About the properties of prime numbers in the form $md^m + 1$ and $d^m + 1$

By Shazly Abdullah

ABSTRACT. In this study we used an algebraic method that uses elementary algebra and binomial theorem. To create series .We used these series to study the prime numbers of the form $p = md^m + 1$ and $q = d^m + 1$, We found several characteristics .for example, we proved If , p prime number and $p = md^m + 1$ where d is odd then $m^{d^m} \equiv -1 (mod\ p)$ .We also obtained several results in finite series.

Key words: binomial theorem, series , prime numbers , finite series

1.INTRODUCTION

Numbers of the form $b_n = k2^n + 1$ called *broth* and $C_n = n2^n + 1$ *called culled number* in There are also other similar formulas, such as Mersenne counter and Fermat, such numbers. It is known that they provide us with large prime numbers, [*see James J Tattersall* 146]

In this paper, elementary algebra and binomial theorem, and difference of tow nth power are used to created finite series in an algebraic method, then we used series to create congruence with specific properties. Through this process, we reached the theorem. 1 We used Theorem 1 to study the prime numbers in the form $p = md^m + 1$ and $q = d^m + 1$ of which we found about these numbers theorem.2 and several results in finite series.

According binomial theorem and difference of tow nth power theorem if n a positive integer and x y real numbers then [*see K.H* 22]

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^j y^{n-j}$$

And

$$x^n - y^n = (x-y)\sum_{j=1}^{n} x^{n-j} y^{j-1}$$

2.basic series

Theorem.1 let a d real numbers where n a positive integer then

$$a^{n-1}(d^n - 1) = \frac{d-1}{a-1}(a^n - 1) + (d-1)\sum_{j=1}^{n-1} \frac{(ad-1)^j}{(a-1)^j}\left(a^n - \binom{n}{0} - \binom{n}{1}(a-1) \dots\dots \binom{n}{j}(a-1)^j\right)$$

Theorem.2 if p q primers numbers and d m a positive integers, where $p = md^m + 1$ and $q = d^m + 1$ then

$$\begin{cases} m^{d^m} \equiv 1 (mod\ pq) & if\ \ d\ in\ even \\ m^{d^m} \equiv -1 (mod\ p) & if\ \ d\ is\ odd \end{cases}$$

In this section first we will create the basic binomial series Then we use the series to prove the first theorem and the other results

Basic binomial series. let $k, g, u,$ *real numbers and m constant then*

$$L_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g)$$

Where

$$L_n\,(k,g,u) = (u-k+g)^n - m(1+g)^n$$

And

$$V_n{}^n(k,g,u) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(u^{n-j}-m\right)(k-g)^j$$

And

$$S_n(k,g) = mk\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n - \left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2 \dots\dots\dots.\binom{n}{j}g^j\right)\right)$$

Proof. let $g\ k\ u$, real numbers then according to difference of tow nth power theorem we have that

$$(k-g)^n - (-g)^n = \mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

$$-(-g)^n = -(k-g)^n + \mathrm{k}\sum_{j=1}^{n} f^{j-1}(k,h)g^{n-j}(-h)$$

let $q \in R\ , n \in N$ where m constant then by multiplying m and adding $u^q(k-g)^n$ from both sides

$$u^q(k-g)^n - m(-g)^n = u^q(k-g)^n - m(k-g)^n + \mathrm{m\,k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

Then

$$(1) \qquad u^q(k-g)^n - m(-g)^n = (u^q - m)(k-g)^n + m\,\mathrm{k}\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

According binomial theorem

$$(u-k+g)^n = u^n - \binom{n}{1}u^{n-1}(k-g) + \binom{n}{2}u^{n-2}(k-g)^2 - \binom{n}{3}u^{n-3}(k-g)^3 \dots\dots\dots\dots\dots\dots\dots (k-g)^n$$

And

$$m(1+g)^n = m + m\binom{n}{1}g + m\binom{n}{2}g^2 + m\binom{n}{3}g^3 \dots\dots\dots\dots\dots.mg^n$$

By subtracting $m(1+g)^n \, from \quad (u-k+g)^n$ then

$$(u-k+g)^n - m(1+g)^n$$
$$= u^n - m - \binom{n}{1} u^{n-1}(k-g) - m\binom{n}{1} g + \binom{n}{2} u^{n-2}(k-g)^2 - m\binom{n}{2} g^2$$
$$- \binom{n}{3} u^{n-3}(k-g)^3 - m\binom{n}{3} g^3 \ldots\ldots\ldots\ldots\ldots..(k-g)^n - mg^n$$

By extracting the common factor $\binom{n}{j}$ between the terms

(2) $$(u-k+g)^n - m(1+g)^n$$
$$= u^n - m - \binom{n}{1}(u^{n-1}(k-g) + mg) + \binom{n}{2}(u^{n-2}(k-g)^2 - mg^2)$$
$$- \binom{n}{3}(u^{n-3}(k-g)^3 + mg^3) \ldots\ldots\ldots\ldots((k-g)^n - mg^n)$$

According equation (1)

$$u^q(k-g)^n - m(-g)^n = (u^q - m)(k-g)^n + m\,k\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

So we note in (2) limit (1) equal $u^n - m \; and$ limit (2) equal $\binom{n}{1}(u^{n-1}(k-g) + mg)$ and limit 2 equal $\binom{n}{2}(u^{n-2}(k-g)^2 - mg^2)$ and 3 equal $\binom{n}{3}(u^{n-3}(k-g)^3 + mg^3)$ $(k\; - g)^n - mg^n$ then

Let

(3) $$W_n{}^q(k,g,u) = \; u^q(k-g)^n - m(-g)^n$$

And

$$Z_n{}^q(\mathrm{k},g,\mathrm{u}) \; = (u^q - m)(k-g)^n$$

and

$$C_n\,(k,g) = \; m\,k\sum_{j=1}^{n}(k-g)^{j-1}(-g)^{n-j}$$

So

(4) $$W_n{}^q(k,g,u) = Z_n{}^q(\mathrm{k},g,\mathrm{u}) + C_n(k,g)$$

From (3) and term (1) in (2)

$$\binom{n}{0}(u^n - m) = \binom{n}{0} W_0{}^{n-0}(k,g,u)$$

From equation (3) and term (2) in equation (2)

$$\binom{n}{1}(u^{n-1}(k-g) + mg) = \binom{n}{1} W_1{}^{n-1}(k,g,u)$$

Limit (3)

$$\binom{n}{2}(u^{n-2}(k-g)^2 - mg^2) = \binom{n}{2} W_2{}^{n-2}(k,g,u)$$

term (4) in equation (2)

$$\binom{n}{3}(u^{n-3}(k-g)^3 + mg^3) = \binom{n}{3} W_3{}^{n-3}(k,g,u)$$

And

Last term

$$\binom{n}{n}\left((k-g)^n - mg^n\right) = \binom{n}{n} W_n{}^{n-n}(k,g,u)$$

so

$$(u-k+g)^n - m(1+g)^n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} w^{n-j}{}_j(k,g,u)$$

And

$$L_n(k,g,u) = (u-k+g)^n - m(1+g)^n$$

Then

$$L_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} w^{n-j}{}_j(k,g,u)$$

From equation(4) $w^q{}_n(k,g,u) = z_n{}^q(k,g,u) + c_n(k,g)$ then we have that

(5) $$L_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} z_j{}^{n-j}(k,g,u) + \sum_{j=0}^{n} (-1)^j \binom{n}{j} c_j(k,g)$$

We note from the equation (4)

$$Z_n{}^q(\mathrm{k},g,\mathrm{u}) = (u^q - m)(k-g)^n$$

And

$$C_n(k,g) = m\,k \sum_{j=1}^{n} (k-g)^{j-1}(-g)^{n-j}$$

Then

$$L_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j + \mathrm{mk} \sum_{j=1}^{n} \sum_{r=1}^{j} (-1)^j \binom{n}{j} (k-g)^{r-1}(-g)^{j-r}$$

Let

$$V^n{}_n(k,g,u) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(u^{n-j} - m\right)(k-g)^j$$

And

$$S_n(k,g) = \mathrm{mk} \sum_{j=1}^{n} \sum_{r=1}^{j} (-1)^j \binom{n}{j} (k-g)^{r-1}(-g)^{j-r}$$

Then we have

(6) $$L_n(k,g,u) = V_n{}^n(k,g,u) + S_n(k,g)$$

we find in $S_n(k,g)$ tow signs $(-1)^j(-1)^{j-r} = (-1)^r$ if r j even or odd so they can by combined in $(-1)^r$ then we find that

$$S_n(k,g) = \mathrm{mk} \sum_{j=1}^{n} \sum_{r=1}^{j} (-1)^r \binom{n}{j} (k-g)^{r-1} g^{j-r}$$

where

$$s_n(k,g) = \text{mk}\left(\sum_{r=1}^{1}(-1)^r\binom{n}{1}(k-g)^{r-1}g^{1-r} + \sum_{r=1}^{2}(-1)^r\binom{n}{2}(k-g)^{r-1}g^{2-r} + \sum_{r=1}^{3}(-1)^r\binom{n}{3}(k-g)^{r-1}g^{3-r}\dots\dots\dots\dots\sum_{r=1}^{n}(-1)^r\binom{n}{n}(k-g)^{r-1}g^{n-r}\right)$$

In $S_n(k,g)$ a all compound terms have been dismantled note if we add for every first term in the complex term we find that $-\left(\binom{n}{1}+\binom{n}{2}g\dots\dots\dots\binom{n}{n}g^{n-1}\right)$ then we adding the terms to include that $(k-\text{g})$ finding that $(k-g)$ $\left(\binom{n}{2}+\binom{n}{3}g\dots\dots\binom{n}{n}g^{n-2}\right)$ then the term that include $(k-g)^2$ we find that $(k-g)^2\left(-\left(\binom{n}{3}+\binom{n}{4}g\dots\dots\binom{n}{n}g^{n-3}\right)\right)$ if the method is equal all the terms can be added $1 \le j \le n-1$ until we reach the last terms $(k-g)^{n-1}$ then

$$S_n(k,g) = \text{mk}\Bigg(-\left(\binom{n}{1}+\binom{n}{2}g+\binom{n}{3}g^2\dots\dots\dots\binom{n}{n}g^{n-1}\right) + (k-\text{g})\left(\left(\binom{n}{2}+\binom{n}{3}g+\binom{n}{4}g^2+\binom{n}{5}g^3\dots\dots\dots\binom{n}{n}g^{n-2}\right)\right) - (k-g)^2\left(\binom{n}{3}+\binom{n}{5}g+\binom{n}{6}g^2+\binom{n}{7}g^3\dots\dots\dots\binom{n}{n}g^{n-3}\right)\dots\dots\dots\binom{n}{n}g^{n-n}\Bigg)$$

Using the binomial theorem it is possible to abbreviate all the terms that include, $(k-g)$ *and* $(k-g)^2$ and $(k-g)^3$ until we reach the last term $(k-g)^{n-1}$, we notice that

$$-\left(\binom{n}{1}+\binom{n}{2}g+\binom{n}{3}g^2\dots\dots\dots\binom{n}{n}g^{n-1}\right) = -\frac{(1+g)^n-\binom{n}{0}}{g}$$

$$\left(\binom{n}{2}+\binom{n}{3}\text{g}\dots\dots\dots\binom{n}{n}g^{n-2}\right) = \frac{(1+g)^n-\binom{n}{0}-\binom{n}{1}g}{g^2}$$

$$-\left(\binom{n}{3}+\binom{n}{4}g\ \dots\dots\dots\binom{n}{n}g^{n-3}\right) = -\frac{(1+g)^n-\binom{n}{0}-\binom{n}{1}g-\binom{n}{2}g^2}{g^3}$$

Then

$$S_n(k,g) = \text{m}k\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2\dots\dots\dots\binom{n}{j}g^j\right)\right)$$

Let

(7) $$L_n(k,g,u) = (u-\text{k}+\text{g})^n - m(1+g)^n$$

(8) $$V^n{}_n(k,g,u) = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\left(u^{n-j}-m\right)(k-g)^j$$

(9) $$S_n(k,g)$$

$$= \text{mk}\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}g+\binom{n}{2}g^2\dots\dots\dots\binom{n}{j}g^j\right)\right)$$

3.proof theorem.1

In this section we will use the basic series $L_n(k,g,u) = V^n{}_n(k,g,u) + S_n(k,g)$ in prove the theorem.1 then according basic infinite series if $u = 1$ in $V_n{}^n(k,g,u)$ we have that

$$V_n{}^n(k,g,1) = \sum_{j=1}^{n}(-1)^j\binom{n}{j}\left((1)^{n-j}-1\right)(k-g)^j = 0$$

Then according equations

$$L_n(k,g,1) = V_n{}^n(k,g,1) + S_n(k,g)$$

Then

$$L_n(k,g,1) = S_n(k,g)$$

Then according to the equations, (,7, 8 ,9) we find that

$$L_n(k,g,1) = S_n(k,g)$$

Then

$$(1-k+g)^n-(1+g)^n = \text{k}\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(k-g)^j}{g^{j+1}}\left((1+g)^n-\left(\binom{n}{0}+\binom{n}{1}\text{g}\dots\dots\binom{n}{j}g^j\right)\right)$$

Let a d a positive integers where

$$g = a-1$$
$$k = -a\text{d}+a$$

Then

$$(1+ad-a+a-1)^n-(1+a-1)^n$$
$$= (-ad+a)\sum_{j=0}^{n-1}(-1)^{j+1}\frac{(-ad+a-a+1)^j}{(a-1)^{j+1}}\left((1+a-1)^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots..\binom{h}{j}(a-1)^j\right)\right)$$

We have that

$$(ad)^n-a^n$$
$$= (-ad+a)\sum_{j=0}^{n-1}(-1)^{j+1+j}\frac{(ad-a)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots..\binom{n}{j}(a-1)^j\right)\right)$$

Then

$$a^{n-1}(d^n-1) = \frac{-d+1}{a-1}(a^n-1)$$
$$+(\text{d}-1)\sum_{j=1}^{n-1}\frac{(-1)^{2j+2}(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots\binom{n}{j}(a-1)^j\right)\right)$$

so we have that

$$a^{n-1}(d^n-1) = \frac{d-1}{a-1}(a^n-1)$$

$$+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^j}\left(a^n-\binom{n}{0}-\binom{n}{1}(a-1)\dots\dots\binom{n}{j}(a-1)^j\right)$$

Note the negative sign in the equation

(10)
$$a^{n-1}(d^n-1)$$
$$=\frac{d-1}{a-1}(a^n-1)+(d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^j}{(a-1)^{j+1}}\left(a^n-\left(\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots\dots\binom{n}{j}(a-1)^j\right)\right)$$

Then

$$a^{n-1}(d^n-1)=\frac{d-1}{a-1}(a^n-1)$$
$$+(ad-1)\left((d-1)\sum_{j=1}^{n-1}\frac{(ad-1)^{j-1}}{(a-1)^j}\left(a^n-\binom{n}{0}+\binom{n}{1}(a-1)\dots\dots\binom{n}{j}(a-1)^j\right)\right)$$

3. proof theorem.2

In this section we will prove theorem.2 using theorem.1 but before that we mention according to Euler's theorem $a^{\varphi(n)}\equiv 1(mod\, n)$ where $(a,n)=1$ and $\varphi(n)$ is Euler function see proof Euler theorem in $[K.M.244]$

Proof theorem.2 if $d=-k$ in theorem.1 we have

(11)
$$a^{n-1}((-k)^n-1)=-\frac{k+1}{a-1}(a^n-1)$$
$$-(ka+1)\left((-k-1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}(ak+1)^{j-1}}{(a-1)^j}\left(a^n-\binom{n}{0}+\binom{n}{1}(a-1)..\binom{n}{j}(a-1)^j\right)\right)$$

let in equation (11) $k=m\ and\ a=n=d^m\ then$

$$(d^m)^{d^m-1}\left((-m)^{d^m}-1\right)=-\frac{m+1}{d^m-1}\left((d^m)^{d^m}-1\right)$$
$$+(md^m+1)\left((m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}(md^m+1)^j}{(d^m-1)^j}\left((d^m)^n-\binom{n}{0}-\binom{n}{1}(d^m-1)\dots\dots\binom{n}{j}(d^m-1)^j\right)\right)$$

Then

(12)
$$d^{m(d^m-1)}\left((-m)^{d^m}-1\right)=-\frac{m+1}{d^m-1}\left(d^{md^m}-1\right)$$
$$+(md^m+1)\left((m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}(md^m+1)^j}{(d^m-1)^j}\left((d^m)^{d^m}-\binom{n}{0}-\binom{n}{1}(d^m-1)\dots\dots\binom{n}{j}(d^m-1)^j\right)\right)$$

Let p q prime number where $p=md^m+1$ and $q=d^m+1$ then according equation (12)

(13)
$$d^{m(q-1)}((-m)^{q-1}-1)=-\frac{m+1}{d^m-1}(d^{p-1}-1)$$

$$+\mathrm{p}\left((m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}\mathrm{p}^{j-1}}{(d^m-1)^j}\left(d^{p-1}-\binom{n}{0}+\binom{n}{1}(d^m-1)\ldots\ldots\binom{n}{j}(d^m-1)^j\right)\right)$$

Let in equation (13) V equal

(14) $$V=(m+1)\sum_{j=1}^{n-1}\frac{(-1)^{j-1}\mathrm{p}^{j-1}}{(d^m-1)^j}\left(d^{p-1}-\binom{n}{0}+\binom{n}{1}(d^m-1)\ldots\ldots\binom{n}{j}(d^m-1)^j\right)$$

Then from equations (14) and (13) we have

(15) $$d^{m\mathrm{q}-\mathrm{m}}((-m)^{q-1}-1)=-\left(\frac{m+1}{d^m-1}\right)(d^{p-1}-1)+pV$$

So according equation (13) $(-m)^{q-1}=(-m)^{d^m}$ then if $d\ is\ even\ in\ equation$ (15) we have

$$d^{m\mathrm{q}-\mathrm{m}}(\mathrm{m}^{q-1}-1)=-\left(\frac{m+1}{d^m-1}\right)(d^{p-1}-1)+pV$$

According Euler theorem

$$d^{p-1}\equiv 1(mod\ p)$$

Then from equation (15) and Euler theorem

$$\mathrm{m}^{q-1}\equiv 1(mod\ p)$$

Then

$$m^{q-1}\equiv 1(mod\ pq)$$

But $q=d^m+1\ \ then$

(16) $$m^{d^m}\equiv 1(mod\ \ pq)$$

And if $d\ is\ odd\ in\ equation$ (15) $we\ have\ that$ $(-m)^{q-1}=-m^{q-1}\ becuse\ \ q=d^m+1$

(17) $$-d^{m\mathrm{q}-\mathrm{m}}(\mathrm{m}^{q-1}+1)=-\left(\frac{m+1}{d^m-1}\right)(d^{p-1}-1)+pV$$

From Euler theorem

$$d^{p-1}\equiv 1(mod\ p)$$

Then from Euler theorem and equation (17) we have that

$$m^{q-1}\equiv -1(mod\ p)$$

But $q=d^m+1\ \ then$

(18) $$m^{d^m}\equiv -1(mod\ p)$$

End of proof

Lemma.1 let $d\ \in\mathbb{N}$ where then

$$(d^{n-1}-1)(d^n-1)=(d-1)\sum_{j=1}^{n-1}\frac{(d^2-1)^j}{(a-1)^{j+1}}\left(d^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

Proof. Let in theorem.1 $a=d\ then$

$$d^{n-1}(d^n-1)=\frac{d-1}{d-1}(d^n-1)$$
$$+(d-1)\sum_{j=1}^{n-1}\frac{(d^2-1)^j}{(d-1)^{j+1}}\left(d^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

Then

$$d^{n-1}(d^n-1)=(d^n-1)+(d-1)\sum_{j=1}^{n-1}\frac{(d^2-1)^j}{(d-1)^{j+1}}\left(d^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

Then

$$d^{n-1}(d^n-1)-(d^n-1)=(d-1)\sum_{j=1}^{n-1}\frac{(d^2-1)^j}{(d-1)^{j+1}}\left(d^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

So

$$(d^{n-1}-1)(d^n-1)=(d-1)\sum_{j=1}^{n-1}\frac{(d^2-1)^j}{(d-1)^{j+1}}\left(d^n-\left(\binom{n}{0}+\binom{n}{1}(d-1)\ldots\ldots\ldots\binom{n}{j}(d-1)^j\right)\right)$$

End of proof

References

1. H. S. Rose . A Course in Number Theory " Oxford Science publications (1988).
2. K. M. Rose, Elementary Number Theory" 4th Edition Addison- Wesley (2000)..
3. James J. Tattersall Elementary Number Theory in Nine Chapters Cambridge University Press 2005

Student :Shazly Abdullah Fdl
Faculty of mathematics sciences & statistics
Aleenlain University Sudan
Shazlyabdullah3@gmail.com