh^{-1}(\operatorname{csch} x)\, dx + \int_u^{\sinh^{-1}(\operatorname{csch} u)} \sinh^{-1}(\operatorname{csch} x)\, dx$$

Entry 20.

$$\frac{\pi^2}{4} = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n+1)!} + \sum_{n=1}^{\infty} n\, \ln\left(\frac{\tanh\frac{n+1}{2}}{\tanh\frac{n}{2}}\right) + 2\sum_{n=0}^{\infty} \frac{e^{-2n-1}\left(1-e^{-2n-1}(2n+2)\right)}{(2n+1)^2(1-e^{-2n-1})}$$

Remark: $B_n$ are the Bernoulli numbers.

Entry 21.

$$\frac{\pi^2}{4} = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2^{2n-1}-1)B_n}{(2n+1)!} + 4\sum_{n=0}^{\infty} \frac{(n+1)e^{-2n-1}}{(2n+1)^2}$$

Remark: $B_n$ are the Bernoulli numbers.

Entry 22. For $u > 0$ , we have

$$\frac{\pi^2}{4} = -\int_0^u \ln\tanh\left(\frac{x}{2}\right) dx + \sum_{n=0}^{\infty} \frac{2^{-n}}{n+1}\left(1-\tanh\frac{u}{2}\right)^{n+1} \sum_{k=0}^{n} \frac{2^k}{k+1}$$

Entry 23. For $u > 0$ , we have

$$\frac{\pi^2}{4} = -\int_0^u \ln\tanh\left(\frac{x}{2}\right) dx + \sum_{n=1}^{\infty} \frac{1}{n^2}\left(1-\tanh\frac{u}{2}\right)^n F\left(1, n, n+1, \frac{1}{2} - \frac{1}{2}\tanh\frac{u}{2}\right)$$

Remark: $F(a, b, c, x)$ is the Gauss hypergeometric function.

Entry 24. For $u > 0$ , we have

$$\frac{\pi^2}{4} = -\int_0^u \ln\tanh\left(\frac{x}{2}\right) dx + \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\ln\left(\frac{2}{1+\tanh\frac{u}{2}}\right) - \sum_{k=1}^{n-1}\binom{n-1}{k}\frac{(-1)^{k-1}}{k}\left(1-\left(\frac{1}{2}+\frac{1}{2}\tanh\frac{u}{2}\right)^k\right)\right)$$

Entry 25.

$$\frac{\pi}{2(e-1)} = 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{e^{2n+1}-1} + \frac{1}{e^{2n+3}-1}\right)\sum_{k=n+1}^{\infty}\frac{(-1)^k}{2k+1} + \sum_{n=1}^{\infty} n\ln\left(\frac{\tanh\frac{n+1}{2}}{\tanh\frac{n}{2}}\right)$$

Entry 26. For $s = 2,3,4, \dots$ , we have

$$\frac{\pi^2}{4} = -2\int_0^s \ln\tanh(x)\, dx - 2\sum_{n=s}^{\infty}\ln\tanh n + 2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\, c_k\left((\tanh(n-k+s))^k - (\tanh(n-k+s))^{-k}\right)$$

where

$$c_{2k}=1-\sum_{m=0}^{k-1}\frac{(\tanh 1)^{2m+1}}{2m+1}\quad ,k=1,2,3,\ldots$$

$$c_{2k-1}=\ln\cosh 1-\sum_{m=1}^{k-1}\frac{(\tanh 1)^{2m}}{2m}\quad ,k=1,2,3,\ldots$$

for $s=2$

$$\frac{\pi^2}{4}=-2\int_0^2\ln\tanh(x)\,dx-2\sum_{n=2}^{\infty}\ln\tanh n$$
$$+2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\;c_k((\tanh(n-k+2))^k-(\tanh(n-k+2))^{-k})$$

Entry 27.

$$\frac{\pi^2}{4}=-2\int_0^1\ln\tanh x\,dx+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{k^2}\left(\frac{1-(\tanh(n-k+1))^k}{(1+\tanh(n-k+1))^k}\right)c_k$$

where

$$c_k=F\left(1,k,k+1,\frac{1}{2}\right)-(1-\tanh 1)^kF\left(1,k,k+1,\frac{1-\tanh 1}{2}\right)$$

Remark: $F(a,b,c,x)$ is the Gauss hypergeometric function.

Entry 28.

$$\frac{\pi}{2(e^2-1)}=-2\sum_{n=1}^{\infty}\ln\tanh n+2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{e^{4n+2}-1}+\frac{1}{e^{4n+6}-1}\right)\sum_{k=n+1}^{\infty}\frac{(-1)^k}{2k+1}$$

$$\frac{\pi}{2(e^2-1)}=-2\sum_{n=1}^{\infty}\ln\tanh n-2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+3}\sum_{k=0}^{n}(-1)^k\left(\frac{1}{e^{4k+2}-1}+\frac{1}{e^{4k+6}-1}\right)$$

Entry 29.

$$\frac{\pi^2}{8}=u^2+\int_u^{\ln 2}\cosh^{-1}\left(\frac{1}{e^x-1}\right)dx+\int_u^{\infty}\ln(1+\operatorname{sech}x)\,dx$$

$$\frac{\pi^2}{8}+u^2=\int_0^{u}\cosh^{-1}\left(\frac{1}{e^x-1}\right)dx+\int_0^{u}\ln(1+\operatorname{sech}x)\,dx$$

where

$$u = \ln\left(\frac{1}{3}\left(1 + \sqrt[3]{19 + 9\sqrt{\frac{11}{3}}} + \sqrt[3]{19 - 9\sqrt{\frac{11}{3}}}\right)\right)$$

Entry 30. For $u \geq 0$ , we have

$$\frac{\pi^2}{8} = u\ln(1 + \operatorname{sech} u) + \int_{\ln(1+\operatorname{sech} u)}^{\ln 2} \cosh^{-1}\left(\frac{1}{e^x - 1}\right) dx + \int_u^\infty \ln(1 + \operatorname{sech} x)\, dx$$

$$\frac{\pi^2}{8} + u\ln(1 + \operatorname{sech} u) = \int_0^{\ln(1+\operatorname{sech} u)} \cosh^{-1}\left(\frac{1}{e^x - 1}\right) dx + \int_0^u \ln(1 + \operatorname{sech} x)\, dx$$

Entry 31. For $u > 0$ , we have

$$\frac{\pi^2}{8} = u\ln 2 - \sqrt{2(1 - \operatorname{sech} u)} \sum_{n=0}^{\infty} \frac{(1 - \operatorname{sech} u)^{n+1}}{2n + 3} \sum_{k=0}^{n} \frac{2^{-k-1}}{k + 1} \sum_{m=0}^{n-k} \binom{2m}{m} 2^{-3m}$$
$$+ \sum_{n=0}^{\infty} \frac{(-1)^n (\operatorname{sech} u)^{n+1}}{n + 1} \sum_{k=0}^{[n/2]} \binom{2k}{k} \frac{2^{-2k}}{n - 2k + 1}$$

## Endnote

Entry 31.

$$\frac{\pi^2}{4} = \int_0^\infty \sinh^{-1}\left(\frac{1}{\sqrt{e^x - 1}}\right) dx$$

$$\frac{\pi^2}{8} = \int_0^{\ln 2} \cosh^{-1}\left(\frac{1}{e^x - 1}\right) dx$$

$$\frac{\pi^2}{8} = \int_0^\infty \ln(1 + \operatorname{sech} x)\, dx$$

$$\frac{\pi^2}{8} = \int_0^\infty \sinh^{-1}\left(\sqrt{\frac{\coth x - 1}{2}}\right) dx$$

$$\frac{\pi^2}{8} = \int_0^{\ln 2} \tanh^{-1}\sqrt{2e^x - e^{2x}}\; dx$$

$$\pi^2 = 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 + 8\int_{\ln\left(1+\sqrt{2}\right)}^{\infty} \sinh^{-1}(\operatorname{csch} x)\, dx$$

$$\pi^2 + 4\left(\ln\left(1+\sqrt{2}\right)\right)^2 = 8\int_{0}^{\ln\left(1+\sqrt{2}\right)} \sinh^{-1}(\operatorname{csch} x)\, dx$$

## References


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