“**Derivation of Characteristic Formulae pertaining to an R-C-L Series Electrical Circuit**.”

by Stephen C. Pearson MRACI; AffilIEAust;
Affiliate of the Royal Society of Chemistry &
Member of the London Mathematical Society,
Consultant Technologist
[T/A S. C. Pearson Technical Services].

Email Address:- *scpearson1952@outlook.com* .

* * * * * * * * * *

Abstract:-

Whilst the aforesaid formulae are essentially algebraic in nature, their derivation, however, cannot be obtained simply by relying upon elementary algebraic techniques, as is evident from the contents of this particular paper. Subsequently, the author envisages that any professional electrical / electronics engineer should in his opinion be capable of understanding the concepts enunciated therein, bearing in mind that the study of real and complex valued analytic functions and their concomitant differential equations constitutes an integral part (inter alia) of his / her theoretical training.

I. PRELIMINARY REMARK.

It should be noted that this paper was originally presented under the heading, **Appendix SCPTS-RCLSEC-1-2021**, as part of an email, dated 6/12/2021 [Subject:- Alternating Current (a.c.) Electrical Circuits.], having been sent by the author to seven (7) individuals, who had either previously been or are currently employed in the engineering and scientific fields.

II. COPY OF AUTHOR’S PAPER.

[**N.B**. Total number of A4 pages to follow after this page = 10.]

Appendix SCPTS-RCLSEC-1-2021.

Re:- Derivation of Characteristic Formulae pertaining to an R-C-L Series Electrical Circuit.

For the purposes of deriving the aforesaid formulae, we shall accordingly enunciate the following theorem and concomitant definition:-

Theorem T-RCLSEC-1.

Let there exist an R-C-L series electrical circuit as indicated by Figs. 1 & 2 below. Hence, it may be proven that the resultant current, $I(t)$, generated by this particular circuit is accordingly expressed by the formula,

$$I(t) = (V_p/Z)\cos(\omega t + \phi),$$

insofar as the constants,

$$Z = \sqrt{(X_c - X_L)^2 + R^2}; \quad \cos(\phi) = \left(\tfrac{1}{C} - L\omega^2\right)\Big/\sqrt{\left(\tfrac{1}{C} - L\omega^2\right)^2 + R^2\omega^2};$$

$$X_c = 1/C\omega = T_0/2\pi C; \quad \sin(\phi) = -R\omega\Big/\sqrt{\left(\tfrac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}.$$

$$X_L = L\omega = 2\pi L/T_0;$$

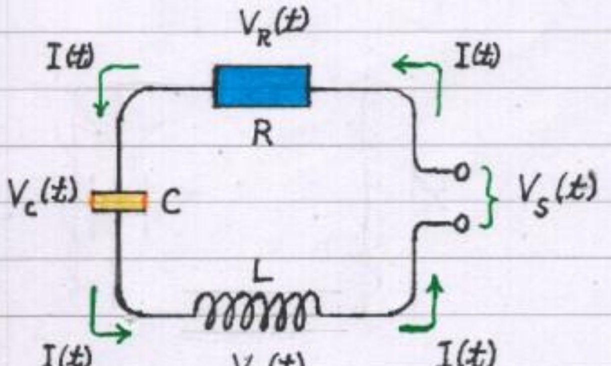


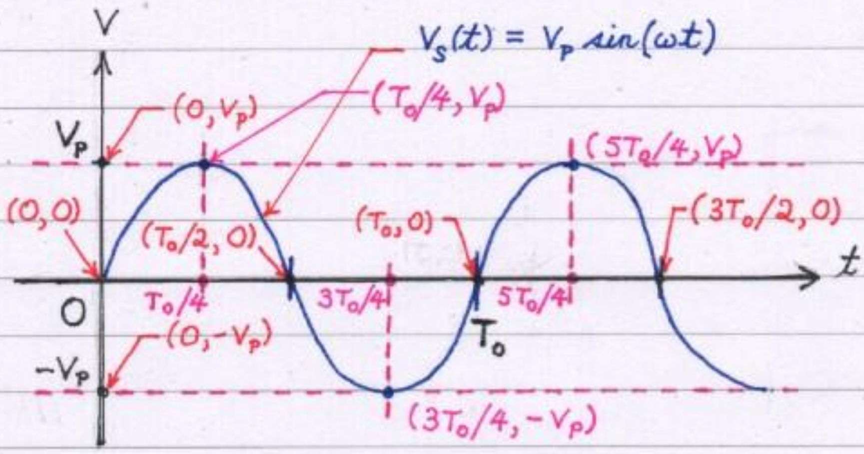
Fig.1.

N.B.

(a) The symbols,
R = resistor (ohms); C = capacitor (farads); L = inductor (henrys).

(b) The voltages, [*]
$V_S(t) = V_p\sin(\omega t)$; $V_R(t) = RI(t)$;
$V_C(t) = Q(t)/C$; $V_L(t) = L\frac{d}{dt}(I(t))$.

[*] By definition, 1 volt = 1 joule/coulomb.



Graph of Supply Voltage, $V_s(t)$, vs. Time, t (secs).

N.B.

(a) The constant, V_p, denotes the peak voltage corresponding to the function, $V_s(t)$.

(b) The constant, T_0, denotes the periodicity of the function, $V_s(t)$.

(c) The constant, $\omega = 2\pi/T_0$, denotes the frequency[*] corresponding to the function, $V_s(t)$.

[*] Expressed in Hertz, Hz (= cycles/sec).

Fig. 2.

* * * * *

PROOF:-

With reference to Figs. 1 & 2 depicted in the preamble to this proof, we recall from Kirchoff's Laws (cf. web-page article [1] & Boyce & DiPrima [2]) the following definitive formula, namely -

The supply voltage of the circuit = the voltage drop over the resistor, R, + the voltage drop over the capacitor, C, + the voltage drop over the inductor, L, in other words -

$$V_S(t) = V_R(t) + V_C(t) + V_L(t)$$

$$\therefore V_p \sin(\omega t) = RI(t) + Q(t)/C + L\tfrac{d}{dt}(I(t))$$

$$\therefore L\tfrac{d}{dt}(I(t)) + RI(t) + Q(t)/C = V_p \sin(\omega t) \qquad (1).$$

Since by definition the <u>current function, $I(t)$</u>, is the <u>first derivative with respect to 't'</u> of the <u>charge function, $Q(t)$</u>, i.e. -

$$I(t) = \tfrac{d}{dt}(Q(t)),$$

it automatically follows, after making the appropriate algebraic substitutions, that Eq.(1) can be rewritten as

$$L\tfrac{d}{dt}(\tfrac{d}{dt}(Q(t))) + R\tfrac{d}{dt}(Q(t)) + Q(t)/C = V_p \sin(\omega t)$$

$$\therefore L\tfrac{d^2}{dt^2}(Q(t)) + R\tfrac{d}{dt}(Q(t)) + Q(t)/C = V_p \sin(\omega t) \qquad (2a),$$

which by definition is a specific example of an <u>inhomogeneous second order linear differential equation</u> with respect to the function, $Q(t)$. (cf. Boyce & DiPrima [2]).

Now, in order to solve this particular differential equation, let us set the <u>charge function</u>,

$$Q(t) = A\sin(\omega t) + B\cos(\omega t) \qquad (2b),$$

where A and B are <u>arbitrary constants</u>, whose values have yet to be determined. Subsequently, in view of Eq.(2b), we deduce that Eq.(2a) can likewise be rewritten as

$$L\frac{d^2}{dt^2}(A\sin(\omega t) + B\cos(\omega t)) + R\frac{d}{dt}(A\sin(\omega t) + B\cos(\omega t)) +$$
$$(1/C)(A\sin(\omega t) + B\cos(\omega t))$$

$$= V_p \sin(\omega t) \qquad (3a).$$

From the established definitions and theorems pertaining to the calculus of real variable functions (cf. Salas & Einar Hille [3]) we accordingly deduce that

(a) the first derivative with respect to 't' of the charge function, $Q(t)$,

$$\frac{d}{dt}(Q(t)) = \frac{d}{dt}(A\sin(\omega t) + B\cos(\omega t)) = \frac{d}{dt}(A\sin(\omega t)) + \frac{d}{dt}(B\cos(\omega t))$$

$$= A\frac{d}{dt}(\sin(\omega t)) + B\frac{d}{dt}(\cos(\omega t)) = A\omega\cos(\omega t) - B\omega\sin(\omega t) \qquad (3b);$$

(b) the second derivative with respect to 't' of the charge function, $Q(t)$,

$$\frac{d^2}{dt^2}(Q(t)) = \frac{d^2}{dt^2}(A\sin(\omega t) + B\cos(\omega t)) = \frac{d}{dt}\left(\frac{d}{dt}(Q(t))\right)$$

$$= \frac{d}{dt}(A\omega\cos(\omega t) - B\omega\sin(\omega t)) = \frac{d}{dt}(A\omega\cos(\omega t)) - \frac{d}{dt}(B\omega\sin(\omega t))$$

$$= A\omega\frac{d}{dt}(\cos(\omega t)) - B\omega\frac{d}{dt}(\sin(\omega t)) = -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t)$$

$$= -(A\omega^2\sin(\omega t) + B\omega^2\cos(\omega t)) \qquad (3c),$$

and hence Eq. (3a) can similarly be rewritten in view of Eqs. (3b) & (3c) as

$$-L(A\omega^2\sin(\omega t) + B\omega^2\cos(\omega t)) + R(A\omega\cos(\omega t) - B\omega\sin(\omega t)) +$$
$$(1/C)(A\sin(\omega t) + B\cos(\omega t))$$

$$= V_p\sin(\omega t)$$

$$\therefore -LA\omega^2\sin(\omega t) - LB\omega^2\cos(\omega t) + RA\omega\cos(\omega t) - RB\omega\sin(\omega t)$$
$$+ (A/C)\sin(\omega t) + (B/C)\cos(\omega t)$$

$= V_p \sin(\omega t)$

$\therefore -LA\omega^2 \sin(\omega t) - RB\omega \sin(\omega t) + (A/C)\sin(\omega t) - LB\omega^2 \cos(\omega t) +$
$RA\omega \cos(\omega t) + (B/C)\cos(\omega t)$

$= V_p \sin(\omega t)$

$\therefore [-LA\omega^2 - RB\omega + A/C]\sin(\omega t) + [-LB\omega^2 + RA\omega + B/C]\cos(\omega t)$

$= V_p \sin(\omega t) = V_p \sin(\omega t) + 0.\cos(\omega t) \quad (4).$

From the left and right hand sides of Eq.(4) we accordingly perceive that the constant coefficients of sin(ωt) & cos(ωt) generate the pair of simultaneous equations,

$-LA\omega^2 - RB\omega + A/C = V_p$

$-LB\omega^2 + RA\omega + B/C = 0$

$\therefore -LA\omega^2 + A/C - RB\omega = V_p$

$RA\omega - LB\omega^2 + B/C = 0$

$\therefore (-L\omega^2 + 1/C)A - R\omega B = V_p$

$R\omega A + (-L\omega^2 + 1/C)B = 0$

$\therefore (1/C - L\omega^2)A - R\omega B = V_p \quad (5a);$

$R\omega A + (1/C - L\omega^2)B = 0 \quad (5b).$

By invoking the well-established Cramer's Rule for determinants from linear algebra, we similarly perceive that the required solutions of Eqs.(5a) & (5b) in terms of the arbitrary constants, A and B, are therefore given by (cf. Floney [4]) –

$$A = \begin{vmatrix} V_p & -R\omega \\ 0 & (1/C - L\omega^2) \end{vmatrix} \mathcal{D}^{-1} \quad \& \quad B = \begin{vmatrix} (1/C - L\omega^2) & V_p \\ R\omega & 0 \end{vmatrix} \mathcal{D}^{-1},$$

where the denominator, $\mathcal{D} = \begin{vmatrix} (1/C - L\omega^2) & -R\omega \\ R\omega & (1/C - L\omega^2) \end{vmatrix}$,

$$\therefore A = (V_p(1/C - L\omega^2))/((1/C - L\omega^2)^2 + R^2\omega^2) \qquad (6a) \quad \&$$

$$B = -(V_p R\omega)/((1/C - L\omega^2)^2 + R^2\omega^2) \qquad (6b).$$

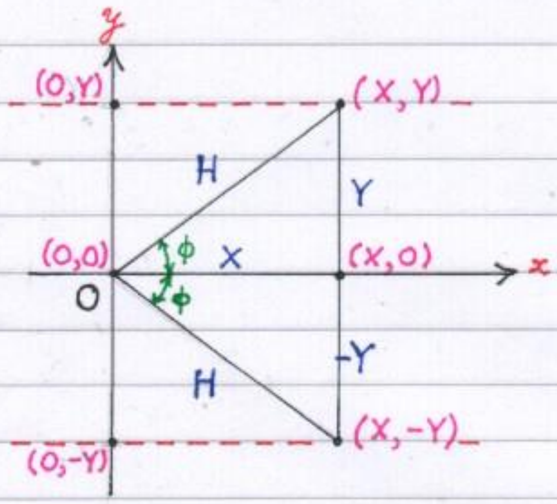


Fig. 3.

N.B.

(a) $X = 1/C - L\omega^2$;

(b) $Y = R\omega \Longrightarrow -Y = -R\omega$;

(c) By virtue of Pythagoras's Theorem, the resultant hypotenuse,
$H = \sqrt{X^2 + Y^2} = \sqrt{X^2 + (-Y)^2}$
$= \sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}$;

(d) By definition, the trigonometric ratios, $\cos(\phi) = X/H$ & $\sin(\phi) = \pm Y/H$.

With reference to Fig. 3 depicted above, it furthermore follows from Eqs. (6a) & (6b) that the constants,

$$A = V_p(1/C - L\omega^2)/(\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})^2$$

$$= (V_p/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})((1/C - L\omega^2)/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})$$

$= (V_p/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})\cos(\phi)$

$= (V_p/\sqrt{\omega^2[(1/\omega^2)(1/C - L\omega^2)^2 + R^2]})\cos(\phi)$

$= (V_p/\omega\sqrt{(1/\omega^2)(1/C - L\omega^2)^2 + R^2})\cos(\phi)$

$= (V_p/\omega\sqrt{[(1/\omega)(1/C - L\omega^2)]^2 + R^2})\cos(\phi)$

$= (V_p/\omega\sqrt{(1/C\omega - L\omega)^2 + R^2})\cos(\phi)$ (7a) &

$B = (V_p(-R\omega))/(\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})^2$

$= (V_p/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})(-R\omega/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})$

$= (V_p/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2})\sin(\phi)$

$= (V_p/\sqrt{\omega^2[(1/\omega^2)(1/C - L\omega^2)^2 + R^2]})\sin(\phi)$

$= (V_p/\omega\sqrt{(1/\omega^2)(1/C - L\omega^2)^2 + R^2})\sin(\phi)$

$= (V_p/\omega\sqrt{[(1/\omega)(1/C - L\omega^2)]^2 + R^2})\sin(\phi)$

$= (V_p/\omega\sqrt{(1/C\omega - L\omega)^2 + R^2})\sin(\phi)$ (7b).

Let there exist a constant,

$Z = \sqrt{(X_c - X_L)^2 + R^2}$

such that the constants,

$X_c = 1/C\omega = T_0/2\pi C$ & $X_L = L\omega = 2\pi L/T_0$,

whereupon we subsequently deduce after making the appropriate algebraic substitutions into Eqs.(7a) & (7b) that the constant coefficients,

$$A = (V_p/Z\omega)\cos(\phi) \quad \& \quad B = (V_p/Z\omega)\sin(\phi).$$

Finally, in view of the preceding statements, we conclude that the charge function,

$$Q(t) = (V_p/Z\omega)\cos(\phi)\sin(\omega t) + (V_p/Z\omega)\sin(\phi)\cos(\omega t)$$

$$= (V_p/Z\omega)(\cos(\phi)\sin(\omega t) + \sin(\phi)\cos(\omega t))$$

$$= (V_p/Z\omega)(\sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi))$$

$$= (V_p/Z\omega)\sin(\omega t + \phi),$$

bearing in mind the well established trigonometric formula for compound angles, namely (cf. Fig. 4) -

$$\sin(\theta_1 + \theta_2) = \sin(\theta_1)\cos(\theta_2) + \cos(\theta_1)\sin(\theta_2),$$

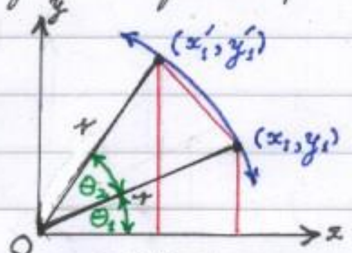


Fig. 4.

and hence the resultant current function,

$$I(t) = \frac{d}{dt}(Q(t)) = \frac{d}{dt}\left[(V_p/Z\omega)\sin(\omega t + \phi)\right] = (V_p/Z\omega)\frac{d}{dt}(\sin(\omega t + \phi))$$

$$= (V_p/Z\omega)\,\omega\cos(\omega t + \phi) = (V_p/Z)\cos(\omega t + \phi),$$

insofar as the constants,

$$Z = \sqrt{(X_c - X_L)^2 + R^2}\,;\; X_c = 1/C\omega = T_0/2\pi C;\; X_L = L\omega = 2\pi L/T_0\,;$$

$$\cos(\phi) = (1/C - L\omega^2)/\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}\,;$$

$$\sin(\phi) = -R\omega/\sqrt{(1/C - L\omega^2) + R^2\omega^2},$$

as required. Q.E.D. [*]

---

[*] N.B.

The acronym 'Q.E.D' denotes the Latin phrase, 'Quod erat demonstrandum', which precisely translates as 'That (thing), which had to be proven' and is commonly utilised by many mathematicians to signify the completion of the proof of any given theorem.

Definition D-RCLSEC-1.

With regard to the contents of Theorem T-RCLSEC-1 we accordingly observe that

(a) the constant,

$$X_c = 1/C\omega = T_0/2\pi C,$$

denotes the capacitive reactance of the circuit depicted in Fig.1 [*];

(b) the constant,

$$X_L = L\omega = 2\pi L/T_0,$$

denotes the inductive reactance of the circuit depicted in Fig.1 [*];

(c) the resultant difference between X_L and X_c,

$$X_{(L,C)} = X_L - X_c,$$

denotes the total reactance of the circuit depicted in Fig.1 [*];

(d) the constant,

$$Z = \sqrt{(X_c - X_L)^2 + R^2} = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X^2_{(L,C)}}$$

$$= |Z[\mathbb{C}]| = |R + jX_{(L,C)}|,$$ (N.B. The imaginary number, $j = \sqrt{-1} \in \mathbb{C}$, the set of complex numbers.)

denotes the impedance of the circuit depicted in Fig.1 [*].

[*] Measurements expressed in ohms as indicated in web-page article [5].

BIBLIOGRAPHY.

[1] Wikipedia; Kirchoff's Circuit Laws (viz. https://en.wikipedia.org/wiki/Kirchoff%27s_circuit_laws).

[2] W. E. Boyce & R. C. DiPrima; Elementary Differential Equations and Boundary Value Problems (3rd Edition); John Wiley & Sons Inc., New York. USA.

[3] S. L. Salas & Einar Hille; Calculus - One and Several Variables (3rd Edition); John Wiley & Sons Inc., New York. USA.

[4] F. G. Florey; Elementary Linear Algebra with Applications; Prentice-Hall Inc., New Jersey. USA.

[5] Wikipedia; Electrical Reactance (viz. https://en.wikipedia.org/wiki/Electrical_reactance).

Stephen C. Pearson
STEPHEN. C. PEARSON.

1st December 2021.

[END OF PAPER – THIS PAGE HAS BEEN INTENTIONALLY LEFT BLANK.]