Collatz conjecture solution approach through the series convergence study

Ruslan Enikeev

ruslan.enkeev@gmail.com

Samara State Technical university, Russia, Samara, Molodogvardeiskaya 244, 443100

Abstract

We propose a solution approach to the so-called Collatz conjecture problem. The iterations of the algorithm are represented through the series which convergence analysis is supposed to confirm the conjecture.

Keywords

“3n+1” problem; Collatz conjecture; Series convergence

Statements and Declarations

There are no competing interests.

No funding was received.

Foreword

This is a draft version of the paper, the so-called preprint. This material had not been reviewed. The purpose of the publication is a constructive discussion of the material: finding errors, sharing the interest, etc.

Thank you for the reading!

Introduction

This is a well-known problem in the mathematical community, usually referred to as the "Collatz conjecture" or "3n+1" (and sometimes by other names: Kakutani's, Ulam's, Syracuse problem or conjecture). The formulation of the problem is attributed to Lothar Collatz, but the exact origin is not clearly known [1, 2].

The full-fledged problematic discussion, as well as analysis and solution approaches, could be found in the works by Jeffrey Lagarias [3-5].

The problem is so famous because it is still considered unsolved [6] despite a very simple formulation. We believe that the popularization of science on YouTube channels [7] has sparked the interest of a wider audience and drawn attention to the problem.

In this particular work, we would like to propose a representation of the algorithm that can be analyzed through a series convergence process. First, we recall the conjecture to be proved: we apply the known algorithm to any positive integer; the process will eventually reach the number 1, regardless of which positive integer was initially chosen.

Discussion

1. Algorithm representation form through the series

For the explanation purpose we will use the following operational notation:

$$n_1 \xrightarrow{3n+1} 3n_1 + 1,$$

$$n_2 \xrightarrow{\frac{1}{2}n} \frac{1}{2}n_2; \quad (1)$$

where n – the integer number at each iteration step the particular operation to be applied, $n \in \mathbb{N}[1;\infty)$, n_1 – an odd number, n_2 – even number. Obviously, $3n_1 + 1$ – even number, as the operation is applied to the odd number only, according to the algorithm rule, while $\frac{1}{2}n_2$ could be odd or even as well.

Let us assume that any positive integer n_0 can be initially represented in the following form:

$$n_0 = a2^{m_0}; \quad (2)$$

where a – rational positive number, $a \in \mathbb{Q}(0;1]$, the numerator of a is an odd integer; m_0 – integer number satisfying the representation form, $m_0 \in \mathbb{N}[0;\infty)$. Examples: $5 = \frac{5}{8} \cdot 2^3$, $27 = \frac{27}{32} \cdot 2^5$, $1 = 1 \cdot 2^0$ etc.

Then we will consider the result of applying $3n+1$ operation:

$$a2^{m_0} \xrightarrow{3n+1} 3a2^{m_0} + 1 = \frac{3}{4}a2^{m_0+2} + 1 = 2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{3}$$

From this notation, we can see that the expression under the brackets is less than or equal to 1 for any positive initial number: $\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \leq 1$, that could be easily shown:

$$4n \geq 3n + 1 \quad ==> \quad 1 \geq \frac{3}{4} + \frac{1}{4n};$$

for any positive integer n, where the equal sign holds only for $n = 1$. From this we notice:

$$2^{m_0+2}[a] \geq 2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \quad ==> \quad \frac{1}{2^2}a \geq \frac{1}{2^{m_0+2}} \quad ==> \quad a \geq \frac{1}{2^{m_0}}; \tag{4}$$

that corresponds to the applied constraints: $a \in \mathbb{Q}(0;1]$, $m_0 \in \mathbb{N}[0;\infty)$.

After operation (3), we must necessarily perform a division operation. We represent this in the following notation, reducing only the power of two while leaving the expression under the brackets unchanged:

$$2^{m_0+2}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \xrightarrow{\frac{1}{2}n} 2^{m_0+1}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right]. \tag{5}$$

We cannot know how many times we need to apply the division operation before we get an odd number, so we call the power of two reached as m_1; we can say with certainty that $0 \leq m_1 \leq m_0 + 1$, or through induction $0 \leq m_i \leq m_{i-1} + 1$. We will say, that we had performed the algorithm cycle, where the index of m refers to the number of applied cycles.

Then we apply operation $3n+1$ again after the first cycle completion:

$$2^{m_1}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] \xrightarrow{3n+1} 2^{m_1+2}\frac{3}{4}\left[\frac{3}{4}a + \frac{1}{2^{m_0+2}}\right] + 1 = 2^{m_1+2}\left[\left(\frac{3}{4}\right)^2 a + \frac{3}{4}\frac{1}{2^{m_0+2}} + \frac{1}{2^{m_1+2}}\right]. \tag{6}$$

To find the overall result of the algorithm application, we reapply the required number of $\frac{1}{2}n$ to complete the second cycle and provide $3n+1$ operation:

$$2^{m_2}\left[\left(\frac{3}{4}\right)^2 a + \frac{3}{4}\frac{1}{2^{m_0+2}} + \frac{1}{2^{m_1+2}}\right] \xrightarrow{3n+1} 2^{m_2+2}\left[\left(\frac{3}{4}\right)^3 a + \left(\frac{3}{4}\right)^2\frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)\frac{1}{2^{m_1+2}} + \frac{1}{2^{m_2+2}}\right]. \tag{7}$$

We consider the sequence of $3n+1$ with the connected $\frac{1}{2}n$ operations as the algorithm cycle. We associate the number of cycles applied with the letter "k" corresponding to the lower index of m in the previous equations. The k is taken to be unboundedly high, $k \in \mathbb{N}[0;\infty)$:

$$a2^{m_0} \xrightarrow{\lim_{k\to\infty} 3n+1,\ \frac{1}{2}n} \lim_{k\to\infty} 2^{m_k+2} \begin{bmatrix} \left(\frac{3}{4}\right)^k a + \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m_0+2}} + \left(\frac{3}{4}\right)^{k-2} \frac{1}{2^{m_1+2}} + \cdots \\ \ldots + \left(\frac{3}{4}\right)^1 \frac{1}{2^{m_{k-2}+2}} + \left(\frac{3}{4}\right)^0 \frac{1}{2^{m_{k-1}+2}} \end{bmatrix}. \tag{8}$$

Importantly to note, that we assume an infinite process that starts from any odd integer (including 1) which is described through the series (8).

We assume the possibility to start with initial even number: in that case, the first cycle consists only $\frac{1}{2}n$ operations. In our notation, we assume cycle operator application for the odd number with the resulting odd number. So, we assume the initial even number is brought to the consequent odd before the series (8) form application.

For the demonstration purpose, we provide several cycles application for the start number 1 $(a = 1,\ m_0 = 0)$; for the notation simplification purpose we will write the cycle number above the operator arrow. For this particular example $m_i = m_{i-1} - 2$, as after $3n+1$ operation application we apply strictly two $\frac{1}{2}n$ operations:

$$2^0[1] \xrightarrow{k=1} 2^{(0-2)+2}\left[\frac{3}{4}1 + \frac{1}{2^{0+2}}\right] \xrightarrow{k=2} 2^{(0-2)+2}\left[\left(\frac{3}{4}\right)^2 1 + \left(\frac{3}{4}\right)^1 \frac{1}{2^{0+2}} + \frac{1}{2^{0+2}}\right] \longrightarrow \cdots \tag{9}$$

The form (9) serves to represent the consequence $\cdots \to 1 \to 4 \to 2 \to 1 \to \cdots$ as each cycle of the algorithm.

We could rephrase the (8) formula to produce a series form for the convenient analysis:

$$a2^{m_0} \xrightarrow{\lim_{k\to\infty} 3n+1,\ \frac{1}{2}n} \lim_{k\to\infty} 2^{m_k+2}\left[\left(\frac{3}{4}\right)^k a + \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{k-i} \frac{1}{2^{m_{i-1}+2}}\right]. \tag{10}$$

The series as algorithm representation form (10), we will use as a basis for the further discourse.

2. The series analysis

If we apply to the initial number representation form (2), we assume it is easy to show through the induction, that algorithm convergence does not depend on the initial power of two, or m_0 equally. Suppose, we consider a number $a_j 2^{m_0}$ and it turns out to converge through the algorithm application, therefore, we let ourselves conclude $a_j 2^{m_0+i}$ converge for any $i \in \mathbb{N}[0;\infty)$: according to the algorithm rule we apply to $a_j 2^{m_0+i}$ i number of $\frac{1}{2}n$ operations, the representation forms become equal, as for m_0, $(m_0+i) \in \mathbb{N}[0;\infty)$. We apply the proposition "n_0 converges through Collatz algorithm to 1" as $C(n)$, while assuming n_0 as a function of a and m_0 to conclude:

$$\exists a \exists m_0 C\big(n_0(a,m_0)\big) \rightarrow \exists a \forall m_0 C\big(n_0(a,m_0)\big). \tag{11}$$

Thus, the Collatz convergence result does not depend from m_0 parameter.

Then we will analyze the series under the square brackets:

$$S_k = \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{k-i} \frac{1}{2^{m_{i-1}+2}}. \tag{12}$$

We can say, that this series monotonically increases and is finite. We know from the algorithm property, that terms in the square brackets do not exceed 1. It could be checked by necessary condition for convergence, where the last term corresponds to $i=1$, as we can choose whatever terms order:

$$\lim_{k\to\infty} \left(\frac{3}{4}\right)^{k-1} \frac{1}{2^{m_{1-1}+2}} = 0, \tag{13}$$

this condition allows considering the series as convergent, which we previously confirmed through the series construction algorithm property. Thus, the limit of the function in square brackets exists and is finite and we can provide the conclusion:

$$\lim_{k\to\infty} \left[\left(\frac{3}{4}\right)^{k} a + \sum_{i=1}^{k} \left(\frac{3}{4}\right)^{k-i} \frac{1}{2^{m_{i-1}+2}}\right] = g, \;\; g \in \mathbb{Q}[0;\; 1]. \tag{14}$$

We consider the case when $g=0$: this situation corresponds to monotonically increasing $m_k = m_{k-1}+1$ which is allowable according to the algorithm formulation. To uncover uncertainty, we provide transformations of the limit (10), assuming $m_k = m_0 + k$:

$$\lim_{k\to\infty} 2^{m_0+k+2} \left[\left(\frac{3}{4}\right)^{k} a + S_k\right] = \lim_{k\to\infty} \left(2^{m_0+2} \left(\frac{3}{2}\right)^{k} a + 2^{m_0+k+2} S_k\right) = \infty, \tag{15}$$

according to the theorems of the divergent sequence, because:

$$\lim_{k\to\infty} 2^{m_0+2} \left(\frac{3}{2}\right)^{k} a = \infty. \tag{16}$$

Analyzing such possibility, we conclude, that the limit diverges to infinity regardless of what the value of a is taken. Important: here we do not provide any conclusions regarding the Collatz algorithm realization.

Then, we consider the case, when $g \neq 0$; the 2^{m_k+2} sequence therefore converge and we could provide the following transformation:

$$\lim_{k\to\infty}\left(2^{m_k+2}\left[\left(\frac{3}{4}\right)^k a+S_k\right]\right)=\lim_{k\to\infty}(2^{m_k+2})\left[\lim_{k\to\infty}\left(\left(\frac{3}{4}\right)^k a\right)+\lim_{k\to\infty}(S_k)\right]. \quad (17)$$

We will consider the limit, associated with a, while other terms converge to nonzero numbers:

$$\lim_{k\to\infty}\left(\frac{3}{4}\right)^k a=0. \quad (18)$$

We conclude, that the limit associated with a converges to zero regardless of what the value of a is taken.

According to the provided argumentation, we conclude, that limit (10) is not affected by the value a. The limit (10) in turn is an unambiguous representation of infinite Collatz algorithm applied to any initial integer, which is represented as $n(a, m_0)$. Then, we conclude that the Collatz algorithm application result is not affected by the value a.

Finally, we apply to the previously provided implication (11) to conclude:

$\exists a \exists m_0 C(n_0(a,m_0)) \rightarrow \forall a \forall m_0 C(n_0(a,m_0))$, or

$$\exists n_0 C(n_0) \rightarrow \forall n_0 C(n_0). \quad (19)$$

Obviously, $\exists n_0 C(n_0) \therefore \forall n_0 C(n_0)$, Q.E.D.

Summary

1. Any initial integer number can be represented through the form (2), $n_0(a, m_0)$.
2. The infinite process of the Collatz algorithm application could be presented through the form with initially known properties, consisting of a series (10).
3. The form result does not depend on parameter m_0.
4. The series convergence analysis shows that convergence result does not depend on parameter a either.
5. The form (10) is an unambiguous representation of the Collatz algorithm applied to a number, therefore, the Collatz algorithm result does not depend on a and m_0.
6. If we accept proposition 1, the Collatz algorithm result does not depend on what initial number is chosen (19).
7. Therefore, if we show, that the Collatz conjecture satisfies any particular number, the Collatz conjecture satisfy any positive integer; that confirm the conjecture.

References

1. Andrei S., Masalagiu C. About the Collatz conjecture. Acta Informatica (1998), Vol. 35, P. 167–179.
2. Garner L.E. On the Collatz 3n+1 algorithm. Proceedings of the American Mathematical Society (1981), Vol. 82 (1), P. 19–22.
3. Lagarias J.C. The Ultimate Challenge: The 3x+1 Problem. American Mathematical Soc., 2010, 344 p.
4. Lagarias J.C. The 3x+1 Problem: An Annotated Bibliography (1963-1999) Unpublished .Preprint on arXiv [arxiv.org/abs/math/0309224]
5. Lagarias J.C. The 3x+1 problem and its generalizations Amer. Math. Monthly 92 (1985) 3-23.
6. Gasull A. Some open problems in low dimensional dynamical systems. SeMA Journal (2021), Vol. 78, P. 233–269.
7. The Simplest Math Problem No One Can Solve - Collatz Conjecture. Veritasium, YouTube channel. URL: https://www.youtube.com/watch?v=094y1Z2wpJg