Collatz conjecture solution through the convergence study

Ruslan Enikeev

ruslan.enkeev@gmail.com

Samara State Technical university, Russia, Samara, Molodogvardeiskaya 244, 443100

Abstract

We propose a solution to the so-called Collatz conjecture problem. The iterations of the algorithm are represented through the sequence which convergence analysis is supposed to confirm the conjecture.

Keywords

“3n+1” problem; Collatz conjecture; Sequence convergence

Statements and Declarations

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Introduction

This is a well-known problem in the mathematical community, usually referred to as the "Collatz conjecture" or "3n+1" (and sometimes by other names: Kakutani's, Ulam's, Syracuse problem or conjecture). The formulation of the problem is attributed to Lothar Collatz, but the exact origin is not clearly known [1, 2]. The full-fledged problematic discussion, as well as analysis and solution approaches, could be found in the works by Jeffrey Lagarias [3-5].

The problem is so famous because it is still considered unsolved [6] despite a very simple formulation. We believe that the popularization of science on YouTube channels [7] has sparked the interest of a wider audience and drawn attention to the problem.

In this particular work, we would like to propose a representation of the algorithm that can be analyzed through a sequence convergence process.

First, we recall the conjecture to be proved: we apply the known algorithm $f(n)$ to any positive integer; the process will eventually reach the number 1, regardless of which positive integer was initially chosen:

$$f(n) = \begin{cases} 3n + 1 \text{ if } n \equiv 1 \pmod 2, \\ n/2 \text{ if } n \equiv 0 \pmod 2. \end{cases}$$

Discussion

1. Algorithm representation form

For the explanation purpose we will use the following operational notation:

$$n_1 \xrightarrow{3n+1} 3n_1 + 1,$$

$$n_2 \xrightarrow{\frac{1}{2}n} \frac{1}{2}n_2; \tag{1.1}$$

where n – the integer number at each iteration step the particular operation to be applied, $n \in \mathbb{N}[1;\infty)$, n_1 – an odd number, n_2 – even number. Obviously, $3n_1 + 1$ – even number, as the operation is applied to the odd number only, according to the algorithm rule, while $\frac{1}{2}n_2$ could be odd or even as well.

Any given even number could be reduced to the lesser odd number by consequent application of the $\frac{1}{2}n_2$ operations; as evidently from the number theory. Thus, we start discussion for the odd number n_0, which was given initially or was produced from the initial even number.

Definition 1

Assume that any positive odd integer n_0 can be initially represented in the following form:

$$n_0 = a2^{m_0}; \tag{1.2}$$

where a – rational positive number, $a \in \mathbb{Q}\left(\frac{1}{2};1\right]$, the numerator of a is an odd integer; m_0 – integer number satisfying the representation form, $m_0 \in \mathbb{N}[0;\infty)$. Examples: $5 = \frac{5}{8}\cdot 2^3$, $27 = \frac{27}{32}\cdot 2^5$, $1 = 1\cdot 2^0$ etc.

The parameters a and m_0 we assume to achieve through the realization of the following recursive functions A and M, respectively:

$$A(n,0) = n,\ A(n,x+1) = \begin{cases} A(n/2\,,x) & \text{if } n > 1, \\ n & \text{if } n \le 1. \end{cases} \tag{1.3}$$

$$M(n,0) = 0,\ M(n,x+1) = \begin{cases} M(n,x) & \text{if } 2^x < n, \\ x & \text{if } 2^x \ge n. \end{cases} \tag{1.4}$$

Then we will consider the result of applying $3n+1$ operation:

$$a2^{m_0} \xrightarrow{3n+1} 3a2^{m_0} + 1 = \frac{3}{4}a2^{m_0+2} + 1 = 2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{1.5}$$

From this notation, we can see that the expression under the brackets is less than or equal to 1 for any positive initial number: $\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \le 1$, that corresponds to the applied constraints: $a \in \mathbb{Q}\left(\frac{1}{2};1\right]$, $m_0 \in \mathbb{N}[0;\infty)$.

After operation (1.5), we must necessarily perform a division operation. We represent this in the following notation, reducing only the power of two while leaving the expression under the brackets unchanged:

$$2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \xrightarrow{\frac{1}{2}n} 2^{m_0+1}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{1.6}$$

We cannot know how many times we need to apply the division operation before we get an odd number, so we call the power of two reached as m_1; we can say with certainty that $0 \leq m_1 \leq m_0 + 1$, or through induction $0 \leq m_{i+1} \leq m_i + 1$. We will say, that we had performed the algorithm cycle, where the index of m refers to the number of applied cycles.

Definition 2

Introduce the function c, which applies the explored algorithm (1.1) for the any odd number n_0 until the output result becomes an odd number. According to the formulae (1.5) and (1.6) this operation is expressed as follows:

$$c(n_0) = 2^{m_1}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{1.7}$$

Let us define a recursive application of the operation c to an odd number through the recursive superposition notation:

$$c^1(n_0) = c(n_0),\ c^{k+1}(n_0) = c\left(c^k(n_0)\right). \tag{1.8}$$

The consequent application of the function c we will consider as a mathematical sequence. Thus, k applications of the operation c to the initial odd number n_0 results the sequence $\{c^k(n_0)\}$.

Example 1

We can consider the application of the algorithm under study as the result of applying the c operation to an odd number. For the $n_0 = 19$ we receive the following sequence: $c^1(19) = 29$, $c^2(19) = c^1(29) = 11$, $c^3(19) = c^2(29) = c^1(11) = 17$, $c^4(19) = 13$, $c^5(19) = 5$, $c^6(19) = 1$, $c^7(19) = 1$ and etc. Therefore, starting from the sixth term of the sequence the following numbers are units.

Definition 3

Introduce the limit C which is defined as a limit of the sequence $\{c^k(n_0)\}$:

$$C(n_0) = \lim_{k\to\infty} c^k(n_0). \tag{1.9}$$

To achieve explicit expression of the operation c we apply it to the result (1.7):

$$c^2(n_0) = 2^{m_2}\left[\left(\frac{3}{4}\right)^2 a + \frac{3}{4}\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}}\right)\right];$$

and to the produced result as well:

$$c^3(n_0) = 2^{m_3}\left[\left(\frac{3}{4}\right)^3 a + \left(\frac{3}{4}\right)^2\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}} + \left(\frac{3}{4}\right)^{-2}\frac{1}{2^{m_2+2}}\right)\right].$$

Finally, we are able to formulate the k member of the sequence:

$$c^k(n_0) = 2^{m_k}\left[\begin{array}{l}\left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{-1}\frac{1}{2^{m_1+2}} + \cdots\right. \\ \left.\ldots + \left(\frac{3}{4}\right)^{-k+0}\frac{1}{2^{m_{k-2}+2}} + \left(\frac{3}{4}\right)^{-k+1}\frac{1}{2^{m_{k-1}+2}}\right)\end{array}\right]. \tag{1.10}$$

The formula (1.10) could be interpreted as consisting a series:

$$c^k(n_0) = 2^{m_k}\left[\left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right]. \tag{1.11}$$

Example 2

The sixth member of the sequence in the Example 1 could be expressed through the formula (1.10) as follows:

$$c^6(19) = 2^3\left[\left(\frac{3}{4}\right)^6\frac{19}{32} + \left(\frac{3}{4}\right)^{6-1}\left(\frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{-2+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-3+1}\frac{1}{2^{5+2}} + \left(\frac{3}{4}\right)^{-4+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-5+1}\frac{1}{2^{6+2}} + \left(\frac{3}{4}\right)^{-6+1}\frac{1}{2^{5+2}}\right)\right] = 1.$$

The conjecture problem overall we will reformulate as exploration of possible results of the following limit:

$$C(n_0) = \lim_{k\to\infty} 2^{m_k}\left[\left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right]. \tag{1.12}$$

The proposed formulation of the conjecture should be considered as equal to the original: $\forall n_0(C(n_0) = 1)$. While the proposed formulation is for odd numbers, the validity for all natural numbers quite obvious: if $\forall n_0(C(n_0) = 1)$, therefore, Collatz conjecture satisfies all $2n_0$ numbers as well.

Note 1

To define the formal possibility of the limit of the sequence (1.12) to be equal to unit we use the following theorem: if all terms of a sequence are equal to the same number, then the sequence converges to that number [8]. Thus, the proposition $C(n_0) = 1$ means there exist segments of the sequence $\{c^k(n_0)\}$ which all terms are equal to the unit. These arguments allow us to apply the theorems of sequences limit according to the formal definition.

2. Limit analysis

Theorem 1

The result of the limit C is totally defined through the parameter a and does not depend from the parameter m_0, which define number n to the conformity of the conjecture.

Indeed, for any initial odd number $n_0(a, m_0)$ we are able to construct sequence of the corresponding even numbers through the parameter i, according to the formula (1.2):

$$2^i n_0 = a2^{m_0+i}, i \in \mathbb{N}[1; \infty). \tag{2.1}$$

Let us pick some numbers n_0^c for that the Collatz conjecture is fulfilled, obviously, this proposition is true [9]:

$$\exists n_0^c: n_0^c \subseteq n_0, C(n_0^c) = 1. \tag{2.2}$$

That allows us to construct infinite sequence of even numbers, for which the conjecture is fulfilled, according to the (2.1) we receive $C\big(n_0^c(a^c, m_0^c + i)\big) = 1$.

The same argumentation line could be applicated for the hypothetical number $n_0^{\neg c}$ for that the Collatz conjecture is not fulfilled: $C\big(n_0^{\neg c}(a^{\neg c}, m_0^{\neg c} + i)\big) \neq 1$.

According to the excluded third rule: $(C(n_0) = 1) \vee (C(n_0) \neq 1)$, or equally $\Big(C\big(n_0(a^c, m_0^c + i)\big) = 1\Big) \vee \Big(C\big(n_0(a^{\neg c}, m_0^{\neg c} + i)\big) \neq 1\Big)$.

If $\exists n_0\Big(C\big(n_0(a^{\neg c}, m_0^{\neg c})\big) \neq 1\Big)$ is true, there could be approved proposition $\exists n_0\Big(C\big(n_0(a^c, m_0^{\neg c})\big) = 1\Big)$. E.g., $C\big(n_0(1,0)\big) = 1$ and $C\big(n_0(1,0 + i)\big) = 1$, where $i \in \mathbb{N}[1; \infty)$ corresponds to the domain of m_0. We conclude:

$$\neg\exists m_0^{\neg c}: (m_0^c \subseteq m_0) \wedge (m_0^{\neg c} \subseteq m_0), C\big(n_0(a, m_0^c)\big) = 1 \wedge C\big(n_0(a, m_0^{\neg c})\big) \neq 1. \tag{2.3}$$

The above argumentation allows us to make an intermediate conclusion on dependency of the propositional function $\left(C\left(n_0(a, m_0)\right) = 1\right)$ result on parameter a exclusively, thus, $\left(C\left(n_0(a)\right) = 1\right)$ is an equal form.

Note 2

It was supposed by the author to exclude the interference of parameter a to the limit result by concerning the limit of the parameter associated sequence $\left\{\left(\frac{3}{4}\right)^k a\right\}$, and consequently exclude the particular number n_0 interference to the Collatz algorithm, such as result of the limit C does not depend from the number as it does not depend from the parameters, which define the number. Alas, such as logical pass was not convincing enough; obviously, the more thorough analysis is required.

Proposition 1

To confirm the $\forall n_0(C(n_0) = 1)$ we equally can exclude the following hypotheses, which reflect the proposition $\exists n_0(C(n_0) \neq 1)$: (1) it could be picked such number n_0 for which the sequence $\{c^k(n_0)\}$ is unbounded, or $\exists n_0(C(n_0) = \infty)$; (2) it could be picked such number n_0 for which the sequence $\{c^k(n_0)\}$ is divergent, or $C(n_0)$ is not exist; (3) it could be picked such number n_0 for which the sequence $\{c^k(n_0)\}$ converges to a number different from the unit, or $\exists n_0(C(n_0) = g)$, $g \in \mathbb{N}[2; \infty)$. According to the previous conclusions (Theorem 1), we are able to reformulate all these conditions relative to the parameter a instead of n_0.

The sequence $\{c^k(n_0)\}$ according to the affordable operations over sequences, could be represented as follows:

$$\{c^k(n_0)\} = \{2^{m_k}\}\left[\left\{\left(\frac{3}{4}\right)^k a\right\} + \left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}\right]. \tag{2.4}$$

Consider the sequence $\{2^{m_k}\}$: it could be unbounded, divergent or could converge.

The sequence $\left\{\left(\frac{3}{4}\right)^k a\right\}$ has a limit without another option:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^k a = 0. \tag{2.5}$$

The series $\sum_{i=1}^{\infty}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}$ could be convergent or divergent, depending on the m_{i-1} parameter values.

Hypotheses 2 and 3

Suppose, the $\{2^{m_k}\}$ is bounded. This case corresponds to the approbation of the conjecture, but and for the hypotheses 2 and 3 as well. For this case the parameter m is defined as follows: $0 \leq m_{i+1} \leq m_i$, which means equally that it does not exceed certain number $j: j \in \mathbb{N}$. The series diverges according to the necessary condition:

$$\lim_{i\to\infty}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}} = \lim_{i\to\infty}\left(\frac{4}{3}\right)^{i-1}\left(\frac{1}{2}\right)^{j+2} = \infty. \tag{2.6}$$

To evaluate the limit of the sequence $\left\{\left(\frac{3}{4}\right)^{k-1}\sum_{i=1}^{k}\left(\frac{3}{4}\right)^{-i+1}\frac{1}{2^{m_{i-1}+2}}\right\}$ consider the partial sum:

$$s_k = \left(\frac{1}{2}\right)^{j+2}\frac{1-\left({}^4/_3\right)^k}{1-\left({}^4/_3\right)}, \tag{2.7}$$

according to the formula for the geometric series sum. The associated limit takes the following form:

$$\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1}s_k = \lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1}\left(\frac{1}{2}\right)^{j+2}\frac{1-\left({}^4/_3\right)^k}{1-\left({}^4/_3\right)} = \lim_{k\to\infty}\left(\frac{1}{2}\right)^{j}\left(1-\left({}^3/_4\right)^k\right) = \left(\frac{1}{2}\right)^{j}. \tag{2.8}$$

The desired limit in this case could be formulated as follows:

$$C(n_0) = \lim_{k\to\infty}2^{m_k}\left[\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k}a + \lim_{k\to\infty}\left(\frac{3}{4}\right)^{k-1}s_k\right] = \lim_{k\to\infty}2^{m_k}\left[0+\left(\frac{1}{2}\right)^{j}\right] = \left(\frac{1}{2}\right)^{j}\lim_{k\to\infty}2^{m_k}. \tag{2.9}$$

This means whatever the number n_0 is taken, if $\{2^{m_k}\}$ is bounded assumed, the sequence associated with Collatz conjecture converges to the above form. But this leads to unambiguous resolution the Collatz algorithm on the unit, which proves the conjecture. The final result $\left(\frac{1}{2}\right)^{j}\lim_{k\to\infty}2^{m_k}$ represents the number two in the certain natural number power; according to the conjecture formulation that follows application of corresponding quantity of $\frac{1}{2}n$ operations, $m_k - j$, specifically, that results unit.

Concluding, assuming the necessary condition for the hypotheses 2 and 3 fulfilment results confirmation of the conjecture.

Note 3

One might argue, that the sequence could infinitely approach the limit (2.9) and does not take the value of the limit. According to the formulation of the limit (1.12) the series takes exactly the value of the limit (see note 1). This approved by the discrete nature of the natural number sequence. The $\left\{\left(\frac{3}{4}\right)^k a\right\}$ decreases monotonically as well as $\left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}$ in case of the bounded $\{2^{m_k}\}$. This form represents finite natural number with finite quantity of associated intervals, to which are applied infinite operations associated with the conjecture. As we consider the result of the operation c can take only odd numbers, the only possible number for $m_k - j$ is equal zero, which results only possible odd result of the power of two.

Hypothesis 1

Suppose, the $\{2^{m_k}\}$ is unbounded, that corresponds the hypothesis 1. In this case we are able to estimate the m_{i-1} value as follows: $m_{i-1} = \max(m_i)$. According to the definitively applied condition $0 \leq m_{i+1} \leq m_i + 1$ we provide $m_{i-1} = i - 1$. The series in this case converges, because satisfies necessary condition:

$$\lim_{i\to\infty} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{(i-1)+2}} = \lim_{i\to\infty} \left(\frac{4}{3}\right)^{i-1} \left(\frac{1}{2}\right)^{i-1+2} = \lim_{i\to\infty} \left(\frac{2}{3}\right)^{i-1} \left(\frac{1}{2}\right)^{2} = 0. \quad (2.10)$$

According this, we can provide sum of the series:

$$\sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{(i-1)+2}} = \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} \left(\frac{1}{2}\right)^{2} = \left(\frac{1}{2}\right)^{2} \frac{1}{1-{}^{2}/_{3}} = \frac{3}{4}. \quad (2.11)$$

Thus, the concerning limit of the sequence $\{c^k(n_0)\}$ results uncertainty of type $\infty[0+0]$, that in first sight does not contradict the acceptance of the hypothesis.

In our current representation from the condition for the expression in the brackets consists not to exceed the unit. Let introduce the following operation on the form: if the expression in the brackets less or equal to $\frac{1}{2}$ we double the expression, thus, it is stabilized in the interval $\left(\frac{1}{2}; 1\right]$ as the parameter a. That allows to interpret operation over some natural number $n_0(a_0, m_0)$ which results another number $n_1(a_1, m_1 - z)$ as follows:

$$c(2^{m_0}[a_0]) = 2^{m_1} 2^{-z} \left[2^z \left(\frac{3}{4} a_0 + \frac{1}{2^{m_0+2}}\right)\right] = 2^{m_1 - z}[a_1]; \quad (2.12)$$

where $z \in \mathbb{N}$. In this case the operation representation form becomes identical with the initial natural number interpretation according to the formula (1.2).

We continue hypothesis consideration, where every consequent number exceeds previous:

$$\{c^k(n_0)\} = \{2^{m_k - z_k}\}\left[2^{z_k}\left(\left\{\left(\frac{3}{4}\right)^k a\right\} + \left\{\left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right\}\right)\right] = \{2^{m_k - z_k}\}[\{a_k\}]. \qquad (2.13)$$

This condition as well corresponds to the unbounded sequence $\{2^{m_k - z}\}$, while the expression in brackets corresponds to the interval $\left(\frac{1}{2}; 1\right]$. We do not know the value of z, because have not estimated the conditions of doubling the expression in brackets yet. We will consider the arbitrarily chosen very high numbers to find out the z estimation. The relation n_k/n_0 in this case could be expressed as follows:

$$\lim_{\substack{n_k \to \infty \\ k \to \infty}} \frac{n_k}{n_0} = \lim_{\substack{m_k \to \infty \\ k \to \infty}} \frac{2^{m_0 + k} \cdot 2^{-z_k}\left[2^{z_k}\left(\left(\frac{3}{4}\right)^k a_0 + \left(\frac{3}{4}\right)^{k-1} \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{-i+1} \frac{1}{2^{m_{i-1}+2}}\right)\right]}{2^{m_0}[a_0]} = \lim_{k \to \infty} 2^{k - z_k}\left[2^{z_k}\left(\frac{3}{4}\right)^k\right]. \qquad (2.14)$$

Last expression (2.14) gives the following estimation of the parameter:

$$k \log_2 \frac{4}{3} - 1 < z_k < k \log_2 \frac{4}{3}, \qquad (2.15)$$

which gives the following approximate relation at high numbers: $z_k/k \approx 0.415$.

The provided transformations still do not obviously contradict the hypothesis; however, we have changed the uncertainty $\infty[0 + 0]$ to consideration to another uncertainty, which could be defined as $\infty[\{a_k\}]$, or $\infty[\infty \cdot 0]$, where $\{a_k\}$ is a sequence of numbers satisfying demands for the parameter a in the Definition 1.

Thus, we continue: from the admittance of the hypothesis 1, there is a number, say a_0^∞, starting from all the produced consequent numbers $\{a_k^\infty\}$ satisfy the hypothesis 1. If there is such a_i^∞ from $\{a_k^\infty\}$ which does not satisfy the hypothesis 1, that leads to contradiction, because a_i^∞ was produced from a_0^∞ by applying the Collatz algorithm, while a_0^∞ should be kept satisfying the hypothesis 1.

In case of unbounded $\{2^{m_k}\}$, the operation c is totally defined, such as parameter in the series takes following values: $m_i = i$.

Definition 4

Suppose we construct reversive to c operation, the operation c^{-1}, which generates parameters before a_0^∞, such as:

$$c^{-1}\left(c\left(n_0^\infty(a_0^\infty)\right)\right) = n_0^\infty(a_0^\infty),\ c^{-k-1}(n_0) = c^{-1}\left(c^{-k}\left(n_0^\infty(a_0^\infty)\right)\right) = n_{-k-1}^\infty(a_{-k-1}^\infty). \tag{2.16}$$

Obviously, $n_{i-1}^\infty < n_i^\infty$. But that means we cannot generate infinite quantity of numbers n_{-k}^∞, because n_0^∞ is a finite natural number. That means there should be such n_i^∞ and the corresponding a_i^∞ that do not produce numbers n_{i-1}^∞ and a_{i-1}^∞ which should give the disproof of the Collatz conjecture. But this leads to the contradiction: we cannot produce that number, because cannot satisfy condition $n_{i-1}^\infty < n_i^\infty$ for the infinite sequence for the natural number domain. If we assume existence of the number $c^{-1}(n_i^\infty)$ it should be such, that approve the conjecture, but that consequently means approving conjecture by the n_i^∞ as well.

Thus, the assumption of existence such number n_0^∞ leads to contradiction.

Finally, it is shown that all the possible hypotheses of disproving the Collatz conjecture result the contradiction, that proves the conjecture. Q.E.D.

3. Summary

1. Any initial integer number can be represented through the form (1.2).
2. The infinite process of the Collatz algorithm application could be presented through the limit of the sequence (1.12).
3. According to the Proposition 1 we can propose Hypotheses 1–3 which only disprove the conjecture.
4. Acceptance of the necessary condition for the hypotheses 2 and 3 and for the approving the conjecture leads to the limit result (2.9) which can only approve the conjecture. The hypotheses 2 and 3, therefore, are not applicable.
5. Acceptance of the hypothesis 1 through the (2.13) leads to acceptance of infinite quantity of numbers satisfying the hypothesis.
6. The definition 4 along with (2.16) allow us to generate numbers prior to the assumed one according to the hypothesis 1.
7. This process cannot be infinite, thus cannot produce infinite quantity, satisfying the hypothesis 1. The hypothesis causes contradiction.
8. Along with denying of the possible disproving hypotheses, the Collatz conjecture is proved.

4. Constructivistic approach

Here we will use the following interpretation: $C(n_0)$ is a propositional function of the conformity of the natural number n_0 to the Collatz algorithm.

Proposition 1

If the odd number n_0 conforms the algorithm, then the following numbers also conform:

$\forall n_0 C(n_0) \to C(3n_0 + 1) \wedge C(2n_0)$.

It is easy to prove: if n_0 is odd and is known to converge, then through the application of the algorithm the following operation will be $n_1 = 3n_0 + 1$ with the resulting number; this number n_1 must converge, because n_0 converges. If we start with $2n_0$, the following operation will be dividing by two with the resulting n_0, which converges.

Proposition 2

Suppose we can find such odd n'_0, that number $3n'_0$ converges:

$\exists n'_0 \big(C(n'_0) \wedge C(3n'_0)\big)$.

This proposition is true, such numbers exist.

For this number we can as well say, that $C(3n'_0 + 1) \wedge C(2n'_0)$ according to the proposition 1.

Then, we can provide algebraic transformations for the propositional functions objects to provide:

$C(3(2n'_0) + 1) \wedge C\big(2(3n'_0)\big)$ or $C(6n'_0 + 1) \wedge C(6n'_0)$,

which allows us to consider n'_0 to be an even as well. The last formula allows to conclude $C(n'_0 + 1) \wedge C(n'_0)$.

Proposition 3

Consider the unit as n'_0: $n'_0 = 1$. This is the induction base.

The $n'_0 + 1$ is an induction step.

This means through the induction process we can construct all the numbers, that satisfy the Collatz conjecture. But this is as well a construction scheme for the natural numbers without zero. The set of the natural numbers satisfying the conjecture coincide with natural numbers set without zero.

Q.E.D. (x2)

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