Tentatives For Obtaining The Proof of The Riemann Hypothesis

- Version 4., December 2022 -

Abstract

This report presents a collection of some tentatives to obtain a final proof of the Riemann Hypothesis. The last paper of the report is submitted to a mathematical journal for review.

Résumé

Ce rapport présente une collection de quelques tentatives pour obtenir une démonstration de l'hypothèse de Riemann.

Le dernier papier du rapport est soumis à une revue de mathématiques pour lecture.

December, 31 2022

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TENTATIVES FOR OBTAINING THE PROOF OF THE RIEMANN HYPOTHESIS - VERSION 4., DECEMBER 2022 -

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Figure 1. Photo of the Author

To the memory of my Parents, to my wife Wahida, my daughter Sinda and my son Mohamed Mazen

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Une Solution de l'Hypothèse de Riemann - A Solution of The Riemann Hypothesis -

Abstract. — In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros) $s = \sigma + it$ of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \quad \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$.

We give proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann Hypothesis.

Résumé. - En 1859, Georg Friedrich Bernhard Riemann avait annoncé la conjecture suivante, dite Hypothèse de Riemann: *Les zéros non triviaux* $s = \sigma + it$ *de la fonction zeta définie par:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ pour } \quad \Re(s) > 1$$

ont comme parties réelles $\sigma = \frac{1}{2}$.

On donne une démonstration que $\sigma = \frac{1}{2}$ en utilisant une proposition équivalente de l'Hypothèse de Riemann.

1.1. Introduction

En 1859, G.F.B. Riemann avait annoncé la conjecture suivante [**1**] :

Conjecture 1.1.1. — *Soit $\zeta(s)$ la fonction complexe de la variable complexe $s = \sigma + it$ définie par le prolongement analytique de la fonction :*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ pour } \Re(s) = \sigma > 1$$

sur tout le plan complexe sauf au point $s = 1$. Alors les zéros non triviaux de $\zeta(s) = 0$ sont de la forme :

$$s = \frac{1}{2} + it$$

Dans cette communication, nous donnons une démonstration que $\sigma = \frac{1}{2}$. Notre idée est de partir d'une proposition équivalente de l'Hypothèse de Riemann et en utilisant la définition de la limite des suites réelles.

1.1.1. La fonction ζ. — Notons par $s = \sigma + it$ la variable complexe de $\mathbb{C}$. Pour $\Re(s) = \sigma > 1$, appelons ζ_1 la fonction définie par :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ avec } \Re(s) = \sigma > 1$$

Nous savons qu'avec la définition précédente, la fonction ζ_1 est une fonction analytique de s. Notons par $\zeta(s)$ la fonction obtenue par prolongement analytique de $\zeta_1(s)$, alors nous rappelons le théorème suivant [**2**] :

Théorème 1.1.1. - *Les zéros de $\zeta(s)$ satisfont :*
1. *$\zeta(s)$ n'a pas de zéros pour $\Re(s) > 1$;*
2. *le seul pôle de $\zeta(s)$ est au point $s = 1$; son résidu vaut 1 et il est simple ;*
3. *les zéros triviaux de $\zeta(s)$ sont déterminés pour les valeurs $s = -2, -4, \dots$;*
4. *les zéros non triviaux se situent dans la région $0 \leq \Re(s) \leq 1$ dite bande critique et ils sont symétriques respectivement par rapport à l'axe vertical $\Re(s) = \frac{1}{2}$ et l'axe des réels $\Im(s) = 0$.*

On sait aussi que les zéros de $\zeta(s)$ dans la bande critique sont tous des nombres complexes $\neq 0$ (voir page 30 de [**3**]).

La conjecture relative à l'Hypothèse de Riemann est exprimée comme suit :

Conjecture 1.1.2. — *(Hypothèse de Riemann,[**2**]) Tous les zéros non triviaux de $\zeta(s)$ sont sur la droite critique* $\Re(s) = \frac{1}{2}$.

En plus des propriétes citées par le théorème cité ci-dessus, la fonction $\zeta(s)$ vérifie la relation fonctionnelle [**2**] pour $s \neq 1$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{1.1}$$

où $\Gamma(s)$ est la fonction définie sur le demi-plan $\Re(s) > 0$ par :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt,$$

Alors, au lieu d'utiliser la fonctionnelle donnée par (7.1), nous allons utiliser celle présentée par G.H. Hardy [**3**] à savoir la fonction eta de Dirichlet [**2**] :

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

La fonction eta est convergente pour tout $s \in \mathbb{C}$ avec $\Re(s) > 0$ [**2**].

1.1.2. Une Proposition équivalente à l'Hypothèse de Riemann. — Parmi les propositions équivalentes à l'Hypothèse de Riemann celle de la fonction eta de Dirichlet qui s'énnonce comme suit [**2**] :

Equivalence 1.1.3. — *L'Hypothèse de Riemann est équivalente à l'énnoncé que tous les zéros de la fonction eta de Dirichlet :*

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

qui se situent dans la bande critique $0 < \Re(s) < 1$, *sont sur la droite critique* $\Re(s) = \frac{1}{2}$.

1.2. Démonstration que les zéros de $\eta(s)$ vérifient $\sigma = 1/2$

Démonstration. — Notons par $s = \sigma + it$ avec $0 < \sigma < 1$. Considérons maintenant un zéro de $\eta(s)$ qui se trouve dans la bande critique et appelons $s = \sigma + it$ ce zéro, nous avons donc $0 < \sigma < 1$ et $\eta(s) = 0 \Longrightarrow (1 - 2^{1-s})\zeta(s) = 0$. Notons $\zeta(s) = A + iB$, et $\theta = tLog2$, alors :

$$(1-2^{1-s})\zeta(s) = \left[A(1-2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta\right] + i\left[B(1-2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta\right]$$

$(1-2^{1-s})\zeta(s) = 0$ donne le système :

$$A(1-2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta = 0$$
$$B(1-2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0$$

Comme les fonctions *sin* et *cos* ne s'annulent pas simultanément, supposons par exemple que $sin\theta \neq 0$, la première équation du système donne $B = \dfrac{A(1-2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}$, la deuxième équation s'écrit :

$$\frac{A(1-2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}(1-2^{1-\sigma}cos\theta)+2^{1-\sigma}Asin\theta = 0 \Longrightarrow A = 0$$

Par suite, $B = 0 \Longrightarrow \zeta(s) = 0$, il s'ensuit que :
(1.2)

$$\boxed{s\ est\ un\ zéro\ de\, \eta(s)\ \ dans\ la\ bande\ critique\ est\ aussi\ un\ zéro\ de\, \zeta(s)}$$

Reciproquement, si s est un zéro de $\zeta(s)$ dans la bande critique, soit $\zeta(s) = A + iB = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, donc s est aussi un zéro de $\eta(s)$ dans la bande critique. Nous pouvons écrire :
(1.3)

$$\boxed{s\ est\ un\ zéro\ de\, \zeta(s)\ \ dans\ la\ bande\ critique\ est\ aussi\ un\ zéro\ de\, \eta(s)}$$

Ecrivons la fonction η :

$$\eta(s) = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

Remarquons que la fonction η est convergente pour tout $s \in \mathbb{C}$ avec $\Re(s) > 0$, mais non absolument convergente. Comme $\eta(s) = 0$, c'est-à-dire :

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s} = 0$$

ou encore :

$$\forall \tau > 0 \quad \exists n_0, \forall \mathscr{N} > n_0,\ \left|\sum_{n=1}^{\mathscr{N}}\frac{(-1)^{n-1}}{n^s}\right| < \tau$$

Définissons la suite de fonctions $((\eta_n)_{n\in\mathbb{N}^*}(s))$, par :

$$\eta_n(s) = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

avec $s = \sigma + it$ et $t \neq 0$.

Soit s un zéro de η dans la bande critique, soit $\eta(s) = 0$, avec $0 < \sigma < 1$. Par suite, on peut écrire $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. Ce qui donne :

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} = 0$$
$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma} = 0$$

Utilisons la définition de la limite d'une suite, on peut écrire :

$(1.4)\forall\epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2 < \epsilon_1^2$
$(1.5)\forall\epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2 < \epsilon_2^2$

Ce qui donne :

$$0 < \sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

En prenant $\epsilon = \epsilon_1 = \epsilon_2$ et $N > max(n_r, n_i)$, on obtient en faisant la somme membre à membre des deux dernières inégalités, on obtient :

$$(1.6) \qquad 0 < \sum_{k=1}^{N}\frac{1}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2$$

1.2.1. Cas $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. — On suppose que $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Commençons par rappeler le théorème de Hardy (1914) [**2**],[**3**] :

> ***Théorème 1.2.1.*** *- Il y'a une infinité de zéros de $\zeta(s)$ sur la droite critique.*

Des propositions (7.5-7.6), nous déduisons la proposition suivante :

> ***Proposition 1.2.1.*** — Il y'a une infinité de zéros de $\eta(s)$ sur la droite critique.

Soit $s_j = \frac{1}{2} + it_j$ un des zéros de la fonction $\eta(s)$ sur la droite critique, soit $\eta(s_j) = 0$. L'équation (7.9) s'écrit pour s_j :

$$0 < \sum_{k=1}^{N}\frac{1}{k} + 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

ou encore :

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Si on fait tendre N vers $+\infty$, la série $\sum_{k=1}^{N} \frac{1}{k}$ est divergente et devient infinie. Soit :

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Par suite, nous obtenons le résultat suivant :

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{1.7}$$

sinon, nous aurons une contradiction avec le fait que :

$$lim_{N\longrightarrow+\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ est convergente pour } s_j = \frac{1}{2} + it_j$$

Comme $t_j \neq 0$, et qu'il y'a une infinité de zéros sur la droite critique, alors le résultat de la formule donnée par (7.11) est indépendant de t_j. Revenons maintenant à $s = \sigma + it$ un zéro de $\eta(s)$ dans la bande critique, soit $\eta(s) = 0$. Prenons $\sigma = \frac{1}{2}$. En partant de la définition de la limite des suites, appliquée ci-dessus, nous obtenons :

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

avec sans aucune contradiction. De la proposition (7.5) il s'ensuit que $\zeta(s) = \zeta(\frac{1}{2} + it) = 0$. Il existe donc des zéros de $\zeta(s)$ sur la droite critique $\Re(s) = \frac{1}{2}$.

1.2.2. Cas $0 < \sigma < \frac{1}{2}$. —

1.2.2.1. *Cas où il n'existe pas de zéros de $\eta(s)$ avec $s = \sigma + it$ et $0 < \sigma < \frac{1}{2}$.* — En utilisant, pour ce cas, le point 4 du théorème (7.1.2), nous déduisons que la fonction $\eta(s)$ n'a pas de zéros avec $s = \sigma + it$ et $\frac{1}{2} < \sigma < 1$. Par suite, d'après la proposition (7.5), il s'ensuit que la fonction $\zeta(s)$ a ses zéros seulement sur la droite critique $\Re(s) = \sigma = \frac{1}{2}$ et **l'Hypothèse de Riemann est vraie**.

1.2.2.2. *Cas où il existe des zéros de $\eta(s)$ avec $s = \sigma + it$ et $0 < \sigma < \dfrac{1}{2}$.*
— Supposons qu'il existe $s = \sigma + it$ un zéro de $\eta(s)$ soit $\eta(s) = 0$ avec $0 < \sigma < \frac{1}{2} \Longrightarrow s \in$ à la bande critique. Nous écrivons l'équation (7.9), :

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

ou :

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

Or $2\sigma < 1$, il s'ensuit que $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \dfrac{1}{k^{2\sigma}}$ tende vers $+\infty$ et nous obtenons par suite :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty \tag{1.8}$$

Là aussi, le résultat ci-dessus est indépendant de t.

1.2.3. Cas $\dfrac{1}{2} < \Re(s) < 1$. — Soit $s = \sigma + it$ le zéro de $\eta(s)$ dans $0 < \Re(s) < \dfrac{1}{2}$, objet du paragraphe précédent. Suivant le point 4 du théorème 7.1.2, le nombre complexe $s' = 1 - \sigma + it = \sigma' + it'$ avec $\sigma' = 1 - \sigma$ et $t' = t$ est aussi un zéro de la fonction $\eta(s)$ dans la bande $\dfrac{1}{2} < \Re(s) < 1$. En appliquant (7.9), nous obtenons :

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{1.9}$$

Comme $\sigma < \frac{1}{2}$, d'où $2\sigma' = 2(1-\sigma) > 1$, alors la série $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ est convergente vers une constante positive non nulle $C(\sigma')$. De l'équation (7.12), nous déduisons que :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} > -\infty$$

Considérons maintenant la fonction $F_N(u,t)$, $N \in \mathbb{N}^* \geq 2$, définie par :

$$F_N(u,t) = \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^u k'^u} =$$

$$= \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} cos(tLog(k/k'))e^{-uLog(kk')}, \quad u \in]0,1[,\ t \in]0,+\infty[$$

La fonction $F_N(u,t)$ est continue pour $\forall N \geq 2$ et $(u,t) \in]0,1[\times]0,+\infty[$, et nous avons obtenu précédement que $\forall t > 0$, pour $N \longrightarrow +\infty$:

$$\begin{cases} \displaystyle\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} \quad \text{pour } u = \sigma' = 1-\sigma > \frac{1}{2} \\ \displaystyle\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty \quad \text{pour } \ u = \frac{1}{2} \\ \displaystyle\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty \quad \text{pour } u = \sigma < \frac{1}{2} \end{cases}$$

Fixons $t = t_0 > 0$ une valeur arbitraire et écrivons que $F_N(u,t_0)$ est continue au point $u = 1/2$, on peut écrire :

$$\forall \epsilon > 0, \exists \delta \text{ tel que } \forall\, u \,/\, |u - 1/2| < \delta \Longrightarrow |F_N(u,t_0) - F_N(1/2,t_0)| < \epsilon$$

Soit $u = \sigma' \in]0,1[$ avec $\sigma' > \frac{1}{2}$ vérifiant $\sigma' - \frac{1}{2} < \delta$, on a alors l'équation :

$$|F_N(\sigma',t_0) - F_N(1/2,t_0)| < \epsilon \Longrightarrow$$

$$-\epsilon + F_N(\sigma',t_0) < \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_0Log(k/k'))}{\sqrt{k}\sqrt{k'}} < \epsilon + F_N(\sigma',t_0)$$

$$\Longrightarrow -\epsilon + \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_0Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_0Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Comme pour t, u fixés, la fonction F_N est définie pour tout entier $N \geq 2$, faisons alors tendre N vers $+\infty$, nous obtenons :

$$-\epsilon + \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma'} k'^{\sigma'}} \leq \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

$$\Longrightarrow -\epsilon - \frac{C(\sigma')}{2} \leq \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty$$

D'où la contradiction avec $C(\sigma')$ bornée. Par suite, l'hypothèse qu'il existe des zéros de $\eta(s)$ dans l'intervalle $\frac{1}{2} < \Re(s) < 1$ étudiée au début de cette section est fausse. Il s'ensuit que la fonction $\eta(s)$ ne s'annule pas dans les intervalles $0 < \Re(s) < \frac{1}{2}$ et $\frac{1}{2} < \Re(s) < 1$ et par suite la fonction $\eta(s)$ a ses zéros non triviaux sur la droite critique $\Re(s) = \frac{1}{2}$ de la bande critique.

□

1.3. Conclusion

En résumé : pour nos démonstrations, nous avons fait usage de la convergence simple de la fonction $\eta(s)$ de Dirichlet :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

dans la bande critique $0 < \Re(s) < 1$, en obtenant :

- $\eta(s)$ s'annule pour $0 < \sigma = \Re(s) = \dfrac{1}{2}$;

- $\eta(s)$ ne s'annule pas pour $0 < \sigma = \Re(s) < \dfrac{1}{2}$ et $\dfrac{1}{2} < \sigma = \Re(s) < 1$.

Par suite, tous les zéros non triviaux de $\eta(s)$ dans la bande critique $0 < \Re(s) < 1$ s'annulent sur la droite critique $\Re(s) = \dfrac{1}{2}$. En appliquant la proposition équivalente à l'Hypothèse de Riemann 7.1.5, tous les zéros non triviaux de la fonction $\zeta(s)$ se trouvent sur la droite critique $\Re(s) = \dfrac{1}{2}$. La démonstration de l'Hypothèse de Riemann est ainsi achevée.

Nous annonçons donc le théorème important :

> ***Théorème 1.3.1.*** - *L'Hypothèse de Riemann est vraie : tous les zéros non triviaux de la fonction $\zeta(s)$ avec $s = \sigma + it$ se situent sur l'axe vertical $\Re(s) = \dfrac{1}{2}$.*

2

Is The Riemann Hypothesis True (v1)?

2.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 2.1.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$ except at most for a finite number of zeros.

2.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 2.1.2. — . *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line. We have also the theorem (see page 16, [**3**]):

Theorem 2.1.3. — . *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]). Then, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

The Riemann Hypothesis is formulated as:

Conjecture 2.1.4. — . *(The Riemann Hypothesis,*[**2**]*) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

In addition to the properties cited by the theorem 7.1.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C} \backslash \{0, 1\}$:

$$\zeta(1 - s) = 2^{1-s} \pi^{-s} cos \frac{s\pi}{2} \Gamma(s) \zeta(s) \tag{2.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt,$$

So, instead of using the functional given by (7.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s}) \zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

2.1.2. A Equivalent statement to the Riemann Hypothesis. — Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 2.1.5. — . The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{2.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

The series (7.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{2.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{2.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2.2. Proof that the zeros of $\eta(s)$ verify $\sigma = 1/2$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longrightarrow (1 - 2^{1-s})\zeta(s) = 0$. Let us denote $\zeta(s) = A + iB$, and $\theta = tLog2$, then :

$$(1-2^{1-s})\zeta(s) = \left[A(1 - 2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta\right] + i\left[B(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta\right]$$

$(1 - 2^{1-s})\zeta(s) = 0$ gives the system:

$$A(1 - 2^{1-\sigma}cos\theta) - 2^{1-\sigma}Bsin\theta = 0$$
$$B(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0$$

As the functions *sin* and *cos* are not equal to 0 simultaneously, we suppose for example that $sin\theta \neq 0$, the first equation of the system gives $B = \dfrac{A(1 - 2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}$, the second equation is written as :

$$\frac{A(1 - 2^{1-\sigma}cos\theta)}{2^{1-\sigma}sin\theta}(1 - 2^{1-\sigma}cos\theta) + 2^{1-\sigma}Asin\theta = 0 \Longrightarrow A = 0$$

Then, $B = 0 \Longrightarrow \zeta(s) = 0$, it follows that:

(2.5)

s is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = A + iB = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:
(2.6)

s is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall \mathcal{N} > n_0, \ \left| \sum_{n=1}^{\mathcal{N}} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$
$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(2.7) $\forall \epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2 < \epsilon_1^2$
(2.8) $\forall \epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2 < \epsilon_2^2$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{2.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{2.10}$$

or $\rho(s) = 0$.

2.2.1. Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. — We suppose that $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 2.2.1.*** — . *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (7.5-7.6), it follows the proposition :

> ***Proposition 2.2.2.*** — . There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (7.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty}\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}=-\infty} \tag{2.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}(-1)^{k-1}\frac{1}{k^{s_j}}=0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j=\frac{1}{2}+it_j$$

Let $s=\sigma+it$ one zero of $\eta(s)$ on the critical line $\Longrightarrow \eta(s)=0$. We take $\sigma=\dfrac{1}{2}$. Starting from the definition of the limit of sequences, applied above, we obtain:

$$\sum_{k=1}^{+\infty}\frac{1}{k}\leq 2\epsilon^2-2\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

with any contradiction. From the proposition (7.5), it follows that $\zeta(s)=\zeta(\frac{1}{2}+it)=0$. There are therefore zeros of $\zeta(s)$ on the critical line $\Re(s)=\dfrac{1}{2}$.

2.2.2. Case $0<\Re(s)<\dfrac{1}{2}$. —

2.2.2.1. *Case there is no zeros of* $\eta(s)$ *with* $s=\sigma+it$ *and* $0<\sigma<\dfrac{1}{2}$. — Using, for this case, point 4 of theorem (7.1.2), we deduce that the function $\eta(s)$ has no zeros with $s=\sigma+it$ and $\dfrac{1}{2}<\sigma<1$. Then, from the proposition (7.5), it follows that the function $\zeta(s)$ has all its nontrivial zeros only on the critical line $\Re(s)=\sigma=\dfrac{1}{2}$ and the **Riemann Hypothesis is true**.

2.2.2.2. *Case where there are zeros of* $\eta(s)$ *with* $s=\sigma+it$ *and* $0<\sigma<\dfrac{1}{2}$. — Suppose that there exists $s=\sigma+it$ one zero of $\eta(s)$ or $\eta(s)=0\Longrightarrow \rho^2(s)=0$ with $0<\sigma<\frac{1}{2}\Longrightarrow s$ lies inside the critical band. We write the equation (7.9):

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}<2\epsilon^2$$

or:

$$\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}<2\epsilon^2-2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}\longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}} = -\infty} \tag{2.12}$$

2.2.3. Case $\frac{1}{2} < \Re(s) < 1$. — Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. According to point 4 of theorem 7.1.2, the complex number $s' = 1-\sigma+it = \sigma'+it'$ with $\sigma' = 1-\sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$, is also a zero of the function $\eta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, that is $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (7.9), we get:

$$0 < \sum_{k=1}^{N}\frac{1}{k^{2\sigma'}} + 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} < 2\epsilon^2 \tag{2.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \frac{\pi^2}{6} = \sum_{k=1}^{+\infty}\frac{1}{k^2} \leq \sum_{k=1}^{+\infty}\frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (7.12), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{2.14}$$

Then, we have the two following cases:

1)- There exists an infinity of complex numbers $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$. For each s'_l, the left member of the equation (7.13) above is finite and depends of σ'_l and t'_l, but the right member is a function only of σ'_l equal to $\zeta(2\sigma'_l)$. Hence the contradiction because for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of σ", therefore, the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l+it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the paragraph (2.2.2.2) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (7.1.5), it follows that **the Riemann hypothesis is verified**.

2)- There is at most a single zero $s_0 = \sigma_0 + it_0$ of $\eta(s)$ with $\sigma_0 \in]0, 1/2[, t_0 > 0$ such that $\eta(s_0) = 0$. Let us call this zero *isolated zero* that we denote by (IZ). Therefore, the interval $]1/2, 1[$ contains a single zero $s'_0 = 1-\sigma_0+it_0$. Since the critical line contains an infinity of zeros of

$\zeta(s) = 0$, it follows that all the nontrivial zeros of $\zeta(s)$ are on the critical line $\sigma = \dfrac{1}{2}$, except the 4 zeros relative to (IZ). Here too, we deduce that **the Riemann Hypothesis holds** except at most for the (IZ) in the critical band. $\square$

2.3. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$ except at most for the (IZ) (with its symmetrical) inside the critical band.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ vanish on the critical line $\Re(s) = \dfrac{1}{2}$ except at most at (IZ) (with its symmetrical). Applying the equivalent proposition to the Riemann Hypothesis 7.1.5, all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$ except at most at (IZ) (with its symmetrical) inside the critical band. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 2.3.1. — . *All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$, except for at most four zeros of respective affixes $(\sigma_0, t_0), (1 - \sigma_0, t_0), (\sigma_0, -t_0), (1 - \sigma_0, -t_0)$, belonging to the critical band.*

3

Is The Riemann Hypothesis True (v2)?

3.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 3.1.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$.

3.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s=1$, then we recall the following theorem [**2**]:

Theorem 3.1.2. — *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s=1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s=-2,-4,\dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 3.1.3. — *(The Riemann Hypothesis,[**2**]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

In addition to the properties cited by the theorem 7.1.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C} \backslash \{0,1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{3.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (7.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 3.1.4. — *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]).

3.1.2. A Equivalent statement to the Riemann Hypothesis.

— Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 3.1.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{3.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$. The series (7.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{3.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{3.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

3.2. Preliminaries of the proof that the zeros of $\eta(s)$ verify $\sigma = 1/2$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:
(3.5)

s, is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:
(3.6)

s, is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall \mathscr{N} > n_0, \ \left| \sum_{n=1}^{\mathscr{N}} \frac{(-1)^{n-1}}{n^s} \right| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} = 0$$
$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(3.7) $\forall\epsilon_1 > 0 \quad \exists n_r, \forall N > n_r \quad |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow |\Re(\eta(s)_N)|^2 < \epsilon_1^2$

(3.8) $\forall\epsilon_2 > 0 \quad \exists n_i, \forall N > n_i \quad |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow |\Im(\eta(s)_N)|^2 < \epsilon_2^2$

Then:

$$0 < \sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2 \tag{3.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{3.10}$$

or $\rho(s) = 0$.

3.3. Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$

We suppose that $\sigma = \dfrac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 3.3.1.*** — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (7.5-7.6), it follows the proposition :

> ***Proposition 3.3.2.*** — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (7.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{3.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}(-1)^{k-1}\frac{1}{k^{s_j}}=0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j=\frac{1}{2}+it_j$$

3.4. Case $0 < \Re(s) < \dfrac{1}{2}$

3.4.1. Case there is no zeros of $\eta(s)$ with $s=\sigma+it$ and $0<\sigma<\dfrac{1}{2}$. — As there is no zeros of $\eta(s)$ with $s=\sigma+it$ and $0<\sigma<\dfrac{1}{2}$, it follows from the proposition (7.5) that $\zeta(s)$ has also no zeros with $0<\sigma<\dfrac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s=\sigma+it$ and $\dfrac{1}{2}<\sigma<1$. We deduce from the proposition (7.6) that the function $\eta(s)$ has no zeros with $s=\sigma+it$ and $\dfrac{1}{2}<\sigma<1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s)=\sigma=\dfrac{1}{2}$ and from the equivalent statement 7.1.5, we conclude that **the Riemann Hypothesis is true**.

3.4.2. Case where there are zeros of $\eta(s)$ with $s=\sigma+it$ and $0<\sigma<\dfrac{1}{2}$. — Suppose that there exists $s=\sigma+it$ one zero of $\eta(s)$ or $\eta(s)=0 \Longrightarrow \rho^2(s)=0$ with $0<\sigma<\frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (7.9):

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}<2\epsilon^2$$

or:

$$\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}<2\epsilon^2-2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}$$

But $2\sigma<1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}\longrightarrow+\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}=-\infty} \tag{3.12}$$

3.5. Case $\dfrac{1}{2} < \Re(s) < 1$

Let $s=\sigma+it$ be the zero of $\eta(s)$ in $0<\Re(s)<\frac{1}{2}$, object of the previous paragraph. From the proposition (7.5), $\zeta(s)=0$. According to point 4 of theorem 7.1.2, the complex number $s'=1-\sigma+it=\sigma'+it'$ with $\sigma'=1-\sigma$,

$t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (7.6) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (7.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} < 2\epsilon^2 \tag{3.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (7.12), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{3.14}$$

Let $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$. For each s'_l, the left member of the equation (7.13) above is finite and depends of $\sigma'_l = 1-\sigma_l$ and $t'_l = t_l$, but the right member is a function only of σ'_l equal to $-\zeta(2\sigma'_l)/2$. Hence the contradiction because for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of σ", then in particular for all σ" with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of $\sigma" \Longrightarrow$ the equation (7.13) is false, then, the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l + it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the second case of the paragraph (6.4) above concerning the case $0 < \Re(s) < \dfrac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (7.1.5), it follows that **the Riemann hypothesis is verified**.

□

3.6. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \dfrac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (7.1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

> ***Theorem 3.6.1.*** — *The Riemann Hypothesis is true:*
> *All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.*

4

Is The Riemann Hypothesis True? Yes, It Is. (V1)

4.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 4.1.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give a proof that $\sigma = \dfrac{1}{2}$.

4.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 4.1.2. — *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 4.1.3. — *(The Riemann Hypothesis,[**2**]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

In addition to the properties cited by the theorem 7.1.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C} \backslash \{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{4.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (7.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 4.1.4. — *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

It is also known that the zeros of $\zeta(s)$ inside the critical strip are all complex numbers $\neq 0$ (see page 30 in [**3**]).

4.1.2. A Equivalent statement to the Riemann Hypothesis. —

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 4.1.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \sigma > 1 \tag{4.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$. The series (7.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{4.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{4.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

4.2. Preliminaries of the proof that the zeros of $\eta(s)$ verify $\sigma = 1/2$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1-2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(4.5)

s, is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(4.6)

s, is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \Big|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}\Big| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$
$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(4.7) $\forall \epsilon_1 > 0 \, \exists n_r, \forall N > n_r, \, |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2$

(4.8) $\forall \epsilon_2 > 0 \, \exists n_i, \forall N > n_i, \, |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}} < 2\epsilon^2 \tag{4.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{4.10}$$

or $\rho(s) = 0$.

4.3. Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$

We suppose that $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 4.3.1.*** — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (7.5-7.6), it follows the proposition :

> ***Proposition 4.3.2.*** — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (7.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N\longrightarrow+\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_jLog(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{4.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

4.4. Case $0 < \Re(s) < \dfrac{1}{2}$

4.4.1. Case there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. — As there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$, it follows from the proposition (7.5) that $\zeta(s)$ has also no zeros with $0 < \sigma < \frac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. We deduce from the proposition (7.6) that the function $\eta(s)$ has no zeros with $s = \sigma + it$ and $\frac{1}{2} < \sigma < 1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \frac{1}{2}$ and from the equivalent statement 7.1.5, we conclude that **the Riemann Hypothesis is true**.

4.4.2. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \frac{1}{2}$. — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (7.9):

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} = -\infty} \tag{4.12}$$

4.5. Case $\dfrac{1}{2} < \Re(s) < 1$

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (7.5), $\zeta(s) = 0$. According to point 4 of theorem 7.1.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$,

$t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (7.6) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (7.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} < 2\epsilon^2 \tag{4.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (7.12), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{4.14}$$

Let $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$.

Firstly, we suppose that $t_l \neq 0$. For each s'_l, the left member of the equation (7.13) above is finite and depends of $\sigma'_l = 1 - \sigma_l$ and $t'_l = t_l$, but the right member is a function only of σ'_l equal to $-\zeta(2\sigma'_l)/2$. Hence the contradiction because for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of σ", then in particular for all σ" with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of $\sigma" \Longrightarrow$ the equation (7.13) is false.

Secondly, we suppose that $t_l = 0 \Longrightarrow t'_l = 0$. The equation (7.13) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'_l}k'^{\sigma'_l}} = -\frac{C(\sigma'_l)}{2} = -\frac{\zeta(2\sigma'_l)}{2} > -\infty \tag{4.15}$$

Then $s'_l = \sigma'_l > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s'_l) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'_l}} = 0 \tag{4.16}$$

Let us define the sequence S_m as:

$$S_m(s'_l) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'_l}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'_l}} = S_m(\sigma'_l) \tag{4.17}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) \tag{4.18}$$

We have also:

$$S_1(\sigma'_l) = 1 > 0 \tag{4.19}$$

$$S_2(\sigma'_l) = 1 - \frac{1}{2^{\sigma'_l}} > 0 \quad \textit{because } 2^{\sigma'_l} > 1 \tag{4.20}$$

$$S_3(\sigma'_l) = S_2(\sigma'_l) + \frac{1}{3^{\sigma'_l}} > 0 \tag{4.21}$$

We proceed by recurrence, we suppose that $S_m(\sigma'_l) > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma'_l) = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'_l}} = S_m(\sigma'_l) + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$, it gives:

$$S_{m+1}(\sigma'_l) = S_m(\sigma'_l) + \frac{(-1)^{2q}}{(m+1)^{\sigma'_l}} = S_m(\sigma'_l) + \frac{1}{(m+1)^{\sigma'_l}} > 0 \Rightarrow S_{m+1}(\sigma'_l) > 0$$

2. $m = 2q+1$, we can write $S_{m+1}(\sigma'_l)$ as:

$$S_{m+1}(\sigma'_l) = S_{m-1}(\sigma'_l) + \frac{(-1)^{m-1}}{m^{\sigma'_l}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$$

We have $S_{m-1}(\sigma'_l) > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'_l}} + \frac{(-1)^m}{(m+1)^{\sigma'_l}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'_l}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'_l}} = \frac{1}{(2q+1)^{\sigma'_l}} - \frac{1}{(2q+2)^{\sigma'_l}} > 0 \tag{4.22}$$

and $S_{m+1}(\sigma'_l) > 0$.

Then all the terms $S_m(\sigma'_l)$ of the sequence S_m are great then 0, it follows that $lim_{m\longrightarrow+\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) > 0$ and $\eta(\sigma'_l) < +\infty$ because $\Re(s'_l) = \sigma'_l > 0$ and $\eta(s'_l)$ is convergent. We deduce the contradiction that s'_l is a zero of $\eta(s)$ and the equation (7.14) is false. Then, the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l + it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the second case of the paragraph (6.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (7.1.5), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following proposition:

> ***Proposition 4.5.1.*** — For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

4.6. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (7.1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 4.6.1. — *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

5

Is The Riemann Hypothesis True? Yes, It Is. (V2)

5.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 5.1.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

5.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 5.1.2. — *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \le \Re(s) \le 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 5.1.3. — *(The Riemann Hypothesis,[**2**]) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

In addition to the properties cited by the theorem 7.1.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C}\backslash\{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s}cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{5.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (7.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 5.1.4. — *For all $t \in \mathbb{R}$, $\zeta(1 + it) \ne 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

5.1.2. A Equivalent statement to the Riemann Hypothesis. — Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 5.1.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \sigma > 1 \tag{5.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (7.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{5.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{5.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

5.2. Preliminaries of the proof that the zeros of $\eta(s)$ verify $\sigma = 1/2$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1-2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(5.5)

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(5.6)

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \Big| \sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s} \Big| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(5.7) $\forall \epsilon_1 > 0 \, \exists n_r, \forall N > n_r, \; |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2$

(5.8) $\forall \epsilon_2 > 0 \, \exists n_i, \forall N > n_i, \; |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2 \tag{5.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{5.10}$$

or $\rho(s) = 0$.

5.3. Case $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$

We suppose that $\sigma = \frac{1}{2} \Longrightarrow 2\sigma = 1$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 5.3.1.*** — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (7.5-7.6), it follows the proposition :

> ***Proposition 5.3.2.*** — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (7.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{5.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}(-1)^{k-1}\frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

5.4. Case $0 < \Re(s) < \dfrac{1}{2}$

5.4.1. Case there is no zeros of $\eta(s)$ with $s = \sigma+it$ and $0 < \sigma < \dfrac{1}{2}$. — As there is no zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$, it follows from the proposition (7.5) that $\zeta(s)$ has also no zeros with $0 < \sigma < \dfrac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s = \sigma+it$ and $\dfrac{1}{2} < \sigma < 1$. We deduce from the proposition (7.6) that the function $\eta(s)$ has no zeros with $s = \sigma+it$ and $\dfrac{1}{2} < \sigma < 1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s) = \sigma = \dfrac{1}{2}$ and from the equivalent statement 7.1.5, we conclude that **the Riemann Hypothesis is true**.

5.4.2. Case where there are zeros of $\eta(s)$ with $s = \sigma + it$ and $0 < \sigma < \dfrac{1}{2}$. — Suppose that there exists $s = \sigma + it$ one zero of $\eta(s)$ or $\eta(s) = 0 \Longrightarrow \rho^2(s) = 0$ with $0 < \sigma < \frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (7.9):

$$0 < \sum_{k=1}^{N}\frac{1}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N}\frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}$$

But $2\sigma < 1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}} = -\infty} \tag{5.12}$$

5.5. Case $\dfrac{1}{2} < \Re(s) < 1$

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the previous paragraph. From the proposition (7.5), $\zeta(s) = 0$. According to point 4 of theorem 7.1.2, the complex number $s' = 1-\sigma+it = \sigma'+it'$ with $\sigma' = 1-\sigma$,

$t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (7.6) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (7.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} < 2\epsilon^2 \tag{5.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} \leq \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (7.12), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.14}$$

Let $s_l = \sigma_l + it_l$ with $\sigma_l \in]0, 1/2[$ such that $\eta(s_l) = 0$.

Firstly, we suppose that $t_l \neq 0$. For each $s'_l = \sigma'_l + it'_l = 1 - \sigma_l + it_l$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'_l Log(k/k'))}{k^{\sigma'_l}k'^{\sigma'_l}} = -\frac{C(\sigma'_l)}{2} = -\frac{\zeta(2\sigma'_l)}{2} > -\infty \tag{5.15}$$

the left member of the equation (7.23) above is finite and depends of σ'_l and t'_l, but the right member is a function only of σ'_l equal to $-\zeta(2\sigma'_l)/2$. But for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$ depends only of σ", then in particular for all σ" with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of σ", it follows that the left term of (7.23) is infinite, then the contradiction with $-\frac{C(\sigma'_l)}{2} = -\frac{\zeta(2\sigma'_l)}{2} > -\infty$.

(5.16) We conclude that the equation (7.23) is false for the cases $t'_l \neq 0$.

Secondly, we suppose that $t_l = 0 \Longrightarrow t'_l = 0$. The equation (7.13) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'_l}k'^{\sigma'_l}} = -\frac{C(\sigma'_l)}{2} = -\frac{\zeta(2\sigma'_l)}{2} > -\infty \tag{5.17}$$

Then $s'_l = \sigma'_l > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s'_l) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'_l}} = 0 \tag{5.18}$$

Let us define the sequence S_m as:

$$S_m(s'_l) = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'_l}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'_l}} = S_m(\sigma'_l) \tag{5.19}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) \tag{5.20}$$

We have also:

$$S_1(\sigma'_l) = 1 > 0 \tag{5.21}$$

$$S_2(\sigma'_l) = 1 - \frac{1}{2^{\sigma'_l}} > 0 \quad because\ 2^{\sigma'_l} > 1 \tag{5.22}$$

$$S_3(\sigma'_l) = S_2(\sigma'_l) + \frac{1}{3^{\sigma'_l}} > 0 \tag{5.23}$$

We proceed by recurrence, we suppose that $S_m(\sigma'_l) > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma'_l) = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'_l}} = S_m(\sigma'_l) + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$, it gives:

$$S_{m+1}(\sigma'_l) = S_m(\sigma'_l) + \frac{(-1)^{2q}}{(m+1)^{\sigma'_l}} = S_m(\sigma'_l) + \frac{1}{(m+1)^{\sigma'_l}} > 0 \Rightarrow S_{m+1}(\sigma'_l) > 0$$

2. $m = 2q+1$, we can write $S_{m+1}(\sigma'_l)$ as:

$$S_{m+1}(\sigma'_l) = S_{m-1}(\sigma'_l) + \frac{(-1)^{m-1}}{m^{\sigma'_l}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'_l}}$$

We have $S_{m-1}(\sigma'_l) > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'_l}} + \frac{(-1)^{m}}{(m+1)^{\sigma'_l}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'_l}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'_l}} = \frac{1}{(2q+1)^{\sigma'_l}} - \frac{1}{(2q+2)^{\sigma'_l}} > 0 \tag{5.24}$$

and $S_{m+1}(\sigma'_l) > 0$.

Then all the terms $S_m(\sigma'_l)$ of the sequence S_m are great then 0, it follows that $lim_{m \longrightarrow +\infty} S_m(s'_l) = \eta(s'_l) = \eta(\sigma'_l) > 0$ and $\eta(\sigma'_l) < +\infty$ because $\Re(s'_l) = \sigma'_l > 0$ and $\eta(s'_l)$ is convergent. We deduce the contradiction that s'_l is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (7.14) is false for the case}\, t'_l = t_l = 0.} \tag{5.25}$$

From (7.26-7.22), we conclude that the function $\eta(s)$ has no zeros for all $s'_l = \sigma'_l + it'_l$ with $\sigma'_l \in]1/2, 1[$, it follows that the second case of the paragraph (6.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (7.1.5), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following known proposition:

> ***Proposition 5.5.1.*** — For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

5.6. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:
- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (7.1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

> ***Theorem 5.6.1.*** — *The Riemann Hypothesis is true:*
> *All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.*

6

Is The Riemann Hypothesis True? Yes, It Is. (V3)

To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
To the memory of my friend Abdelkader Sellal (1947 - 2017)

6.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

Conjecture 6.1.1. — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

6.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 6.1.2. — *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \ldots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \frac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \frac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 6.1.3. — *(The Riemann Hypothesis,*[**2**]*) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$.*

In addition to the properties cited by the theorem 7.1.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C} \backslash \{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{6.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (7.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 6.1.4. — *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

6.1.2. A Equivalent statement to the Riemann Hypothesis. —

Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 6.1.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{6.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (7.2) is convergent, and represents $(1 - 2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{6.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{6.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

6.2. Preliminaries of the proof that the zeros of $\eta(s)$ verify $\sigma = 1/2$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1 - 2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(6.5)

s, is one zero of $\eta(s)$ that falls in the critical strip, is also one zero of $\zeta(s)$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1 - 2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(6.6)

s, is one zero of $\zeta(s)$ that falls in the critical strip, is also one zero of $\eta(s)$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \Big|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}\Big| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty} \eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n\longrightarrow+\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(6.7) $\forall \epsilon_1 > 0\, \exists n_r, \forall N > n_r,\ |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2$

(6.8) $\forall \epsilon_2 > 0\, \exists n_i, \forall N > n_i,\ |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'} sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} < 2\epsilon^2 \tag{6.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{6.10}$$

or $\rho(s) = 0$.

6.3. Case $\sigma = \frac{1}{2}$

We suppose that $\sigma = \frac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 6.3.1.*** — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (7.5-7.6), it follows the proposition :

> ***Proposition 6.3.2.*** — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (7.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{6.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}(-1)^{k-1}\frac{1}{k^{s_j}}=0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j=\frac{1}{2}+it_j$$

6.4. Case $0 < \Re(s) < \dfrac{1}{2}$

6.4.1. Case there is no zeros of $\eta(s)$ with $s=\sigma+it$ and $0<\sigma<\dfrac{1}{2}$. — As there is no zeros of $\eta(s)$ with $s=\sigma+it$ and $0<\sigma<\dfrac{1}{2}$, it follows from the proposition (7.5) that $\zeta(s)$ has also no zeros with $0<\sigma<\dfrac{1}{2}$. Using the symmetry of $\zeta(s)$, there is no zeros of $\zeta(s)$ with $s=\sigma+it$ and $\dfrac{1}{2}<\sigma<1$. We deduce from the proposition (7.6) that the function $\eta(s)$ has no zeros with $s=\sigma+it$ and $\dfrac{1}{2}<\sigma<1$. Then, the function $\eta(s)$ has all its nontrivial zeros only on the critical line $\Re(s)=\sigma=\dfrac{1}{2}$ and from the equivalent statement 7.1.5, we conclude that **the Riemann Hypothesis is true**.

6.4.2. Case where there are zeros of $\eta(s)$ with $s=\sigma+it$ and $0<\sigma<\dfrac{1}{2}$. — Suppose that there exists $s=\sigma+it$ one zero of $\eta(s)$ or $\eta(s)=0 \Longrightarrow \rho^2(s)=0$ with $0<\sigma<\frac{1}{2} \Longrightarrow s$ lies inside the critical band. We write the equation (7.9):

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}<2\epsilon^2$$

or:

$$\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}<2\epsilon^2-2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}$$

But $2\sigma<1$, it follows that $lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}\longrightarrow+\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^{\sigma}k'^{\sigma}}=-\infty} \tag{6.12}$$

6.5. Case $\dfrac{1}{2} < \Re(s) < 1$

Let $s=\sigma+it$ be the zero of $\eta(s)$ in $0<\Re(s)<\frac{1}{2}$, object of the previous paragraph. From the proposition (7.5), $\zeta(s)=0$. According to point 4 of theorem 7.1.2, the complex number $s'=1-\sigma+it=\sigma'+it'$ with $\sigma'=1-\sigma$,

$t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (7.6) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (7.9), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{6.13}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (7.12), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{6.14}$$

Firstly, we suppose that $t = 0 \Longrightarrow t' = 0$. The equation (7.13) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{1}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{6.15}$$

Then $s' = \sigma' > 1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s') = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^{s'}} = 0 \tag{6.16}$$

Let us define the sequence S_m as:

$$S_m(s') = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{s'}} = \sum_{n=1}^{m} \frac{(-1)^{n-1}}{n^{\sigma'}} = S_m(\sigma') \tag{6.17}$$

From the definition of S_m, we obtain :

$$lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') \tag{6.18}$$

We have also:

$$S_1(\sigma') = 1 > 0 \tag{6.19}$$

$$S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad because\ 2^{\sigma'} > 1 \tag{6.20}$$

$$S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0 \tag{6.21}$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$, it gives:

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \Rightarrow S_{m+1}(\sigma') > 0$$

2. $m = 2q + 1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^m}{(m+1)^{\sigma'}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0 \tag{6.22}$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (7.14) is false for the case } t' = t = 0.} \tag{6.23}$$

Secondly, we suppose that $t \neq 0$. For each $s' = \sigma' + it' = 1 - \sigma + it$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{6.24}$$

the left member of the equation (7.23) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$. But for all σ" so that $2\sigma" > 1$, we have $\zeta(2\sigma")$:

$$\zeta(2\sigma") = \zeta_1(2\sigma") = \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma"}} < +\infty$$

It depends only of σ", then in particular for all σ" with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of σ", then the result giving by the equation (7.23) is false:

$$\boxed{\text{It follows that the equation (7.23) is false for the cases } t' \neq 0.} \tag{6.25}$$

From (7.22-7.26), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the second case of the paragraph (6.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (7.1.5), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following known proposition:

Proposition 6.5.1. — For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

6.6. Conclusion

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;
- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \frac{1}{2}$ and $\frac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \frac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (7.1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 6.6.1. — *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \frac{1}{2}$.

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Is The Riemann Hypothesis True? Yes, It Is. (V4)

To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
To the memory of my friend Abdelkader Sellal (1946 - 2017)

7.1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [**1**]:

> ***Conjecture 7.1.1.*** — *Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:*
>
> $$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \textit{ for } \Re(s) = \sigma > 1$$
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> *over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :*
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> $$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

7.1.1. The function ζ. — We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [**2**]:

Theorem 7.1.2. — *The function $\zeta(s)$ satisfies the following :*
1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
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The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

The Riemann Hypothesis is formulated as:

Conjecture 7.1.3. — *(The Riemann Hypothesis,*[**2**]*) All nontrivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.*

In addition to the properties cited by the theorem 7.1.2 above, the function $\zeta(s)$ satisfies the functional relation [**2**] called also the reflection functional equation for $s \in \mathbb{C} \backslash \{0, 1\}$:

$$\zeta(1-s) = 2^{1-s}\pi^{-s} cos\frac{s\pi}{2}\Gamma(s)\zeta(s) \tag{7.1}$$

where $\Gamma(s)$ is the *gamma function* defined only for $\Re(s) > 0$, given by the formula :

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, \quad \Re(s) > 0$$

So, instead of using the functional given by (7.1), we will use the one presented by G.H. Hardy [**3**] namely Dirichlet's eta function [**2**]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [**2**].

We have also the theorem (see page 16, [**3**]):

Theorem 7.1.4. — *For all $t \in \mathbb{R}$, $\zeta(1+it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

7.1.2. A Equivalent statement to the Riemann Hypothesis. — Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet function eta which is stated as follows [**2**]:

Equivalence 7.1.5. — The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \sigma > 1 \tag{7.2}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \frac{1}{2}$.

The series (7.2) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([**3**], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{7.3}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{7.4}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

7.2. Preliminaries of the proof that the zeros of $\eta(s)$ verify $\sigma = 1/2$

Proof. — . We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1-2^{1-s})\zeta(s) = 0$. We verifies easily the two propositions:

(7.5)

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of } \zeta(s)}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

(7.6)

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of } \eta(s)}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-sLogn} = \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-(\sigma+it)Logn} =$$

$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}.e^{-itLogn}$$

$$= \sum_{n=1}^{+\infty}(-1)^{n-1}e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. Let s be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = 0$$

or:

$$\forall \epsilon' > 0 \quad \exists n_0, \forall N > n_0, \Big|\sum_{n=1}^{N} \frac{(-1)^{n-1}}{n^s}\Big| < \epsilon'$$

We definite the sequence of functions $((\eta_n)_{n\in\mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} - i\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let s be one zero of η that lies in the critical strip, then $\eta(s) = 0$, with $0 < \sigma < 1$. It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s) = 0 = \eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma} = 0$$

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma} = 0$$

Using the definition of the limit of a sequence, we can write:

(7.7) $\forall \epsilon_1 > 0\, \exists n_r, \forall N > n_r,\, |\Re(\eta(s)_N)| < \epsilon_1 \Longrightarrow \Re(\eta(s)_N)^2 < {\epsilon_1}^2$

(7.8) $\forall \epsilon_2 > 0\, \exists n_i, \forall N > n_i,\, |\Im(\eta(s)_N)| < \epsilon_2 \Longrightarrow \Im(\eta(s)_N)^2 < {\epsilon_2}^2$

Then:

$$0 < \sum_{k=1}^{N} \frac{cos^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma} < \epsilon_1^2$$

$$0 < \sum_{k=1}^{N} \frac{sin^2(tLogk)}{k^{2\sigma}} + 2\sum_{k,k'=1;k<k'}^{N} \frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma} < \epsilon_2^2$$

Taking $\epsilon = \epsilon_1 = \epsilon_2$ and $N > max(n_r, n_i)$, we get by making the sum member to member of the last two inequalities:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^{\sigma} k'^{\sigma}} < 2\epsilon^2 \tag{7.9}$$

We can write the above equation as :

$$0 < \rho_N^2 < 2\epsilon^2 \tag{7.10}$$

or $\rho(s) = 0$.

7.3. Case $\sigma = \frac{1}{2}$

We suppose that $\sigma = \frac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([**2**], page 24):

> ***Theorem 7.3.1.*** — *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (7.5-7.6), it follows the proposition :

> ***Proposition 7.3.2.*** — There are infinitely many zeros of $\eta(s)$ on the critical line.

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (7.9) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{7.11}$$

if not, we will have a contradiction with the fact that :

$$lim_{N\longrightarrow+\infty}\sum_{k=1}^{N}(-1)^{k-1}\frac{1}{k^{s_j}}=0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j=\frac{1}{2}+it_j$$

7.4. Case $\frac{1}{2} < \Re(s) < 1$.

Let $s=\sigma+it$ be the zero of $\eta(s)$ in $0<\Re(s)<\frac{1}{2}$, object of the previous paragraph. From the proposition (7.5), $\zeta(s)=0$. According to point 4 of theorem 7.1.2, the complex number $s'=1-\sigma+it=\sigma'+it'$ with $\sigma'=1-\sigma$, $t'=t$ and $\frac{1}{2}<\sigma'<1$ verifies $\zeta(s')=0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2}<\Re(s)<1$, it follows from the proposition (7.6) that $\eta(s')=0 \Longrightarrow \rho(s')=0$. By applying (7.9), we get:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}<2\epsilon^2 \tag{7.12}$$

As $0<\sigma<\frac{1}{2}\Longrightarrow 2>2\sigma'=2(1-\sigma)>1$, then the series $\sum_{k=1}^{N}\frac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2<1/k^{2\sigma'}$ for all $k>0$, then :

$$0<\zeta(2)=\frac{\pi^2}{6}=\sum_{k=1}^{+\infty}\frac{1}{k^2}<\sum_{k=1}^{+\infty}\frac{1}{k^{2\sigma'}}=C(\sigma')=\zeta_1(2\sigma')=\zeta(2\sigma')$$

From the equation (7.12), it follows that :

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{cos(t'Log(k/k'))}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}=-\frac{\zeta(2\sigma')}{2}>-\infty \tag{7.13}$$

7.4.0.1. *Case* $t=0$. — We suppose that $t=0\Longrightarrow t'=0$. The equation (7.13) becomes:

$$\sum_{k,k'=1;k<k'}^{+\infty}(-1)^{k+k'}\frac{1}{k^{\sigma'}k'^{\sigma'}}=-\frac{C(\sigma')}{2}=-\frac{\zeta(2\sigma')}{2}>-\infty \tag{7.14}$$

Then $s'=\sigma'>1/2$ is a zero of $\eta(s)$, we obtain :

$$\eta(s')=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^{s'}}=0 \tag{7.15}$$

Let us define the sequence S_m as:

$$S_m(s')=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n^{s'}}=\sum_{n=1}^{m}\frac{(-1)^{n-1}}{n^{\sigma'}}=S_m(\sigma') \tag{7.16}$$

From the definition of S_m, we obtain :

$$lim_{m\longrightarrow+\infty}S_m(s')=\eta(s')=\eta(\sigma') \tag{7.17}$$

We have also:

$$S_1(\sigma') = 1 > 0 \tag{7.18}$$

$$S_2(\sigma') = 1 - \frac{1}{2^{\sigma'}} > 0 \quad \textit{because } 2^{\sigma'} > 1 \tag{7.19}$$

$$S_3(\sigma') = S_2(\sigma') + \frac{1}{3^{\sigma'}} > 0 \tag{7.20}$$

We proceed by recurrence, we suppose that $S_m(\sigma') > 0$.

1. $m = 2q \Longrightarrow S_{m+1}(\sigma') = \sum_{n=1}^{m+1} \frac{(-1)^{n-1}}{n^{s'}} = S_m(\sigma') + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$, it gives:

$$S_{m+1}(\sigma') = S_m(\sigma') + \frac{(-1)^{2q}}{(m+1)^{\sigma'}} = S_m(\sigma') + \frac{1}{(m+1)^{\sigma'}} > 0 \Rightarrow S_{m+1}(\sigma') > 0$$

2. $m = 2q+1$, we can write $S_{m+1}(\sigma')$ as:

$$S_{m+1}(\sigma') = S_{m-1}(\sigma') + \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^{m+1-1}}{(m+1)^{\sigma'}}$$

We have $S_{m-1}(\sigma') > 0$, let $T = \frac{(-1)^{m-1}}{m^{\sigma'}} + \frac{(-1)^m}{(m+1)^{\sigma'}}$, we obtain:

$$T = \frac{(-1)^{2q}}{(2q+1)^{\sigma'}} + \frac{(-1)^{2q+1}}{(2q+2)^{\sigma'}} = \frac{1}{(2q+1)^{\sigma'}} - \frac{1}{(2q+2)^{\sigma'}} > 0 \tag{7.21}$$

and $S_{m+1}(\sigma') > 0$.

Then all the terms $S_m(\sigma')$ of the sequence S_m are great then 0, it follows that $lim_{m \longrightarrow +\infty} S_m(s') = \eta(s') = \eta(\sigma') > 0$ and $\eta(\sigma') < +\infty$ because $\Re(s') = \sigma' > 0$ and $\eta(s')$ is convergent. We deduce the contradiction with the hypothesis s' is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (7.14) is false for the case } t' = t = 0.} \tag{7.22}$$

7.4.0.2. *Case $t \neq 0$.* — We suppose that $t \neq 0$. For each $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t' Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{7.23}$$

the left member of the equation (7.23) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$. But for all $\sigma"$ so that $2\sigma" > 1$, we have $\zeta(2\sigma")$:

$$\zeta(2\sigma") = \zeta_1(2\sigma") = \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma"}} < +\infty$$

It depends only of $\sigma"$, then in particular for all $\sigma"$ with $2 > 2\sigma" > 1$, $\zeta(2\sigma")$ depends only of $\sigma"$. Let $\lambda > 0$ be an arbitrary real number very infinitesimal so that $\sigma' + \lambda \in]1/2, 1[$ is not the real part of a zero of $\eta(s)$. We can write to the first order:

$$\zeta(2\sigma' + 2\lambda) = \zeta(2\sigma') + 2\lambda.\zeta'(2\sigma') \tag{7.24}$$

$\zeta'(2\sigma')$ is given by:

$$\zeta'(2\sigma') = -\sum_{k=2}^{+\infty} \frac{Logk}{k^{2\sigma'}} > -\infty \tag{7.25}$$

because we can choose $\alpha > 0$ so that $\sigma' > 1/2 + \alpha \Longrightarrow 2\sigma' - 2\alpha > 1$ and we obtain:

$$|\zeta'(2\sigma')| \leq \frac{1}{2\alpha} \sum_{k=2}^{+\infty} \frac{Logk^{2\alpha}}{k^{2\alpha}} \frac{1}{k^{2(\sigma'-\alpha)}} \leq \frac{1}{2\alpha} \sum_{k=2}^{+\infty} \frac{1}{k^{2(\sigma'-\alpha)}} < +\infty$$

Numerically, the left member of the equation (7.24) is independent of t', the preponderant term of the right member $\zeta(2\sigma')$ depends of t' using the equation (7.23), then the contradiction and we conclude that the result giving by the equation (7.23) is false.

$$\boxed{\text{It follows that the equation (7.23) is false for the case } t' \neq 0.} \tag{7.26}$$

From (7.22-7.26), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the paragraph (6.4) above concerning the case $0 < \Re(s) < \frac{1}{2}$ is false. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \frac{1}{2}$. From the equivalent statement (7.1.5), it follows that **the Riemann hypothesis is verified**. □

From the calculations above, we can verify easily the following known proposition:

> ***Proposition 7.4.1.*** — For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.

7.5. Conclusion.

In summary: for our proofs, we made use of Dirichlet's $\eta(s)$ function:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad s = \sigma + it$$

on the critical band $0 < \Re(s) < 1$, in obtaining:

- $\eta(s)$ vanishes for $0 < \sigma = \Re(s) = \frac{1}{2}$;

- $\eta(s)$ does not vanish for $0 < \sigma = \Re(s) < \dfrac{1}{2}$ and $\dfrac{1}{2} < \sigma = \Re(s) < 1$.

Consequently, all the zeros of $\eta(s)$ inside the critical band $0 < \Re(s) < 1$ are on the critical line $\Re(s) = \dfrac{1}{2}$. Applying the equivalent proposition to the Riemann Hypothesis (7.1.5), we conclude that **the Riemann hypothesis is verified** and all the nontrivial zeros of the function $\zeta(s)$ lie on the critical line $\Re(s) = \dfrac{1}{2}$. The proof of the Riemann Hypothesis is thus completed.

We therefore announce the important theorem as follows:

Theorem 7.5.1. — *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$.

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