THE LEGENDRE CONJECTURE –

A PROPOSED PROOF.

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ABSTRACT.

The Legendre Conjecture is herein proved by analysing the difference between the Prime Number Theorem for adjacent squares, and also by estimating the number of composites between adjacent squares using a slight variation of the Prime Number Theorem.

CONTENTS.

1.0 Introduction.

Adrien Marie Legendre, (1752 – 1833), was a French mathematician who made many contributions to the development of mathematics in the 18th and 19th centuries. In 1798 he proposed the following conjecture.

> "There is at least one prime number between all adjacent squares in the Natural Numbers".

This conjecture has remained unproven or otherwise since then, (223 years). It is the purpose of this paper to prove Legendre's conjecture by comparing $\pi(n)$ for adjacent squares, and also by estimating the number of composites between adjacent squares.

2.0 Proof of the Conjecture.

2.1 The Difference Between the Values of π(n) for Adjacent Squares.

Using the Prime Number Theorem, if Legendre's Conjecture is true, then for all n.

$$\pi(n)^2 < \pi(n+1)^2 \tag{2.1}$$

which is

$$\frac{n^2}{\ln(n)^2} < \frac{(n+1)^2}{\ln(n+1)^2} \tag{2.2}$$

Re-arranging, this becomes

$$\left(1+\frac{2}{n}+\frac{1}{n^2}\right) > \frac{\ln(n+1)^2}{\ln(n)^2} \tag{2.3}$$

and re-writing this as

$$\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(\frac{\ln(n)^2}{\ln(n+1)^2}\right) = X \tag{2.4}$$

where X must be > 1.

Both terms in (2.4) are uni-directional, i.e. as n increases, $(1+2/n+1/n^2)$ continuously decreases towards unity from a value greater than unity, (2.25 at $n = 2$), whereas $\ln(n)^2/\ln(n+1)^2$ continuously increases towards unity from a value less than unity, (0.630929753 at $n = 2$).

Thus it is clear that at n = 2, (2.4) gives

$$X = 1.419591046 \tag{2.5}$$

and as n increases, X slowly decreases towards unity. Note that if $X = 1$ for some finite value of n, Legendre's Conjecture would be false.

The relationship of (2.4) can never be less than unity because this would mean that there would be more primes from zero to n^2 than there were from zero to $(n+1)^2$, i.e. $\pi(n)^2$ would be greater than $\pi(n+1)^2$.
Also, X can never equal unity for a single finite value of n because this would incur a discontinuity in X at that value of n and from the nature (2.4) this is seen to be impossible, i.e. (2.4) is linear for all n. Consequently, if X were unity at some finite value of n, it would have to be unity at all subsequent values of n to avoid the discontinuity. However, this would mean that the total number of primes within the Natural Numbers was finite, and it has been proved by Euclid, circa 300BC, (and many others since), [1], that this is not the case. Therefore X can only reach the value of unity as $n \to \infty$.

2.2 Estimation of the Number of Composites Between Adjacent Squares.

The number of odd numbers between adjacent squares is simply

$$Odds = \frac{(n+1)^2 - n^2 - 1}{2} = n \qquad (2.6)$$

With regard to the Prime Number Theorem, Tchebychev proved in 1852, [2], that

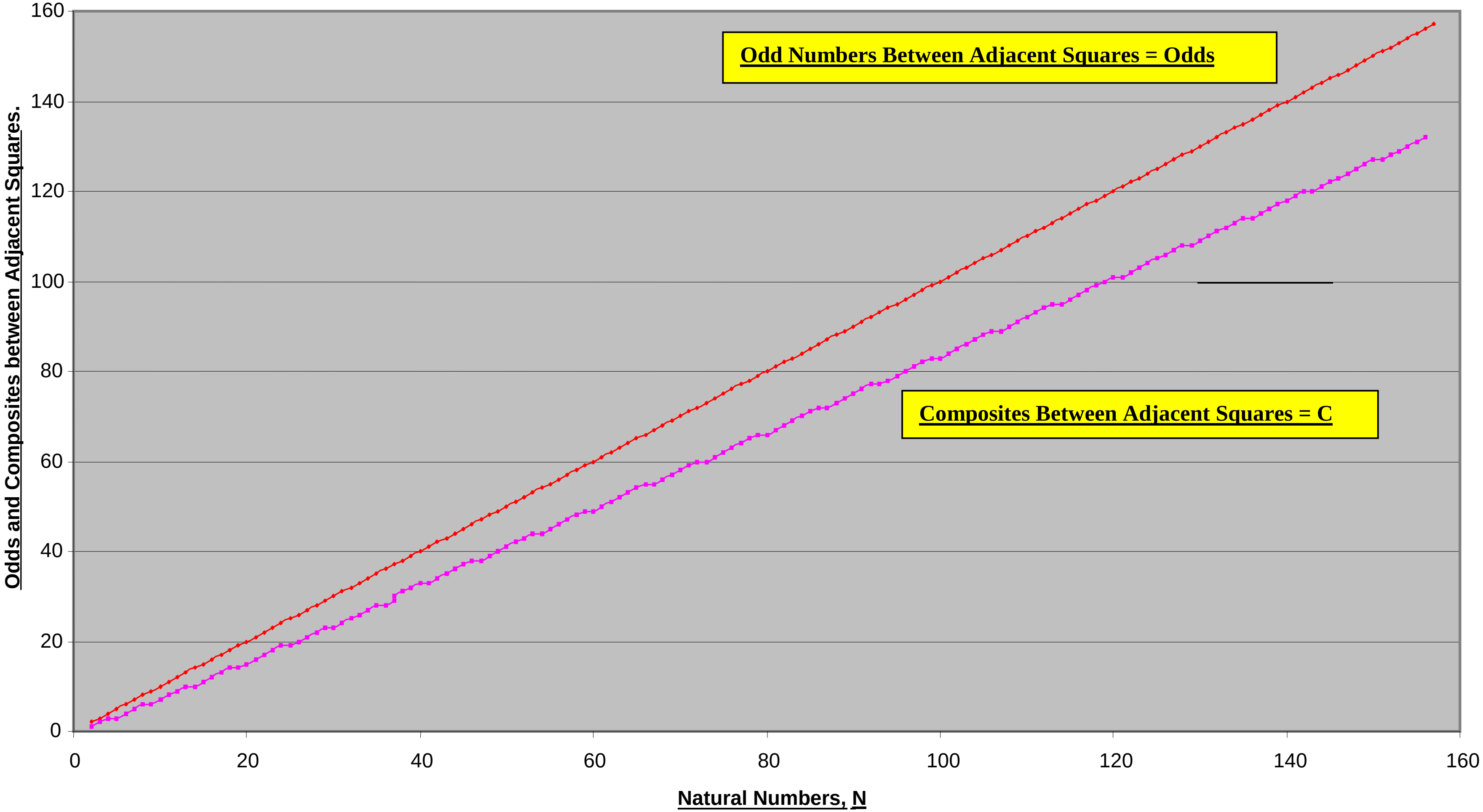
$$\frac{(0.92...)x}{\ln x} < \pi(x) < \frac{(1.105...)x}{\ln x} \qquad (2.7)$$

Taking the lower value, then the maximum number of composites between adjacent squares can be estimated as

$$C = n - 0.92\left[\frac{(n+1)^2}{\ln(n+1)^2} - \frac{n^2}{\ln(n)^2}\right] \qquad (2.8)$$

A plot of (2.8) together with the *ODDS* of (2.6) against the Natural Numbers is shown below as Fig. 2.1. It is clear from this figure that the gap between *ODDS* and C continues to widen as N increases, and in view of Tchebychev's proof of (2.7) this must continue for all N as $N \to \infty$. Of course this gap represents an estimate of the minimum number of primes between adjacent squares.

Fig. 2.1 - Legendre Conjecture Proof - Corrolory.



3.0 Conclusions.

The use of the Prime Number Theorem in this paper may be considered suspect because this theorem is only an approximation to $\pi(n)$, the distribution of prime numbers throughout the Natural Numbers. However, the Prime Number Theorem consistently underestimates $\pi(n)$ thereby overestimating the number of composites in any range of numbers. the analysis leading to Eq.(2.4) and its discussion is therefore superior to what could be achieved using Gaus's Li(n) or Riemann's R(n) both of which explore over and under estimates of $\pi(n)$.

Also, in the second Section taking the lower value of Tchebychev's proof of Eq.(2.7), further accentuates the overestimate of composites between adjacent squares. Consequently, while the proof here cannot be claimed to be a 'definitive' proof, it is sufficient to provide an excellent circumstantial proof of Legendre's Conjecture.

It is believed that a fully definitive proof of this conjecture will only be possible when a complete analytical expression for the distribution of primes within the Natural Numbers is discovered.

REFERENCES.

[1] Wikipedia, www.wikipedia.com.

[2] *David Wells*, Prime Numbers, John Wiley and Sons, Inc.