# The relationship between the $\varphi(n)$ function and solutions of Diophantine equations

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**ABSTRACT.** In this work we used an algebraic method that uses elementary algebra . To create series. We used the series and Euler function $\varphi(n)$ to find solutions to some types of Diophantine equations such as $p = dn - n + 1$ . We found a relationship between the solutions of the Diophantine equations and solutions of some types of congruences that use the $\varphi(n)$ function. This relationship is the results that relate the solutions of congruence to the solution of the equations.

**Key word:** series, Diophantine equation , congruences, Euler function

## 1.INTRODUCTION

According binomial theorem and difference of tow nth power theorem if n a positive integer and x y real numbers then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^j y^{n-j}$$

And

$$x^n - y^n = (x-y)\sum_{j=1}^{n} x^{n-j} y^{j-1}$$

## 2.basic series

**Theorem.1** let $k\ and\ g\ real\ numbers\ where\ n\ is\ odd$ then

$$\frac{1+(k-g)^n}{1+k-g} - \frac{g^n-1}{g-1} = -\mathrm{k}\left(\frac{g^{n-1}-1}{g-1}\right) + \mathrm{k}\sum_{j=1}^{n-2}(-1)^{j-1}(k-g)^j\left(\frac{g^{n-1}-1}{g-1} - \left(g^{n-2}+g^{n-3}\ldots\ldots g^{n-j-1}\right)\right)$$

**Theorem.2** let $\varphi(n)$ $Euler\ function\ where$ $\varphi(m) = d(n-1)$ where n in an odd where $a^d \not\equiv 1(mod\ m)$ , $(a,m) = 1$ , $\forall a \in \mathbb{N}$ then

$$\frac{m^n+1}{m+1} \equiv \frac{a^{dn}-1}{a^d-1}(md\ m)$$

**Theorem.3** if $p$ $prime\ number$ $and$ $p = dn - n + 1$ where n is odd $(p,a) = 1$ then

$$\frac{p^n+1}{p+1} \equiv \frac{a^{dn}-1}{a^d-1}(mod\ \ p)$$

**Theorem.4** let p prime number and $a$ a positive integer $a^{p^{m-1}} \not\equiv 1 mod\ p^m$ then

$$\frac{p^{mp}+1}{p^m+1} \equiv \frac{a^{p^m}-1}{a^{p^{m-1}}-1}(mod\ \ p^m)$$

In this section we will create the basic series

**Basic series**. Let n is an odd $k, g, u,$ real numbers then

$$L^n{}_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g)$$

Where

$$L_n{}^n(k,g,u) = \frac{u^n+(k-g)^n}{u+k-g} - m\left(\frac{g^n-1}{g-1}\right)$$

And

$$V_n{}^n(k,g,u) = \sum_{j=0}^{n-1}\left(u^{n-j-1}-m\right)(k-g)^j$$

And

$$S_n(k,h) = -km\left(\frac{g^{n-1}-1}{g-1}\right) + km\sum_{j=1}^{n-2}(-1)^{j-1}(k-g)^j\left(\frac{g^{n-1}-1}{g-1} - \left(g^{n-2}+g^{n-3}\ldots\ldots\ldots.g^{n-j-1}\right)\right)$$

**Proof.** let $k, g, u$ real number then according to difference of tow nth power theorem we have that

$$(k-g)^n - (-g)^n = \mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

$$-(-g)^n = -(k-g)^n + \mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

let $q \in R, n \in N$ where m constant then by multiplying m and adding $u^q(k-g)^n$ from both sides

$$u^q(k-g)^n - m(-g)^n = u^q(k-g)^n - m(k-g)^n + \mathrm{km}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

$$(1) \qquad u^q(k-g)^n - m(-g)^n = (u^q-m)(k-g)^n + m\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

According difference nth power theorem if n is odd we have

$$\frac{u^n+(k-g)^n}{u+k-g}$$
$$=u^{n-1}-u^{n-2}(k-\mathrm{g})+u^{n-3}(k-g)^2-u^{n-4}(k-g)^3\ldots\ldots\ldots\ldots\ldots\ldots\ldots(k-g)^{n-1}$$

And

$$m\left(\frac{g^n-1}{g-1}\right)=g^{n-1}+g^{n-2}+g^{n-1}\ldots\ldots\ldots\ldots\ldots 1$$

By subtracting $m\left(\frac{g^n-1}{g-1}\right)$ $from$ $\left(\frac{u^n+(k-g)^n}{u+k-g}\right)$ $then$

$$\frac{u^n+(k-g)^n}{u+(k-g)}-m\left(\frac{g^n-1}{g-1}\right)$$

$$=u^{n-1}-m-u^{n-2}(k-\mathrm{g})-mg+u^{n-3}(k-g)^2-mg^2-u^{n-4}(k-g)^3$$
$$-mg^3\ldots\ldots\ldots\ldots\ldots..(k-g)^{n-1}-mg^{n-1}$$

By extracting the common factor between the terms we find that

(2) $$\frac{u^n+(k-g)^n}{u+k-g}-m\left(\frac{g^n-1}{g-1}\right)$$
$$=u^{n-1}-m-(u^{n-2}(k-\mathrm{g})+mg)+(u^{n-3}(k-g)^2-mg^2)$$
$$-(u^{n-4}(k-g)^3+mg^3)\ldots\ldots\ldots\ldots((k-g)^{n-1}-mg^{n-1})$$

So we note in equation (2) term (1) equal $u^{n-1}-m$ and term(2) equal $u^{n-2}(k-\mathrm{g})+mg$ and tem (3) equal $u^{n-3}(k-g)^2-mg^2$ so From equation (1) we have

$$u^q(k-g)^n-m(-g)^n=(u^q-m)(k-g)^n+m\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Let

$$W_n{}^q(k,g,u)=u^q(k-g)^n-m(-g)^n$$

And

$$Z_n{}^q(k,g,u)=(u^q-m)(k-g)^n$$

And

$$C_n(k,g)=m\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

So

(3) $$W_n{}^q(k,g,u)=Z_n{}^q(k,g,u)+C_n(k,g)$$

From equation (3) and term (1) in equation (2)

$$u^{n-1}-m=W_0{}^{n-1-0}(k,g,u)$$

From equation (3) and term (2) in equation (2)

$$u^{n-2}(k-\mathrm{g})+mg=W_1{}^{n-2}(k,g,u)$$

Term (3) and equation (2)

$$u^{n-3}(k-g)^2-mg^2=W^{n-3}{}_2(kgu)$$

Last term in equation (2)

$$(k-g)^{n-1} - mg^{n-1} = W^{n-1-n+1}{}_{n-1}(k,g,u)$$

Then we have that

$$(4) \qquad \frac{u^n + (k-g)^n}{u+k-g} - m\left(\frac{g^n - 1}{g-1}\right) = \sum_{j=0}^{n-1} (-1)^j W^{n-1-j}{}_j(k,g,u)$$

We note from equation (3)

$$W_n{}^q(k,g,u) = Z_n{}^q(k,g,u) + C_n(k,g)$$

Where

$$Z_n{}^q(k,g,u) = (u^q - m)(k-g)^n$$

And

$$C_n(k,g) = m\mathrm{k} \sum_{j=1}^{n} (k-g)^{j-1}(-g)^{n-j}$$

From equation (3) and (4) we have

$$(5) \qquad \frac{u^n + (k-g)^n}{u+k-g} - m\left(\frac{g^n - 1}{g-1}\right)$$

$$= \sum_{j=0}^{n-1} \left(u^{n-1-j} - m\right)(k-g)^j + km \sum_{j=1}^{n-1} \sum_{r=}^{j} (-1)^j (k-g)^{r-1}(-g)^{j-r}$$

Let

$$L_n(k,g,u) = \frac{u^n + (k-g)^n}{u+k-g} - m\left(\frac{g^n - 1}{g-1}\right)$$

And

$$V^n{}_n(k,g,u) = \sum_{j=0}^{n-1} (-1)^j \left(u^{n-j-1} - m\right)(k-g)^j$$

And

$$S_n(k,g) = \mathrm{km} \sum_{j=1}^{n-1} \sum_{r=1}^{j} (-1)^j (k-g)^{r-1}(-g)^{j-r}$$

Then we have

(6) $$L_n(k\mathrm{g},u) = {V_n}^n(k,g,u) + S_n(k,g)$$

Note $g^{j-r}(-h) = (-1)^{j-r}g^{j-r}(h)$ and $(-1)^j(-1)^{j-r} = (-1)^{2j-r} = (-1)^r$ if j and r is odd or even note we find in $s_n(k,h)$

$$S_n(k,g) = \mathrm{km}\sum_{j=1}^{n-1}\sum_{r=1}^{j}(-1)^r(k-g)^{r-1}(-g)^{j-r}$$

Then we have

$$s_n(k,g) = \mathrm{km}\left(\sum_{r=1}^{1}(-1)^r(k-g)^{r-1}g^{1-r} + \sum_{r=1}^{2}(-1)^r(k-g)^{r-1}g^{2-r} + \sum_{r=1}^{3}(-1)^r(k-g)^{r-1}g^{3-r}\ \ldots\ldots\ldots\ldots\sum_{r=1}^{n-1}(-1)^r(k-g)^{r-1}g^{n-r}\right)$$

By analyzing all the complex terms of the $S_n(k,g)$ we find that

$$S_n(k,h) = km\Big((-1) + \big(-g + (k-g)\big) + (-g^2 + g(k-g) - (k-g)^2) - (-g^3 + g^2(k-\mathrm{g}) - g(k-g)^2 + (k-g)^3)\ \ldots\ldots\ldots(-g^{n-1} + g^{n-2}(k-\mathrm{g}) - g^{n-3}(k-g)^2 + g^{n-4}(k-g)^3\ \ldots\ldots\ldots\ldots(k-g)^{n-2})\Big)$$

In $S_n(k,h)$ a all compound terms have been dismantled note if we add for every first term in the complex term we find that $-(-1+g\ \ldots\ldots\ldots g^{n-2})$ then we adding the terms to include that $(k-\mathrm{g})$ finding that $(1+g\ \ldots\ldots g^{n-2})$ then the terms that include $(k-g)^2$ we find that $\big(-(1+g\ \ldots\ldots g^{j-3})\big)$ if the method is equal all the terms can be added $1 \le j \le n-1$ until we reach the last terms $(k-g)^{n-1}$ then

$$s_n(k,h) = \mathrm{km}\big(-(1+g+g^2\ \ldots\ldots\ldots g^{n-2}) + (k-g)\big((1+g+g^2+g^3\ \ldots\ldots\ldots g^{n-3})\big) - (k-g)^2(1+g+g^2+g^3\ \ldots\ldots\ldots g^{n-4})\ \ldots\ldots\ldots(k-g)^{n-1}\big)$$

Using the binomial theorem it is possible to abbreviate all the terms that include, $(k-\mathrm{g})$ *and* $(k-g)^2$ and $(k-g)^3$until we reach the last term $(k-g)^{n-1}$, we notice that

$$-(1+g+g^2\ \ldots\ldots\ldots g^{n-2}) = \frac{g^{n-1}-1}{g-1}$$

$$(k-g)(1+\mathrm{g}\ \ldots\ldots\ldots g^{n-3}) = (k-g)\left(\frac{g^{n-1}-1}{g-1} - g^{n-2}\right)$$

$$(k-g)^2(1+g\ \ \ldots\ldots\ldots g^{n-4}) = (k-g)^2\left(\frac{g^{n-1}-1}{g-1} - g^{n-2} - g^{n-3}\right)$$

Then we have that

$$S_n(k,h) = \mathrm{km}\left(\frac{g^{n-1}-1}{g-1}\right) + \mathrm{km}\sum_{j=1}^{n-2}(-1)^{j-1}(k-g)^j\left(\frac{g^{n-1}-1}{g^n-1} - \big(g^{n-2}+g^{n-3}\ \ldots\ldots g^{n-j-1}\big)\right)$$

Then

(7) $$L_n(k,g,u) = \frac{u^n + (k-g)^n}{u+k-g} - m\left(\frac{g^n-1}{g-1}\right)$$

(8) $$V^n{}_n(k,g,u)=\sum_{j=0}^{n-1}(-1)^j\left(u^{n-j-1}{}_j-m\right)(k-g)^j$$

(9) $$S_n(k,h)$$
$$=-\mathrm{km}\left(\frac{g^{n-1}-1}{g-1}\right)+\mathrm{km}\sum_{j=1}^{n-2}(-1)^{j-1}(k-g)^j\left(\frac{g^{n-1}-1}{g-1}-\left(g^{n-2}+g^{n-3}\ldots..g^{n-j-1}\right)\right)$$

## 3.proof theorem.1

In this section we will use the basic series $L_n(u,k,g)=V^n{}_n(u,k,g)+S_n(k,g)$ in prove the theorem.1 and use the theorem.1 to prove theorem.2 let in $V_n{}^n(u,k,g), u=1\ and\ m=1$ then we find

$$V_n{}^n(k,h,1)=\sum_{j=1}^{n-1}(-1)^j\left((1)^{n-j}-1\right)(k-g)^j=0$$

Then

$$L_n(u,k,1)=V_n{}^n(u,k,1)+S_n(k,g)$$

Then

$$L_n(u,k,1)=0+S_n(k,g)$$

According to the equations, (2,7, 2.8 ,2.9) we find that

$$\frac{1+(k-g)^n}{1+k-g}-\frac{g^n-1}{g-1}$$
$$=-\mathrm{k}\left(\frac{g^{n-1}-1}{g-1}\right)$$
$$+\mathrm{k}\sum_{j=1}^{n-2}(-1)^{j-1}(k-g)^j\left(\frac{g^{n-1}-1}{g-1}-\left(g^{n-2}+g^{n-3}\ldots\ldots g^{n-j-1}\right)\right)$$

## Proof.theorem.2 and theorem.3

According to Euler 's theorem $(\mathrm{a},\mathrm{n})=1$ where $\varphi(\mathrm{n})$ Euler function then $\mathrm{a}^{\varphi(\mathrm{n})}\equiv 1(\mathrm{mod}\ \mathrm{n})$ see $[\mathrm{K.M}\ 244]$

**proof. Theorem.2** from theorem.1 if n is odd and k g real number we have

$$\frac{1+(k-g)^n}{1+k-g}-\frac{g^n-1}{g-1}$$
$$=-\mathrm{k}\left(\frac{g^{n-1}-1}{g-1}\right)$$
$$+\mathrm{k}\sum_{j=1}^{n-2}(-1)^{j-1}(k-g)^j\left(\frac{g^{n-1}-1}{g-1}-\left(g^{n-2}+g^{n-3}\ldots\ldots g^{n-j-1}\right)\right)$$

Let in theorem.1 $k=a^d+m\ \ and\ g=a^d$ then $k-g=m$ so we have

$$\frac{1+m^n}{1+m}-\frac{a^{dn}-1}{a^d-1}$$
$$=-(a^d+m)\left(\frac{a^{d(n-1)}-1}{g-1}\right)$$
$$+(a^d+m)\sum_{j=1}^{n-2}(-1)^{j-1}m^j\left(\frac{a^{dn-d}-1}{a^d-1}-\left(a^{dn-2d}+a^{dn-3d}\ldots\ldots a^{dn-jd-1d}\right)\right)$$

Then

(10) $$\frac{1+m^n}{1+m}-\frac{a^{dn}-1}{a^d-1}=-(a^d+m)\left(\frac{a^{d(n-1)}-1}{a^d-1}\right)$$
$$+m\left(\sum_{j=1}^{n-2}(-1)^{j-1}m^{j-1}\left(\frac{a^{dn-d}-1}{a^d-1}-\left(a^{dn-2d}+a^{dn-3d}\ldots\ldots a^{dn-jd-1d}\right)\right)\right)$$

Let $V\ equal$

(11) $$V=(a^d+m)\sum_{j=1}^{n-2}(-1)^{j-1}m^{j-1}\left(\frac{a^{dn-d}-1}{a^d-1}-\left(a^{dn-2d}+a^{dn-3d}\ldots\ldots a^{dn-jd-1d}\right)\right)$$

From equation (10) and (11) we have that

( 12) $$\frac{1+m^n}{1+m}-\frac{a^{dn}-1}{a^d-1}=-(a^d+m)\left(\frac{a^{d(n-1)}-1}{a^d-1}\right)+mV$$

Let $\varphi(m)=d(n-1)\ where\ \varphi(m)\ Euler\ function\ then$ we note in rigor side equation

(13) $$\frac{1+m^n}{1+m}-\frac{a^{dn}-1}{a^d-1}=-(a^d+m)\left(\frac{a^{\varphi(m)}-1}{a^d-1}\right)+mV$$

According Euler theorem

$$a^{\varphi(m)}\equiv 1(mod\ m)$$

From equation (13) and Euler theorem if $a^d\not\equiv 1(mod\ m)$ we have

$$\frac{m^n+1}{m+1}\equiv\frac{a^{dn}-1}{a^d-1}(mod\ m)$$

**Proof. Theorem.3** from equation (13) we have that

$$\frac{1+m^n}{1+m}-\frac{a^{dn}-1}{a^d-1}=-(a^d+m)\left(\frac{a^{\varphi(m)}-1}{a^d-1}\right)+mV$$

Let $m=p\ where\ p\ prime\ number\ \ according\ Euler\ function\ \varphi(p)=p-1=d(n-1)\ and\ n\ is\ odd\ then\ we\ have$

$$\frac{1+p^n}{1+p}-\frac{a^{dn}-1}{a^d-1}=-(a^d+p)\left(\frac{a^{p-1}-1}{a-1}\right)+pV$$

Then If $p-1=d(n-1)$ we n is odd we have

$$\frac{p^n+1}{p+1} \equiv \frac{a^{dn}-1}{a^d-1} (mod\ p)$$

**Proof. Theorem.4** according theorem.2 if $\varphi(m) = d(n-1)$ $where\ n\ is\ odd\ we\ have$

$$\frac{m^n+1}{m+1} \equiv \frac{a^{dn}-1}{a^d-1} (mod\ m)$$

let in theorem.1 $m = p^m$ $and\ n = p\ then\ according\ Eulere\ function\ \varphi(p^m) = p^{m-1}(p-1)$ $so$ $d = p^{m-1}$ $and$ $dn = p^m$ we have that

$$\frac{p^{mp}+1}{p^m+1} \equiv \frac{a^{p^m}-1}{a^{p^{m-1}}-1} \left(mod\ p^m\left(a^{p^{m-1}}+p^m\right)\right)$$

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