“3n+1 problem” solution approach through the series convergence study

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Abstract

We propose a solution approach to the so-called ”3n+1” problem. The iterations of algorithm are represented by a series which convergence analysis gives us confirmation of the conjecture.

1. *Introduction*

This is a well-known problem in the mathematical community, usually referred to as the ”Collatz conjecture” or ”3n+1” (and sometimes by other names: Kakutani’s, Ulam’s, Syracuse problem or conjecture). The formulation of the problem is attributed to Lothar Collatz, but the exact origin is not clearly known [**1**, **2**].

The problem is so famous because it is still considered unsolved [**3**] despite a very simple formulation. We believe that the popularization of science on YouTube channels has sparked the interest of a wider audience [**4**] and drawn attention to the problem.

In this particular work, we would like to propose a representation of the algorithm that can be analyzed through a series convergence process. First, we recall the conjecture to be proved: we apply the known algorithm to any positive integer; the process will eventually reach the number 1, regardless of which positive integer was initially chosen.

2. *Discussion*

For the explanation purpose we will use the following operational notation:

$$n_1 \xrightarrow{3n+1} 3n_1 + 1,$$

$$n_2 \xrightarrow{1/2\ n} \frac{1}{2}n_2;$$

where n – the integer number at each iteration step the particular operation to be applied, n_1 – odd number, n_2 – even number. Obviously, $3n_1 + 1$ – even number.

Let us assume that any positive integer can be initially represented in the following form:

$$a2^m; \tag{1}$$

where a – rational positive number, $a \in (0; 1]$; m – integer number satisfying the representation form. Examples: $5 = \frac{5}{8} \cdot 2^3$, $27 = \frac{27}{32} \cdot 2^5$, $1 = 1 \cdot 2^0$ etc. For simplicity, we will start with an odd number and assume that any initial even number can be reduced to an odd number with certain $\frac{1}{2}n$ initial operations and reach the m_0 value.

Then we will consider the result of applying $3n+1$ operation:

$$a2^{m_0} \xrightarrow{3n+1} 3a2^{m_0}+1 = \frac{3}{4}a2^{m_0+2}+1 = 2^{m_0+2}\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right]. \tag{2}$$

From this notation, we can see that the expression under the brackets $\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right]$ is less than or equal to 1 for any positive initial number, which can be easily shown:

$$4n \geqslant 3n+1 \quad \Rightarrow \quad 1 \geqslant \frac{3}{4}+\frac{1}{4n};$$

for any positive integer n, where the equal sign holds only for $n=1$. From this we conclude:

$$2^{m+2}\,[a] \geqslant 2^{m+2}\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right] \quad \Rightarrow \quad \frac{1}{2^2}a \geqslant \frac{1}{2^{m+2}} \quad \Rightarrow \quad a \geqslant \frac{1}{2^m}; \tag{3}$$

corresponding to the applied constraints: $a \in \mathbb{Q}\,(0;1]\,,\, m \in \mathbb{N}\,[0;\infty)\,.$

After the previous operation (2), we must necessarily perform a division operation. We represent this in the following notation, reducing only the power of two while leaving the expression under the brackets unchanged:

$$2^{m_0+2}\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right] \xrightarrow{1/2\ n} 2^{m_0+1}\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right]. \tag{4}$$

We cannot know how many times we need to apply the division operation before we get an odd number, so we call the power of two reached as m_1; we can say with certainty that $m_1 \leqslant m_0+1$ and $m_1 \geqslant 0$. Then we apply operation $3n+1$ again:

$$2^{m_1}\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right] \xrightarrow{3n+1} 2^{m_1+2}\frac{3}{4}\left[\frac{3}{4}a+\frac{1}{2^{m_0+2}}\right]+1 = 2^{m_1+2}\left[\left(\frac{3}{4}\right)^2 a+\frac{3}{4}\frac{1}{2^{m_0+2}}+\frac{1}{2^{m_1+2}}\right]. \tag{5}$$

To find the overall result of the algorithm application, we reapply the required number of $\frac{1}{2}n$ with the last $3n+1$ operation:

$$2^{m_2}\left[\left(\frac{3}{4}\right)^2 a+\frac{3}{4}\frac{1}{2^{m_0+2}}+\frac{1}{2^{m_1+2}}\right] \xrightarrow{3n+1} 2^{m_2+2}\left[\left(\frac{3}{4}\right)^3 a+\left(\frac{3}{4}\right)^2\frac{1}{2^{m_0+2}}+\left(\frac{3}{4}\right)\frac{1}{2^{m_1+2}}+\frac{1}{2^{m_2+2}}\right]. \tag{6}$$

We consider the sequence of $3n+1$ with the connected $\frac{1}{2}n$ operations. We associate the number of operations applied with a letter "k" corresponding to the lower index of m in the previous equations. The k is taken to be unboundedly high, assuming that the progression as a whole, hypothetically still, has proceeded to the targeted cycle sequence $(\ldots 1 \to 4 \to 2 \to 1 \ldots)$:

$$a2^{m_0} \xrightarrow{\lim_{k\to\infty}\ 3n+1,\ 1/2\ n} 2^{m_k+2}\left[\begin{array}{c}\left(\frac{3}{4}\right)^{k+1} a+\left(\frac{3}{4}\right)^{k}\frac{1}{2^{m_0+2}}+\left(\frac{3}{4}\right)^{k-1}\frac{1}{2^{m_1+2}}+\ldots \\ \cdots+\left(\frac{3}{4}\right)\frac{1}{2^{m_{k-1}+2}}+\frac{1}{2^{m_k+2}}\end{array}\right]. \tag{7}$$

If the resulting number can be uniquely represented by an integer power of two, the problem can be considered solved. According to the applied algorithm, m_k+2 is a positive integer, including zero. The number under the square brackets refers to the interval $(0;1]$, according to the applied algorithm. Thus, to solve the problem, we should show that the mathematical series under the square brackets converges to the number that gives the cycle $1 \to 4 \to 2 \to 1$.

The influence of the initial non-(integer power of two), which we call a, is neglected since the associated term in the series converges to zero:

$$\lim_{k\to\infty}\left(\frac{3}{4}\right)^{k+1} a \;=0. \tag{8}$$

That is, to solve the problem, we should examine the last terms of the series for convergence.

Let us compare two adjacent terms. We will use the sign "><" to show that initially, we have no information about the ratio of the terms. We take the values of the series from the z position from the beginning, $z \in [0; k-1]$:

$$\left(\frac{3}{4}\right)^{k-z} \frac{1}{2^{m_z+2}} >< \left(\frac{3}{4}\right)^{k-z-1} \frac{1}{2^{m_{z+1}+2}} \quad \Rightarrow \quad \frac{1}{2^{m_z}} >< \frac{4}{3}\frac{1}{2^{m_{z+1}}}; \tag{9}$$

where, as the algorithm realization, we know that $0 \leqslant m_{z+1} \leqslant m_z + 1$. First, examine the case for $m_{z+1} = m_z + 1$:

$$1\frac{1}{2^{m_z}} > \frac{2}{3}\frac{1}{2^{m_z}}\bigg|_{m_{z+1}=m_z+1}. \tag{10}$$

According to the so-called d'Alembert's ratio test (proposed and proved by Cauchy), the series converges in this case because it is positive. That is, if we assume the sequence of an equal number of $3n+1$ and $\frac{1}{2}n$ operations, which speculatively could possibly generate infinity, we obtain the above series realization where a successive term is two-thirds of the previous one. Assuming we observe such a progression from position z, we conclude that:

$$\lim_{k\to\infty} \frac{1}{2^{m_z+2}} \left[\left(\frac{2}{3}\right)^0 + \left(\frac{2}{3}\right)^1 + \left(\frac{2}{3}\right)^2 + \cdots + \left(\frac{2}{3}\right)^{k-z-1} + \left(\frac{2}{3}\right)^{k-z}\right] = \frac{1}{2^{m_z+2}}[3] = \frac{1}{2^{m_z}}\left[\frac{3}{4}\right], \tag{11}$$

where the series in square brackets converge to three. This means that in the case of a continuous increase of the initial number during the sequence of $3n+1$ and $\frac{1}{2}n$ operations, the resulting number converges to $3 \cdot 2^{m_k}$, where m_k is a positive integer; and then to be reduced to 3, which is known to converge to $1 \to 4 \to 2 \to 1$ cycle.

Then we will discuss the alternative case for $m_{z+1} \leqslant m_z$; to obtain the necessary result, we assume $m_{z+1} = m_z$ and compare according to (9):

$$1\frac{1}{2^{m_z}} < \frac{4}{3}\frac{1}{2^{m_z}}\bigg|_{m_{z+1}=m_z}, \quad 1\frac{1}{2^{m_z}} < \frac{8}{3}\frac{1}{2^{m_z}}\bigg|_{m_{z+1}=m_z-1}, \quad etc. \tag{12}$$

Assuming d'Alembert's ratio test, the series is divergent. This means that the series in square brackets (7) is continuously increasing in this case. But the sum of the series is also constrained by the algorithm rule (3): it cannot be greater than 1. So, the moment it reaches 1, we assume that the generated number converges to 1 and the series result will produce the cycle $1 \to 4 \to 2 \to 1$.

The hypothesized situation, which presupposes an alternative cycle, cannot be satisfied because the series of previous terms continues to decrease with each new iteration. The only stable cycle in the proposed representation becomes the series:

$$\lim_{k\to\infty} \frac{1}{2^{m_k+2}} \left[\cdots + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^0\right] = \frac{1}{2^{m_k+2}}[4] = \frac{1}{2^{m_k}}[1]. \tag{13}$$

Thus, any positive integer converges to the sequence $1 \to 4 \to 2 \to 1$ after applying a finite number of the assumed operations.

Thank you for the reading.

3. *Summary*

(i) Any initial number can be represented by the form (1).
(ii) We apply an infinite set of the known iterations and obtain the result described by (7), which consists of 2 of a certain power and a series.
(iii) The sum of the series according to the applied expression form is not greater than 1 (3).
(iv) The series in expression (7) is of interest: if it converges, the produced sum of the series provides information for the result of the algorithm application.
(v) According to the compartment of the series terms (9), the series may converge or diverge.
(vi) If the series converges, it converges to $\frac{3}{4}$ (11); therefore, the convergence result at a certain iteration will give 3, while it is known that 3 leads to the iteration cycle $1 \to 4 \to 2 \to 1$.
(vii) When the series diverges (12), the series keeps increasing and does not converge to a certain number, but the applied expression form constrains the series not to exceed 1, hence the series converges to 1 and the approved cycle.

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