Verification of the Collatz Conjecture by Methods of Sequence Theory

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Summary

A solution is proposed for the so-called Collatz hypothesis, it is also a "problem $3n + 1$". The idea of the proof is to represent the operation of the algorithm as an infinite sequence of numbers, for which hypotheses about convergence or divergence are considered, corresponding to hypotheses about the output values of the algorithm.

In a separate section, other possibilities for constructing a proof are presented.

Keywords

problem $3n + 1$; the Collatz hypothesis; sequence convergence

Introduction

There is a well-known problem in the mathematical community for quite a long time, commonly called the "Collatz conjecture" or "$3n + 1$ problem". The problem statement is attributed to Lothar Collatz; it was not published by him, since he could not provide its solution, however, it was voiced by him many times [1]. A full review of the problem and related problems, including existing approaches to analysis and solution, can be found in the works of D. Lagarias [1-3].

This conjecture has become widely known in mathematical circles due to the fact that, despite its elementary formulation, it is still considered unsolved [4]. This circumstance is also of interest to a wider audience, including through the popularization of mathematics on specialized YouTube channels [5].

In this paper, we want to propose an approach to proving this conjecture. It is quite obvious that the problem itself is not trivial, and therefore verification of the proof is also a task, including for the author. Thus, we point out that the work *can only* contain a proof of the conjecture.

Actually, let's formulate the problem itself: the well-known algorithm, presented here as a recursive function $f(n)$, is applied to any natural number n, while the process will come to the unit, regardless of which natural number was originally chosen; if the number is odd, then a transformation is performed on it $3n + 1$, and if it is even, then $n/2$:

$$f(n) = \begin{cases} 3n + 1 \text{ if } n \equiv 1 \pmod 2, \\ n/2 \text{ if } n \equiv 0 \pmod 2. \end{cases}$$

Discussion

1. Algorithm presentation form

The purpose of this section is to formulate the area of interpretation in the context of which subsequent logical statements will be evaluated.

We denote the operations of the algorithm as follows:

$$n_1 \xrightarrow{3n+1} 3n_1 + 1,$$

$$n_2 \xrightarrow{\frac{1}{2}n} \frac{1}{2}n_2; \qquad (1.1)$$

where n is a natural number at any iteration of the algorithm, $n \in \mathbb{N}[1;\infty)$; n_1 – the same, only it is noted that the number is odd; n_2- an even number. It is quite obvious that the number expressed as $3n_1 + 1$ is even, while it $\frac{1}{2}n_2$can be either even or odd.

Any initial even number, based on the properties of the algorithm, will be reduced to the corresponding odd number, so the algorithm can only be considered in the context of odd numbers.

Definition 1.1

Any initial odd natural number can be expressed by the following formula:

$$n_0 = a2^{m_0}; \tag{1.2}$$

where a is a rational positive number, $a \in \mathbb{Q}\left(\frac{1}{2};1\right]$; $m_0$is a natural number corresponding to the proposed formula, $m_0 \in \mathbb{N}[0;\infty)$. For example, $5 = \frac{5}{8}\cdot 2^3$, $27 = \frac{27}{32}\cdot 2^5$, $1 = 1\cdot 2^0$etc.

If the initial number is even, it can be represented as:

$$n_0 = a2^{m_0+k};$$

where $k$is a natural number corresponding to the number of initial operations $\frac{1}{2}n_2$leading to an odd number expressed by the formula $a2^{m_0}$.

To uniquely determine the transformation of a number to the form (1.2), we define the recursive functions $A$and M; the representation of these functions is given in a way different from the classical representation of recursive functions. The initial number is passed as a parameter $n$for the zero iteration, then at some iteration of the $x$desired value is obtained:

$$a = A(n,0),\, A(n,x+1) = \begin{cases} A(n/2\,,x) & \text{if } n > 1, \\ n & \text{if } n \le 1. \end{cases} \tag{1.3}$$

$$m_0 = M(n,0),\, M(n,x+1) = \begin{cases} M(n/2\,,x) & \text{if } n > 1, \\ x & \text{if } n \le 1. \end{cases} \tag{1. 4}$$

For example, $a(5) = A(5{,}0) = A\left(\frac{5}{2},1\right) = A\left(\frac{5}{4},2\right) = A\left(\frac{5}{8},3\right) = \frac{5}{8}$; $m_0(5) = M(5{,}0) = M\left(\frac{5}{2},1\right) = M\left(\frac{5}{4},2\right) = M\left(\frac{5}{8},3\right) = 3$.

Consider the result of applying the operation $3n + 1$ to the original number:

$$a2^{m_0} \xrightarrow{3n+1} 3a2^{m_0} + 1 = \frac{3}{4}a2^{m_0+2} + 1 = 2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{1.5}$$

In the proposed wording, the expression in square brackets does not exceed one: $\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \le 1$, which corresponds to the applied restrictions for the parameters: $a \in \mathbb{Q}\left(\frac{1}{2}; 1\right]$, $m_0 \in \mathbb{N}[0; \infty)$.

After the implementation of operation (1.5), proceeding from the definition of the algorithm, the operation of division follows. We will consider it in the form of the following representation:

$$2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \xrightarrow{\frac{1}{2}n} 2^{m_0+1}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{1.6}$$

For an arbitrary number, we cannot say how many divisions will be required to obtain an odd number, so we will say that the number two in formula (1.6) acquires a certain power m_1. Let's estimate this number: $0 \le m_1 \le m_0 + 1$, – and at an arbitrary step of the algorithm: $0 \le m_{i+1} \le m_i + 1$, by induction.

We will say that the execution of operations (1.5) and subsequent (1.6) until the next odd number is obtained corresponds to the cycle of the algorithm. Thus, the lower index of the parameter m corresponds to the number of implemented cycles of the algorithm.

Definition 1.2

Let us introduce a function c that, when an arbitrary odd number is passed to it, produces an odd number obtained as a result of the implementation of one cycle of the algorithm under study, so that:

$$c(n_0) = 2^{m_1}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{1.7}$$

We will use the generally accepted notation for the corresponding recursive transformations:

$$c^1(n_0) = c(n_0),\ c^{k+1}(n_0) = c\left(c^k(n_0)\right). \tag{1.8}$$

Successive application of a function c to a number we consider as a mathematical sequence. Thus, the successive application k of the cycles of the algorithm under study to the initial number n_0 will give the corresponding sequence of odd numbers $\{c^k(n_0)\}$.

Example 1.1

For the initial number, $n_0 = 19$ we get the following sequence: $c^1(19) = 29$, $c^2(19) = c^1(29) = 11$, $c^3(19) = c^2(29) = c^1(11) = 17$, $c^4(19) = 13$, $c^5(19) = 5$, $c^6(19) = 1$, $c^7(19) = 1$ etc. Thus, starting from the sixth member of the sequence, $\{c^k(19)\}$ all subsequent members are equal to the unit.

Definition 1.3

We introduce a function $C(n_0)$ that outputs the value of the limit of the sequence $\{c^k(n_0)\}$ when the number of cycles of the algorithm tends to infinity, so that:

$$C(n_0) = \lim_{k\to\infty} c^k(n_0). \tag{1.9}$$

To explicitly express the function C, we first consider the sequential application of the cycles of the algorithm to an arbitrary number:

$$c^2(n_0) = 2^{m_2}\left[\left(\frac{3}{4}\right)^2 a + \frac{3}{4}\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}}\right)\right];$$

and then:

$$c^3(n_0) = 2^{m_3}\left[\left(\frac{3}{4}\right)^3 a + \left(\frac{3}{4}\right)^2\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}} + \left(\frac{3}{4}\right)^{-2}\frac{1}{2^{m_2+2}}\right)\right].$$

As a result, we express an arbitrary kth member of the sequence:

$$c^k(n_0) = 2^{m_k}\begin{bmatrix}\left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}} + \cdots\right. \\ \left.\ldots + \left(\frac{3}{4}\right)^{-k+0}\frac{1}{2^{m_{k-2}+2}} + \left(\frac{3}{4}\right)^{-k+1}\frac{1}{2^{m_{k-1}+2}}\right)\end{bmatrix}. \tag{1.10}$$

This formula (1.10) can be written as containing a number series:

$$c^k(n_0) = 2^{m_k}\left[\left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right]. \tag{1.11}$$

Example 1.2

The sixth member of the sequence from example 1.1 using formula (1.10) can be expressed as follows:

$$c^6(19) = 2^3\left[\left(\frac{3}{4}\right)^6\frac{19}{32} + \left(\frac{3}{4}\right)^{6-1}\left(\frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{-2+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-3+1}\frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{-4+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-5+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-6+1}\frac{1}{2^{5+2}}\right)\right] = 1.$$

It follows from this representation that the study of the Collatz algorithm is identical to the study of possible values of the function $C(n_0)$, which is the limit of the sequence:

$$C(n_0) = \lim_{k\to\infty} 2^{m_k}\left[\left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right]. \tag{1.12}$$

The propositional form of the hypothesis in the context of the described model can be expressed as follows: $\forall n_0(C(n_0) = 1)$. Note that the proposed formulation applies only to all odd numbers, meanwhile, we indicated earlier that the proof of this statement implies the truth of the statement for all even numbers according to the scheme $\forall n_0(C(n_0) = 1) \supset \left(C\left(n_0 2^k\right) = 1\right): k \in \mathbb{N}[1; \infty)$.

Remark 1.1

Let us determine the value of the limit of the sequence expressed by formula (1.12) by stating the following theorem on the limit of the sequence: if all members of the sequence are equal to the same number, then the sequence converges to this number [6]. Thus, the statement $C(n_0) = 1$ means that for a sequence defined as $\{c^k(n_0)\}$ there are segments of this sequence, all members of which are equal to one. This argument allows us to consider the result of the Collatz algorithm in terms of the formal definitions of sequence theory.

2. Proof by examining the range of possible values of the function C

Proposition 2.1

Th statement $\forall n_0(C(n_0) = 1)$ that is equivalent to $\neg\exists n_0(C(n_0) \neq 1)$ using the law of the excluded middle, so that we are going to exclude the truth of the statement $\exists n_0(C(n_0) \neq 1)$. The last statement, which is a counterhypothesis, is equivalent to the following subject statements and only to them (recall that the domain of definition and the range of values of a function C are given on the set of natural numbers; odd, to be more precise): (1) one can find such n_0 for which the sequence $\{c^k(n_0)\}$ is bounded, but at the same time the limit does not exist; (2) it is possible to find such n_0 for which the sequence $\{c^k(n_0)\}$ converges to a number different from one, or $\exists n_0(C(n_0) = g)$, $g \in \mathbb{N}[3; \infty)$; (3)

one can find such n_0 for which the sequence $\{c^k(n_0)\}$ is divergent and unbounded, or $\exists n_0(C(n_0) = \infty)$.

The sequence of numbers $\{c^k(n_0)\}$, according to the theorems of sequence theory, can be represented as follows:

$$\{c^k(n_0)\} = \{2^{m_k}\}\left[\left\{\left(\frac{3}{4}\right)^k a\right\} + \left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}\right]. \tag{2.1}$$

Consider the sequence $\{2^{m_k}\}$: by itself, it can converge or diverge, while being both limited and unlimited.

The sequence $\left\{\left(\frac{3}{4}\right)^k a\right\}$ has a unique limit:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^k a = 0. \tag{2.2}$$

The number series $\sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}$ can be both convergent and divergent, depending on what values the parameter takes m_{i-1}; accordingly, nothing concrete can be said about the limit of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}$ either.

Hypotheses 1 and 2

Suppose the sequence $\{2^{m_k}\}$ limited. This assumption satisfies the conditions of hypotheses (1) and (2) and, in fact, is identical to them, because there are no other subject options that satisfy the fulfillment of these hypotheses. In this case, we can evaluate the parameter $m$as follows: $0 \leq m_{i+1} \leq m_i$, which can also be interpreted as m not exceeding some number j: $j \in \mathbb{N}$.

In this case, we get a divergent numerical series, since the necessary condition for this is met:

$$\lim_{i\to\infty} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}} = \lim_{i\to\infty} \left(\frac{4}{3}\right)^{i-1} \left(\frac{1}{2}\right)^{j+2} = \infty. \tag{2.3}$$

To estimate the limit of the sequence, $\left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}$we write the partial sum of the series, according to the formula for the partial sum of the geometric series:

$$s_k = \left(\frac{1}{2}\right)^{j+2} \frac{1-\left({}^4/_3\right)^k}{1-\left({}^4/_3\right)}, \tag{2.4}$$

Then the limit of the considered sequence can be expressed as follows:

$$\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1}s_k=\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{2}\right)^{j+2}\frac{1-\left({}^4\!/_3\right)^k}{1-\left({}^4\!/_3\right)}=\lim_{k\to\infty}\left(\frac{1}{2}\right)^j\left(1-\left({}^3\!/_4\right)^k\right)=\left(\frac{1}{2}\right)^j, \tag{2.5}$$

which means the convergence of the considered sequence.

It turns out, under the conditions of hypotheses 1 and 2, we can consider the value of the function C on the basis of general theorems on the limits of sequences as follows:

$$C(n_0)=\lim_{k\to\infty}2^{m_k}\left[\lim_{k\to\infty}\left(\frac{3}{4}\right)^k a+\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1}s_k\right]=\lim_{k\to\infty}2^{m_k}\left[0+\left(\frac{1}{2}\right)^j\right]=\left(\frac{1}{2}\right)^j\lim_{k\to\infty}2^{m_k}. \tag{2.6}$$

It is quite obvious that for any natural bounded, based on the requirements of hypotheses 1 and 2, values j and m_k the value of the function C is unique and equals to one. But this leads to a contradiction, since conjectures 1 and 2 are stated for cases where the Collatz conjecture is refuted. It follows from this that hypotheses 1 and 2 are not applicable.

Remark 2.1

Let us admit the following objection: the sequence $\{c^k(n_0)\}$ infinitely approximates the limit (2.6), but there are no terms equal to the unit. Such an assumption is not admissible, since the function is defined on natural numbers. The sequences in square brackets, namely $\left\{\left(\frac{3}{4}\right)^k a\right\}$ and $\left\{\left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right\}$, converge and decrease monotonically, despite the fact that the sequence $\{2^{m_k}\}$ is limited, according to the requirements of hypotheses 1 and 2. This mathematical construction describes some finite natural odd number, therefore, to infinitely approximate the limit, while not reaching it, a monotone decreasing sequence defined on the natural numbers cannot.

Hypothesis 3

The condition of hypothesis 3 assumes the existence of such a number, let's call it n_0^∞, the sequence $\{c^k(n_0^\infty)\}$ formed by which is monotonically increasing; likewise, $C(n_0^\infty)=\infty$.

It is quite obvious that such a number, when the operation is applied to it, $c$also forms a number with the same property, so that $c^{k+1}(n_0^\infty)>c^k(n_0^\infty)$, and:

$$c^{k+1}(n_0^\infty)=\frac{3}{2}c^k(n_0^\infty)+\frac{1}{2}, \tag{2.7}$$

since in this case the cycle of the algorithm consists of an operation $3n + 1$ and only one operation $n/2$, which is the condition for the fulfillment of hypothesis 3.

The operation on such a number is uniquely defined, therefore, we can define the inverse to the c operation.

Definition 2.1

Let us express the operation inverse to c in the context of hypothesis 3 explicitly:

$$c^{-1}\left(c(n_0^\infty)\right) = n_0^\infty, \quad (2.8)$$

$$c^{k-1}(n_0^\infty) = \tfrac{2}{3}\left(c^k(n_0^\infty) - \tfrac{1}{2}\right). \quad (2.9)$$

Now we apply the inverse operation to the hypothetical number corresponding to hypothesis 3 and consider the result: $c^{-1}(n_0^\infty)$. If the operation c^{-1} is not applicable to a given number, then it was not obtained by formula (2.7), which entails: $c^{-1}(n_0^\infty) > n_0^\infty$, - but this contradicts the property of the number n_0^∞, since it is assumed that n_0^∞we can take any arbitrarily large number $c^k(n_0^\infty)$ exceeding $c^{-1}(n_0^\infty)$. If the operation c^{-1} is applicable to a number, this means that for any finite number n_0^∞ we can propose a sequence $\{c^{-k}(n_0^\infty)\}$, which must be monotonically decreasing. This sequence, as well as all considered operations on it, are defined on the set of odd numbers. Accordingly, we can define the limit of this sequence:

$$\lim_{k\to\infty} c^{-k}(n_0^\infty) = 1, \quad (2.10)$$

which also leads to a contradiction, since the unit does not have the hypothetical number property of n_0^∞.

Thus, the conditions of hypothesis 3 are not feasible.

All hypotheses that could justify the statement $\exists n_0(C(n_0) \neq 1)$ lead to a contradiction in the context of the interpretation under consideration, therefore, it is false. The statement is confirmed $\forall n_0(C(n_0) = 1)$, which proves the Collatz conjecture. Q. E. D.

3. Proof by examining the properties of numbers obtained by applying the algorithm to an arbitrary number

The approach to the proof of the conjecture presented below is intended to logically supplement the idea of the previous section outside of sequence theory. In particular, the very idea of the proof is similar to the trick realized by formula (2.10).

Definition 3.1

Let there be an expressive function C applied to an object variable n denoting some natural number. Thus, the formula $C(n)$ will mean that the number n fulfills the Collatz algorithm, i.e. sequential implementation of the algorithm for a finite number of iterations leads to the unit. The number n may be odd; in this case we will use the term n_1, or even, n_2, respectively. The term n_2 is defined as follows: $n_2 = n_1 + 1$, i.e. for an arbitrary odd number, one can put in a one-to-one correspondence some even number, moreover, $n_1 = n_2 - 1$.

Let us define some initial statements that are valid in the context of the interpretation under consideration:

$$\forall n_1 \Big(\big(C(n_1) \equiv C(3n_1 + 1) \big) \land \big(C(n_1) \equiv C(2n_1) \big) \Big), \tag{3.1}$$

which means if n_1 the algorithm executes, then and only then $3n_1 + 1$ and $2n_1$ also execute, which is obvious;

$$\forall n_2 \left(\left(C(n_2) \equiv C\left(\frac{1}{2} n_2\right) \right) \land \big(C(n_2) \equiv C(2n_2) \big) \right), \tag{3.2}$$

which means if n_2 the algorithm executes, then and only if $\frac{1}{2} n_2$ and $2n_2$ also execute.

$C(1)$ identically true. (3.3)

These formulas are essentially axioms in the interpretation under consideration.

Transformations of a term inside an expressive function obey elementary arithmetic rules.

Further reasoning will go, considered as hypotheses.

$$\forall n \Big(C(n) \equiv \big(C(n_1) \lor C(n_2) \big) \Big), \tag{3.4}$$

which means that by n we simultaneously mean a pair of adjacent natural numbers such that at least one of these numbers executes the algorithm. This formula defines an arbitrary n, convenient for applying reasoning by induction. For the second part of the formula, we assume the corresponding closure according to the generalization rule: $\forall n_1 \forall (n_1+1)\big(C(n_1) \vee C(n_1+1)\big)$.

Since the formula is $\forall n\big(C(n) \equiv C(3n+1)\big)$ derivable, we assume that the following formulas are also derivable:

$\forall n\big(C(n+1) \equiv C(3n+4)\big)$, $\forall n\big(C(n+2) \equiv C(3n+7)\big)$, $\forall n\big(C(n+3) \equiv C(3n+10)\big)$... etc.

At the same time, we also derive:

$\forall n\big(C(n-1) \equiv C(3n-2)\big)$, $\forall n\big(C(n-2) \equiv C(3n-5)\big)$, $\forall n\big(C(n-3) \equiv C(3n-8)\big)$... etc.

Let's continue with the last line of reasoning:

... $\forall n\Big(C\big(n-(n-2)\big) \equiv C\big((3n-3n)+7\big)\Big)$, $\forall n\Big(C\big(n-(n-1)\big) \equiv C\big((3n-3n)+4\big)\Big)$.

Of particular interest is the statement $\forall n C(n) \equiv C(3n+1)$ and its corresponding $\forall n C(n-n) \equiv C\big((3n-3n)+1\big)$, however, we will refuse to consider this case for the time being, since the expression $C(0)$ deserves special attention.

By the deduction theorem, we adopt the following formulas:

$\forall n\big(C(n+1) \equiv C(3n+4)\big) \supset \forall n\Big(C\big(n-(n-1)\big) \equiv C\big((3n-3n)+4\big)\Big)$,

$$\forall n\big(C(n+2) \equiv C(3n+7)\big) \supset \forall n\Big(C\big(n-(n-2)\big) \equiv C\big((3n-3n)+7\big)\Big); \tag{3.5}$$

the conclusions in the presented implications are true, since $C(7) \equiv C(4) \equiv C(1)$ the premises must also be true. It means:

$$\forall n\big(C(n+1) \wedge C(n+2) \supset C(1)\big), \tag{3.6}$$

which, in essence, completes the proof, since it means that for any natural pair of numbers $(n+1)$ consisting of a natural number and its successor, the Collatz algorithm is performed for at least one number, and the Collatz algorithm is performed for the successor of this number; by the distributivity property, we obtain for an even or odd number, respectively:

$\big(C(n_1) \vee C(n_1+1)\big) \wedge \big(C(n_1+1) \vee C(n_1+2)\big) \equiv C(n_1+1) \vee \big(C(n_1) \wedge C(n_1+2)\big)$,

$\big(C(n_1+1) \vee C(n_1+2)\big) \wedge \big(C(n_1+2) \vee C(n_1+3)\big) \equiv C(n_1+2) \vee \big(C(n_1+1) \wedge C(n_1+3)\big)$; (3.7)

which confirms (3.6), Q.E.D.

Remark 3.1

For any formula of the form $C(n+k)$ within the framework of the Collatz algorithm, or the interpretation used, the formula is derivable $C(3k+1)$, or: $C(n+k) \vdash C(3k+1)$, which allows deriving $\vdash C(n+k) \supset C(3k+1)$. We do not have to consider the entire set of formulas $C(3k+1)$, but only those in which $k=1$ and $k=2$, which corresponds to a pair of any consecutive natural numbers with their followers $n+1$ and $n+2$.

Remark 3.2

Consider the expression separately $\forall n\left(C(n-n) \equiv C\big((3n-3n)+1\big)\right)$. It gives the formula $C(0)$, and also confirms it, since $C(0) \equiv C(1)$. The condition of the problem does not provide for the application of the algorithm to the number zero, however, if we apply the operation $3n+1$ for $n=0$, we will obtain an identity. On the other hand, in the modern interpretation of Peano's axiomatics, but not in the original one, we note [6], the unit - an odd number - is a follower of zero, considered as a natural number, so the operation $3n+1$ is not applicable to zero. But, again, on the other hand, if we apply the operation $\frac{1}{2}n$ to zero, we get a constant function. We believe that it will not be superfluous to consider the interpretation of the zero argument of a function in the context of the problem under consideration.

4. About missing function argument

In this section, we will conduct arguments based on a constructive approach. We will check the resulting definitions for consistency using generally accepted logical methods and interpretation in the context of mathematical analysis. The following arguments are directly related to the Collatz conjecture, since can serve as a basis for the proof, which will be demonstrated.

Definition 4.1

To simplify the reasoning, we will consider only real arguments of ranges and function definitions.

Let some one-place function be given $y = f^1(x)$. We will say that this function is identical to some two-place function such that $y = f^2(x, \emptyset)$, respectively $\forall x\left(f^1(x) = f^2(x, \emptyset)\right)$. Here we use the symbol " ∅" to denote zero in the interpretation of the absence of something, and we do not use the symbol " 0" in order to avoid confusion associated with the interpretation of zero as the value of an argument, for example, $y(0) = f(x) \mid x = 0$.

A trivial example could be as follows: $y = 2x$ is equal with $y = 2x + 0z$.

Similarly for n-place and $n+1$-place functions:

$$f^n(x_1, \dots, x_n) = f^{n+1}(x_1, \dots, x_n, \emptyset). \tag{4.1}$$

By $f(\emptyset)$ we will, accordingly, understand some constant function.

Let's also assume that:

$$f^{n+1}(x_1, \dots, x_n, \emptyset) = f^{n+1}(x_1, \dots, x_n, x_{n+1}). \tag{4.2}$$

Denote the formula for the identity relation of terms (4.1) and (4.2) as $\mathcal{A}$ and $\mathcal{B}$, respectively. According to the axiom $\forall x(\mathcal{A} \supset \mathcal{B}) \supset (\mathcal{A} \supset \forall x\mathcal{B})$, under the condition that $\mathcal{A}$ does not contain free occurrences x, we get:

$$\forall x_{n+1}\left(f^{n+1}(x_1, \dots, x_n, \emptyset) = f^{n+1}(x_1, \dots, x_n, x_{n+1})\right); \tag{4.3}$$

while from formula (4.1) by the generalization rule (Gen) we can get:

$$\forall x_{n+1}\left(f^n(x_1, \dots, x_n) = f^{n+1}(x_1, \dots, x_n, \emptyset)\right). \tag{4.4}$$

The last equalities allow us to make a conclusion about the property of the function:

$$\forall x_{n+1}\left(f^n(x_1, \dots, x_n) = f^{n+1}(x_1, \dots, x_n, x_{n+1})\right). \tag{4.5}$$

This formula can be interpreted as follows: for an arbitrary given function, we can introduce an additional argument that does not affect the value of the function itself; the converse is also true, if the values of some argument do not affect the value of the function for all values of the remaining arguments, then it can be discarded. The last sentence can be expressed:

$$\forall x_{n+1}\forall x_n \dots \forall x_1\left(f^{n+1}(x_1, \dots, x_n, x_{n+1}) = f^{n+1}(x_1, \dots, x_n, 0)\right) \supset \left(f^{n+1}(x_1, \dots, x_n, x_{n+1}) = f^n(x_1, \dots, x_n)\right). \tag{4.6}$$

For a one-place function, we thus obtain the axiomatically true relation:

$$\forall x_1(f^1(x_1) = a) \supset (f^1(x_1) = a), \quad (4.7)$$

where a is some subject constant. $f^1(x_1)$in this case, it is a constant function.

Example 4.1

Let some function be given, such that $y = f^1(x)$. Obviously, it can be represented as $f^2(x, y) = const$. Let's add an arbitrary variable to this function: $\forall x \forall y \forall z\big(f^3(x, y, z) = f^2(x, y)\big)$. The last equality also has a simple geometric interpretation: if a set of points is formed by adding an additional coordinate to a point located in a space of lower dimension, then this point is a projection of all new points formed from it in this way in a space of higher dimension.

Building a proof of the Collatz conjecture

The function defined by formula (1.12) can also be expressed as follows: $C(n_0) = C\big(a, m_0(a)\big)$. Here the parameters a and m_0 are obtained from the initial number $n_0$by the above formulas (1.3) and (1.4), and the parameter is $m_0$assumed to be obtained from $a$so that $n_0 = a2^{m_0}$by (1.2). Analyzing formula (1.12) by methods of sequence theory, it can be reduced to an analysis of formula (2.1). Moreover, the only sequence associated with the parameter $a$has a unique limit by (2.2), i.e. does not depend on the parameter itself a, while the parameter m_0 can be excluded as being defined through the parameter a, i.e. $C\big(a, m_0(a)\big) = C(a)$. Next we have:

$$\forall a\big(C(n_0) = C(a)\big),$$

$$\forall a\big(C(n_0) = C(a)\big) \supset (C(n_0) = const), \text{ (according to formula 4.7)} \quad (4.8)$$

which essentially leads to a proof by the generalization rule $C(n_0) \supset \forall n_0 C(n_0)$.

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