On the Last Numbers of Positive Integers

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Abstract: In this note, we are interested in the last numbers of positive integers; for example, for 20211206, the last number is 6, typically we note that for any positive integer a, the last numbers of a^5 and a are the same.

Key Words: The last number of a positive integer, Fermat's small theorem, series of integers, Yamane's problem, repeated series.

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1 Introduction

In this note, we are interested in the last numbers of positive integers; for example, for 20211206, the last number is 6, typically we note that for any positive integer a, the last numbers of a^5 and a are the same.

2 Main results

For any natural numbers (positive integers) a and b we consider the series

$$\{(a-n, b+n)\}_{n=\infty}^{n=-\infty},$$

as in

$$,,,(a-2, b+2),(a-1, b+1),(a, b),(a+1, b-1),(a+2, b-2),,,.$$

Then, we have

Theorem 2.1 *In the series*

$$\left\{(a-n)^3+(b+n)^3\right\}_{n=\infty}^{n=-\infty},$$

the last numbers are the repeated series composing at most 4 *numbers.*

Indeed, we note the identity

$$S_5 = (a-5)^3 + (b+5)^2$$

$$= (a+b)\left(a^2 - ab + b^2 - 15(a-b-5)\right).$$

Note that all the cases

$$(a+b)15(a-b-5)$$

is a multiply of 10; that is, the last numbers of $S_0 = a^3 + b^3$ and S_5 are the same. Therefore, in general, we have the theorem.

In general, let Ln denote the last number of an integer n.

Corollary 2.1 *If* $L(a+b) = 0$*, then* $L(a^3+b^3) = 0$.

Theorem 2.1 states that the series is repeated with at most 4 numbers. By examining the details, we have

Corollary 2.2 *For*

$$L(a+b) = 1,$$

the series is repeated as

$$1, 7, 9, 7, 1.$$

Siminary,

$$2: \quad 6, 8, 2, 8, 6.$$

$$
\begin{aligned}
3: &\quad 9,7,3,7,9. \\
4: &\quad 4,8,6,8,4. \\
5: &\quad 5,5,5,5,5. \\
6: &\quad 6,2,4,2,6. \\
7: &\quad 1,3,7,3,1. \\
8: &\quad 4,2,8,2,4. \\
9: &\quad 9,3,1,3,9. \\
0: &\quad 0,0,0,0,0.
\end{aligned}
$$

3 An application of the Fermat's small theorem

In connection with the last numbers of positive integers, we obtain the pleasant theorem:

Theorem 3.1 *For any positive integer a, the last numbers of a^5 and a are the same.*

Indeed, in the identity

$$a^5 - a = a\left(a^{5-1} - 1\right),$$

from the Fermat's small theorem ([1, 2]), if a is not a multiply of 5, since

$$\left(a^{5-1} - 1\right) \equiv 0, \quad (\bmod \quad 5),$$

we see that for the both cases of even a and odd a, $a^5 - a$ is a multiply of 10 and so, we have the desired result.

If a is a multiply of 5, then the result is trivial.

4 Open problems

In this note we are interested in the last number of positive integers. How will be such an problem? It seems that this type problem will be a new type one. Therefore, we would like to propose a general problem as the Yamane's problem:

Yamane's Problem: Discuss or derive the results about the last numbers of positive integers.

In particular, we can propose

(Y1): What is the general theorem of Theorem 2.1?
(Y2): Could we derive the result of Theorem 3.1 type?

References

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