Elementary Formulas For $\Gamma\left(\frac{1}{3}\right)^3$

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December 10 , 2021

Abstract. In this note we give some formulas for $\Gamma\left(\frac{1}{3}\right)^3$.

Introduction

The (complete) gamma function $\Gamma(n)$ is defined to be an extension of the factorial to complex and real number arguments.it is related to the factorial by $\Gamma(n)=(n-1)!$.

The gamma function can be defined as a definite integral for $\Re(z)>0$ (Euler's integral)

$$\Gamma(z)=\int_0^\infty x^{z-1}e^{-x}\,dx=\int_0^1(-\ln x)^{z-1}\,dx$$

The gamma function can also be defined by an infinite product form (Weiertrass)

$$\Gamma(z)=\left(z\,e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right)^{-1}$$

where γ is the Euler-Mascheroni constant.

In this note we give some formulas for $\Gamma\left(\frac{1}{3}\right)^3$.

Formulas for $\Gamma\left(\frac{1}{3}\right)^3$

Entry 1.

$$\Gamma\left(\frac{1}{3}\right)^3=\frac{2\pi}{\sqrt{3}}\int_0^1(x-x^2)^{-2/3}\,dx$$

Entry 2.

$$\Gamma\left(\frac{1}{3}\right)^3=\frac{4\pi\sqrt[3]{2}}{\sqrt{3}}\int_0^\infty\frac{1}{(\cosh x)^{2/3}}\,dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{4\pi\sqrt[3]{2}}{\sqrt{3}} \int_0^1 \cosh^{-1}\left(\frac{1}{x^{3/2}}\right) dx$$

Entry 3.

$$\Gamma\left(\frac{1}{3}\right)^3 = 2\pi\sqrt{3}\,\sqrt[3]{2} \int_0^1 \frac{1}{\sqrt{1-x^3}} dx$$

$$\Gamma\left(\frac{1}{3}\right)^{-3} = \frac{3}{4\pi^2\sqrt[3]{2}} \int_0^1 \frac{x}{\sqrt{1-x^3}} dx$$

Entry 4.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{6\pi\,\sqrt[3]{2}}{\sqrt{3}} + \frac{4\pi\,\sqrt[3]{2}}{\sqrt{3}} \int_0^1 \ln\left(1+\sqrt{1-x^3}\right) dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi\,\sqrt[3]{2}}{\sqrt{3}}\left(\frac{3}{4} + \int_0^1 x\ln\left(1+\sqrt{1-x^6}\right) dx\right)$$

Entry 5.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi\sqrt[3]{2}}{\sqrt{3}} \int_1^\infty \frac{\cosh^{-1} x^3}{x^3} dx$$

Entry 6.

$$\Gamma\left(\frac{1}{3}\right)^3 \left(\frac{3\sqrt{3}}{16\pi}\right) = -\int_0^\infty \frac{\ln x}{x^{1/3}(1+x^2)^{5/3}} dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 \left(\frac{6\pi - 9\sqrt{3}}{48\pi}\right) = \int_0^\infty \frac{\ln(1+x^2)}{x^{1/3}(1+x^2)^{5/3}} dx$$

Entry 7.

$$\Gamma\left(\frac{1}{3}\right)^{-3} = \frac{1}{\pi^2\sqrt[3]{2}\,\sqrt{3}} \int_0^{\pi/2} \sin x^{2/3}\, dx$$

$$\Gamma\left(\frac{1}{3}\right)^{-3} = \frac{1}{\pi^2\sqrt[3]{2}\,\sqrt{3}} \left(\frac{\pi}{2} - \int_0^1 \sin^{-1} x^{3/2}\, dx\right)$$

Entry 8.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi\,\sqrt[3]{2}}{\sqrt{3}} \int_0^1 \sqrt[3]{\frac{x}{(1-x)^2(1+x)^5}}\, dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi\sqrt[3]{2}}{\sqrt{3}}\int_1^\infty \frac{1}{\left(x+\sqrt{x^2-1}\right)\sqrt[3]{x^2-1}}\,dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi\sqrt[3]{2}}{\sqrt{3}}\int_0^\infty \frac{\sqrt[3]{x}}{1+x^2+x\sqrt{1+x^2}}\,dx$$

Entry 9.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi\sqrt[3]{2}}{\sqrt{3}}\int_0^\infty e^{-x}\sqrt[3]{\sinh x}\,dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{6\pi\sqrt{3}}{\sqrt[3]{2}}\int_0^\infty e^{-2x}\sqrt[3]{\sinh 3x}\,dx$$

Entry 10.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi}{\sqrt{3}}\int_0^\infty \sqrt[3]{e^{-2x}-e^{-4x}}\,dx = \frac{4\pi}{\sqrt{3}}\int_0^\infty \sqrt[3]{e^{-x}-e^{-2x}}\,dx$$

Entry 11.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{2\pi\sqrt[3]{2}}{\sqrt{3}}\int_0^1 \ln\left(\frac{1+\sqrt{1-x^3}}{1-\sqrt{1-x^3}}\right)dx$$

Entry 12.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{10\pi\sqrt[3]{2}}{\sqrt{3}}\int_0^1 \sqrt[3]{1-x^2}\,dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{10\pi\sqrt[3]{2}}{\sqrt{3}}\int_0^1 \sqrt{1-x^3}\,dx$$

Entry13. For $0 < a < 1$, we have

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{10\pi\sqrt[3]{2}}{\sqrt{3}}\left(a\,F\left(-\frac{1}{3},\frac{1}{2},\frac{3}{2},a^2\right)+\frac{3(1-a)^{4/3}}{2\sqrt[3]{4}}\,F\left(-\frac{1}{3},\frac{4}{3},\frac{7}{3},\frac{1-a}{2}\right)\right)$$

where $F(u,v,w,x)$, is the Gauss hypergeometric function.

Entry 14.

$$\Gamma\left(\frac{1}{3}\right)^3 = 24\int_0^1 \frac{x}{(1+x^6)^{2/3}}\ln\left(\frac{1+x^6}{x^3}\right)dx$$

Entry 15.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{20\pi\sqrt[3]{2}}{\sqrt{3}} \int_0^1 x\sqrt{1-x^6}\,dx$$

Entry 16.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{4\pi\sqrt[3]{2}}{\sqrt{3}} \int_0^1 \sinh^{-1}\sqrt{\frac{1}{x^3}-1}\,dx$$

Entry 17.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{4\pi\sqrt[3]{2}}{\pi-\sqrt{3}\ln 2} \int_0^\infty \frac{\ln\cosh x}{(\cosh x)^{2/3}}\,dx$$

Entry 18.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{4\pi\sqrt[3]{2}}{\sqrt{3}} \int_0^\infty \frac{1}{(1+x^2)^{5/6}}\,dx$$

Entry 19.

$$\Gamma\left(\frac{1}{3}\right)^3 = 2\sqrt[3]{2} \int_{-\infty}^\infty \frac{e^{-x}\ln(2\cosh x)}{\sqrt[3]{(\cosh x)^5}}\,dx = 2\sqrt[3]{2} \int_{-\infty}^\infty \frac{e^{x}\ln(2\cosh x)}{\sqrt[3]{(\cosh x)^5}}\,dx$$

Entry 20.

$$\Gamma\left(\frac{1}{3}\right)^3 = 2\sqrt[3]{2} \int_0^\infty \frac{\ln\left(\sqrt{x}+\sqrt{x+1}\right)}{\sqrt{x(x+1)}\,\sqrt[3]{x}}\,dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 12\sqrt[3]{2} \int_0^\infty \frac{\ln\left(x^3+\sqrt{x^6+1}\right)}{\sqrt{x^6+1}}\,dx$$

Entry 21.

$$\Gamma\left(\frac{1}{3}\right)^3 = -\int_0^1 \frac{\ln(x-x^2)}{(x-x^2)^{\frac{2}{3}}}\,dx = -2\int_0^{1/2} \frac{\ln(x-x^2)}{(x-x^2)^{\frac{2}{3}}}\,dx$$

Entry 22.

$$\Gamma\left(\frac{1}{3}\right)^3 = 6\ln 2\,\sqrt[3]{2}\sum_{n=0}^\infty \frac{(2/3)_n\,2^{-2n}}{n!\,(3n+1)} + 9\sqrt[3]{2}\sum_{n=0}^\infty \frac{(2/3)_n\,2^{-2n}}{n!\,(3n+1)^2}$$
$$+ 12\sum_{n=1}^\infty \frac{2^{-n}}{n(3n+1)} F1\left(n+\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, n+\frac{4}{3}, \frac{1}{2}, \frac{1}{4}\right)$$

where $F1$ is the Appell hypergeometric function.

Entry 23.

$$\Gamma\left(\frac{1}{3}\right)^3 = 4\sqrt[3]{2}\int_0^{\pi/2} (\tan x)^{-2/3} \sec x \ln(\tan x + \sec x)\, dx$$

Entry 24.

$$\Gamma\left(\frac{1}{3}\right)^3 = 4\sqrt[3]{2}\int_0^{1} \frac{\tanh^{-1} x}{(1-x^2)^{2/3}\, x^{2/3}}\, dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 4\sqrt[3]{2}\int_1^{\infty} \frac{\cosh^{-1} x}{(x^2-1)^{5/6}}\, dx$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 12\sqrt[3]{2}\int_0^{\infty} \frac{\cosh^{-1}\sqrt{1+x^6}}{\sqrt{1+x^6}}\, dx$$

Entry 25.

$$\Gamma\left(\frac{1}{3}\right)^3 + 4\pi\sqrt{3}\sqrt[3]{2} = 36\sqrt[3]{2}\int_0^{\infty} \frac{x^6 \sinh^{-1} x^3}{\sqrt{(1+x^6)^3}}\, dx$$

Entry 26.

$$\Gamma\left(\frac{1}{3}\right)^3 = 2\pi\sqrt{3}\sqrt[3]{2}\left(1 + \int_0^{1} \frac{1}{1+\sqrt[3]{1-x^2}+\sqrt[3]{(1-x^2)^2}}\, dx\right)$$

Entry 27.

$$\Gamma\left(\frac{1}{3}\right)^{-3} = \frac{3\sqrt[3]{2}}{4\pi^2}\int_0^{\infty} \frac{\left(1+\sqrt{1+x^2}\right)^{1/3} - 2^{1/3}}{x^2}\, dx$$

Entry 28.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{4\pi\sqrt[3]{2}}{\sqrt{3}}\left(\sum_{n=0}^{\infty} \frac{(-1)^n (2/3)_n}{(n+1)(4/3)_n} - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{3}{3n-2}\right)\right)$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 6\pi\sqrt{3}\sum_{n=0}^{\infty} (-1)^n (2n+1)\left(\frac{(2/3)_n}{(4/3)_n}\right)^2$$

Entry 29.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{8\pi^3}{9\sqrt{3}}\sum_{n=0}^{\infty} \frac{((1/3)_n)^4 (7/6)_n}{(n!)^4 (1/6)_n}$$

Entry 30.

$$\ln\Gamma\left(\frac{1}{3}\right)^3+\ln\frac{1}{2\pi\sqrt{3}}=\int_0^\infty\frac{(1-e^{-2x})e^{-x}}{x(1+e^{-x}+e^{-2x})}\,dx$$

Entry 31.

$$\Gamma\left(\frac{1}{3}\right)^3=\frac{10\pi\,\sqrt[3]{2}}{\sqrt{3}}\int_0^{\pi/4}\sqrt[3]{\frac{\cos 2x}{(\cos x)^8}}\,dx$$

Entry 32.

$$\Gamma\left(\frac{1}{3}\right)^3=\frac{4\pi\sqrt[3]{2}}{\sqrt[4]{3}}\left(1+\int_1^\infty\left(1-\sqrt{\frac{8(x^2-1)}{x^2\left(6-\sqrt{3}\right)+x\sqrt{16\left(2-\sqrt{3}\right)+x^2\left(2+\sqrt{3}\right)^2}}}\right)dx\right)$$

Entry 33.

$$\Gamma\left(\frac{1}{3}\right)^3=4\pi\sqrt{3}\int_0^\infty\left(\sqrt[3]{1+x^3}-x\right)dx$$

$$\Gamma\left(\frac{1}{3}\right)^3=4\pi\sqrt{3}\int_0^\infty\frac{1}{x^2+x\sqrt[3]{1+x^3}+\sqrt[3]{(1+x^3)^2}}\,dx$$

Entry 34.

$$\Gamma\left(\frac{1}{3}\right)^3=-8\int_0^\infty\sqrt[3]{\coth x-\tanh x}\,\ln\tanh x\,dx$$

$$\Gamma\left(\frac{1}{3}\right)^3=-8\int_0^\infty\sqrt[3]{\coth x-\tanh x}\,\ln\operatorname{sech} x\,dx$$

Entry 35.

$$\Gamma\left(\frac{1}{3}\right)^3=8\sqrt[3]{2}\int_0^{\pi/2}\frac{-\ln\sin x}{\sqrt[3]{\sin 2x}}\,dx=8\sqrt[3]{2}\int_0^{\pi/2}\frac{-\ln\cos x}{\sqrt[3]{\sin 2x}}\,dx$$

Entry 36.

$$\Gamma\left(\frac{1}{3}\right)^3=4\sqrt[3]{2}\int_0^\infty\frac{x}{(\sinh x)^{2/3}}\,dx$$

Entry 37.

$$\Gamma\left(\frac{1}{3}\right)^3=18\sum_{n=0}^\infty\frac{(2/3)_n}{(3n+1)^2\,n!}$$

Entry 38.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{12}{5}\pi\sqrt{3}\sqrt[3]{2}\sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{3}{10}\right)^{2n} F\left(2n+1, \frac{1}{3}, \frac{4}{3}, -\frac{1}{5}\right)$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 3\pi\sqrt{3}\sqrt[3]{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-3n}}{3n+1} F\left(\frac{1}{2}, n+\frac{1}{3}, n+\frac{4}{3}, -\frac{1}{8}\right)$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 3\pi\sqrt{3}\sqrt[3]{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-5n}}{3n+1} F\left(\frac{1}{2}, n+\frac{1}{3}, n+\frac{4}{3}, -\frac{1}{2}\right)$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 3\pi\sqrt{3}\sqrt[3]{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-6n}}{6n+1} F\left(n+\frac{1}{2}, 2n+\frac{1}{3}, 2n+\frac{4}{3}, -\frac{5}{8}\right)$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 3\pi\sqrt{3}\sqrt[3]{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{(-1)^n 2^{-5n}}{3n+1} F\left(2n+1, n+\frac{1}{3}, n+\frac{4}{3}, -\frac{1}{4}\right)$$

where F is the Gauss hypergeometric function.

Entry 39.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{12\pi\sqrt{3}\sqrt[3]{2}}{5}\sum_{n=0}^{\infty}(-5)^{-n}\sum_{k=0}^{n}\binom{2k}{k}\binom{n+k}{n-k}\frac{(-9/20)^k}{3n-3k+1}$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 3\pi\sqrt{3}\sqrt[3]{2}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{-3n}}{3n+1}\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}2^{-2k}$$

Entry 40.

$$\Gamma\left(\frac{1}{3}\right)^3 = 12\pi\sqrt{3}\sqrt[3]{2}\left(\frac{1}{3\sqrt{3}} + \int_{1/(3\sqrt{3})}^{1/4}\sqrt[3]{\sqrt{9+x^{-2}}-5}\;dx\right)$$

Entry 41.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{20\pi}{\sqrt{3}}\int_1^e \frac{1}{x}\sqrt[3]{(\ln x)(1-\ln x)}\;dx$$

Entry 42.

$$\Gamma\left(\frac{1}{3}\right)^3 = \frac{12\pi\sqrt{2}\sqrt{3}\sqrt[3]{2}}{\sqrt{43}}\sum_{n=0}^{\infty}\left(\frac{11}{172}\right)^n\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{20}{11}\right)^k\sum_{m=0}^{k}\binom{k}{m}\frac{10^{-m}}{3k+3m+1}$$

Endnote

$$\Gamma\left(\frac{1}{3}\right)^3 = 19.22596945259569369138\ldots$$

$$\Gamma\left(\frac{1}{3}\right)^3 = 19 + \cfrac{1}{4 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{14 + \cdots}}}}}}}}$$

References

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