Set Theory $NC^{\#}_{\infty^{\#}}$ Based on Bivalent Infinitary Logic with Restricted Modus Ponens Rule.
Basic Real Analysis on External Non-Archimedean Field $\mathbb{R}^{\#}_{c}$.
Basic Complecs Analysis on External Field $\mathbb{C}^{\#}_{c} = \mathbb{R}^{\#}_{c}[\mathbf{i}]$.

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Abstract: In this paper we deal with set theory $\mathbf{NC}^{\#}_{\infty}$ based on gyper infinitary logic with Restricted Modus Ponens Rule [1]-[3].
The main goal of this paper is to present basic analysis on external non Archimedean field $\mathbb{R}^{\#}_{c}$.
The non Archimedean external field $\mathbb{R}^{\#}_{c}$ consist of Cauchy hyperreals.The non Archimedean external field $\mathbb{R}^{\#}_{c} \neq {}^{*}\mathbb{R}$ is obtained as generalized Cauchy completion of non-Archimedean field $\mathbb{Q}^{\#}$ or ${}^{*}\mathbb{Q}$. In order to obtain such completion we deal with external hyper infinite Cauchy sequences $\{x_n\}_{n\in\mathbb{N}^{\#}}, \{x_n\}_{n\in{}^{*}\mathbb{N}}$. We have emphasised that such external Cauchy sequences defined external hyperreal numbers in natural way.
Basic Analysis on External Non-Archimedean Field $\mathbb{R}^{\#}_{c}$ is considered.

Keywords: Infinitary logic;non Afchimedian field.
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Contents

1.Introduction

In this paper we deal with set theory $\mathbf{NC}_{\infty}^{\#}$ based on gyper infinitary logic with Restricted Modus Ponens Rule [1]-[3].
The main goal of this paper is to present basic analysis on non Archimedean field $\mathbb{R}_c^{\#}$. The non Archimedean field $\mathbb{R}_c^{\#}$ consist of Cauchy hyperreals.The non Archimedean field $\mathbb{R}_c^{\#}$ is obtained as generalized Cauchy completion of non Archimedean field $\mathbb{Q}^{\#}$ or ${}^{*}\mathbb{Q}$. In order to obtain such completion we deal with external hyper infinite Cauchy sequences $\{x_n\}_{n\in\mathbb{N}^{\#}}, \{x_n\}_{n\in|{}^{*}\mathbb{N}|}$.
Note that analysis on a non-Archimedean field $\mathbb{R}_c^{\#}$ is essentially different in comparison with analysis on non Archimedean field ${}^{*}\mathbb{R}$ [4]-[5] known in literature as nonstandard analysis, see for example [4]-[5].
Remind that Robinson nonstandard analysis (RNA) many developed using set-theoretical objects called superstructures [5]. A superstructure $\mathbf{V}(S)$ over a set S is

defined in the following way:

$$\mathbf{V}_0(S) = S, \mathbf{V}_{n+1}(S) = \mathbf{V}_n(S) \cup (P(\mathbf{V}_n(S)), \mathbf{V}(S) = \bigcup_{n\in\mathbb{N}} \mathbf{V}_n(S). \tag{1.1}$$

Superstructures of the empty set consist of sets of infinite rank in the cumulative hierarchy and therefore do not satisfy the in…nity axiom. Making $S = \mathbb{R}$ will suffice for virtually any construction necessary in analysis.

Bounded formulas are formulas where all quantifiers occur in the form

$$\forall x(x \in y \Rightarrow \cdots), \exists x(x \in y \Rightarrow \cdots). \tag{1.2}$$

A nonstandard embedding is a mapping

$$* : \mathbf{V}(X) \to \mathbf{V}(Y) \tag{1.3}$$

from a superstructure $\mathbf{V}(X)$ called the standard universum, into another superstructure $\mathbf{V}(Y)$, called nonstandard universum, satisfying the following postulates:

1. $Y = {}^*X$

2.**Transfer Principle**.For every bounded formula $\Phi(x_1,\ldots,x_n)$ and elements $a_1,\ldots,a_n \in \mathbf{V}(X)$, the property Φ is true for $a_1,\ldots,a_n$ in the standard universum if and only if it is true for ${}^*a_1,\ldots,{}^*a_n$ in the nonstandard universum:

$$\langle \mathbf{V}(X), \in \rangle \models \Phi(a_1,\ldots,a_n) \Leftrightarrow \langle \mathbf{V}(Y), \in \rangle \models \Phi({}^*a_1,\ldots,{}^*a_n). \tag{1.4}$$

3.**Non-triviality**.For every infinite set A in the standard universum, the set $\{{}^*a | a \in A\}$ is a proper subset of ${}^*A$.

Definition 1.1.[5].A set x is <u>internal</u> if and only if x is an element of ${}^*A$ for some element A of $\mathbf{V}(\mathbb{R})$. Let X be a set with $A = \{A_i\}_{i\in I}$ a family of subsets of X. Then the collection A has the infinite intersection property, if any infinite subcollection $J \subset I$ has non-empty intersection. Nonstandard universum is κ-saturated if whenever $\{A_i\}_{i\in I}$is a collection of internal sets with the infinite intersection property and the cardinality of I is less than or equal to κ, $\bigcap_{i\in I} A_i \neq \varnothing$.

Definition 1.2.[2]-[3].A set $S \subset {}^*\mathbb{N}$is a <u>hyper inductive</u> if the following statement holds

$$\bigwedge_{\alpha\in{}^*\mathbb{N}} (\alpha \in S \Rightarrow \alpha^+ \in S), \tag{1.5}$$

where $\alpha^+ \triangleq \alpha + 1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.As we see later there is just one hyper inductive subset of ${}^*\mathbb{N}$, namely ${}^*\mathbb{N}$itself.

In this paper we apply the following hyper inductive definitions of a sets [2]-[3]

$$\exists S \forall \beta \left[\beta \in S \Leftrightarrow \bigwedge_{0\leq\alpha<\beta} (\alpha \in S \Rightarrow \alpha^+ \in S) \right], \tag{1.6}$$

We extend up Robinson nonstandard analysis (**RNA**) by adding the following postulate:

4.Any hyper inductive set S is internal.

Remark 1.1.The statement **4** is not provable in ZFC but provable in set theory $\mathbf{NC}^{\#}_{\infty}$, see [2]-[3].Thus postulates 1-4 gives an nonconservative extension of RNA and we denote such extension by NERNA.

Remark 1.2.Note that NERNA of course based on the same gyper infinitary logic with Restricted Modus Ponens Rule as set theory $\mathbf{NC}^{\#}_{\infty}$ [1]-[3].

Remind that in RNA the following induction principle holds.
Theorem 1.1.[6]. Assume that $S \subset {}^*\mathbb{N}$ is internal set, then

$$(1 \in S) \wedge \forall x[x \in S \Rightarrow x+1] \Rightarrow S = {}^*\mathbb{N}. \quad (1.7)$$

In NERNA Theorem 1.1also holds.
Remark 1.3.It follows from postulate 4 and Theorem 1.1 that any hyper inductive set S is equivalent to ${}^*\mathbb{N}$: $S \equiv {}^*\mathbb{N}$.
Remark 1.4. Note that the following statement is provable in $\mathbf{NC}^{\#}_{\infty}$ [2-3]:
4′Axiom of hyper infinite induction

$$\forall S(S \subset {}^*\mathbb{N})\left\{\forall\beta(\beta \in {}^*\mathbb{N})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in S \Rightarrow \alpha^+ \in S)\right] \Rightarrow S = {}^*\mathbb{N}\right\}. \quad (1.8)$$

Thus postulate **4 of the theory** NERNA is provable in $\mathbf{NC}^{\#}_{\infty^{\#}}$.
Remark 1.5. Note that ${}^*\mathbb{N}$ is an model related to hypernaturals $\mathbb{N}^{\#}$ which we introduce axiomatically in [3], see subsection 2.1.
The paper is structured as follows.
In Sec. 2 set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$is formulated as a system
of axioms based on bivalent hyper infinitary logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule [1]-[3],[9].In Subsec.2.1.Axiom of nonregularity and axiom of hyperinfinity is formulated.
In Sec.3 nonstandard arithmetic $\mathbf{A}^{\#}$ related to hypernaturals $\mathbb{N}^{\#}$ [2-3] is formulated axiomatically.
In Sec.4 hyper inductive definitions in general case is considered.
In Sec.5.Fundamental examples of the hyper inductive definitions is considered.
In Sec.6.Nonstandard arithmetic $\mathbf{A}^{\#}$ is formulated by using finitary logic.
In Sec.7 defining hyperintegers $\mathbb{Z}^{\#}$ and hyperrational $\mathbb{Q}^{\#}$ numbers are given [2].
In Sec.8 Cauchy hyperreals $\mathbb{R}^{\#}_{\mathbf{c}}$ via generalized Cauchy completion is formulated.
In Sec.9 Extended Hyperreal Number System $\hat{\mathbb{R}}^{\#}_{c}$ is considered.
In Sec.11 Basic analisys on external non Archimedean field $\mathbb{R}^{\#}_{\mathbf{c}}$ is considered.

2.Set Theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ Based on Bivalent Gyper Infinitary Logic with Restricted Modus Ponens Rule.

Set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ is formulated as a system of axioms based on bivalent hyper infinitary logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule [1]-[3],see Appendix A. The language of set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ is a first-order hyper infinitary language $L^{\#}_{\infty^{\#}}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg(x = y)$ and $x \notin y$ for $\neg(x \in y)$. Individual variables $x,y,z,\ldots$ of $L^{\#}_{\infty^{\#}}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists!x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x)\ \&\ \forall y(\varphi(y) \Rightarrow x = y)]$. $L^{\#}_{\infty^{\#}}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called non-classical sets; we shall use upper case letters $A, B, \ldots$ for such sets. For each non-classical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x,A)]$ is called the defining axioms for the non-classical set A.

Remark 2.1.Remind that in logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ does not always guarantee that

$$\alpha, \alpha \Rightarrow \beta \vdash_{RMP} \beta \tag{2.1}$$

since for some α and β possible

$$\alpha, \alpha \Rightarrow \beta \nvdash_{RMP} \beta \tag{2.2}$$

even if the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ holds [1].

Abbreviation 2.2.We often write for the sake of brevity instead (2.1) by

$$\alpha \Rightarrow_s \beta \tag{2.3}$$

and we often write instead (2.2) by

$$\alpha \Rightarrow_w \beta. \tag{2.4}$$

Remark 2.2.Let A be an nonclassical set.Note that in set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$the following true formula

$$\exists A \forall x[x \in A \Leftrightarrow \varphi(x,A)] \tag{2.5}$$

does not always guarantee that

$$x \in A, x \in A \Rightarrow \varphi(x,A) \vdash_{RMP} \varphi(x,A) \tag{2.6}$$

even if $x \in A$ holds and (or)

$$\varphi(x,A), \varphi(x,A) \Rightarrow x \in A \vdash_{RMP} x \in A; \tag{2.7}$$

even $\varphi(x,A)$ holds, since for nonclassical set A for some y possible

$$y \in A,\ y \in A \Rightarrow \varphi(y,A) \nvdash_{RMP} \varphi(y,A) \tag{2.8}$$

and (or)

$$\varphi(y,A),\ \varphi(y,A) \Rightarrow y \in A \nvdash_{RMP} y \in A. \tag{2.9}$$

Remark 2.3.Note that in this paper the formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x) \wedge x \in u] \tag{2.10}$$

and more general formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \tag{2.11}$$

is considered as the defining axioms for the classical set a.

Remark 2.4.Let a be a classical set. Note that in $\mathbf{NC}^{\#}_{\infty^{\#}}$: (i) the following true formula

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \tag{2.12}$$

always guarantee that

$$x \in a, x \in a \Rightarrow \varphi(x,a) \vdash_{RMP} \varphi(x) \tag{2.13}$$

if $x \in a$ holds and

$$\varphi(x), \varphi(x) \Rightarrow x \in a \vdash_{RMP} x \in a; \tag{2.14}$$

if $\varphi(x)$ holds;

In order to emphasize this fact mentioned above in Remark 2.1-2.3,
we rewrite the defining axioms in general case for the nonclassical sets in the following
form

$$\exists A\forall x\{[x \in A \Leftrightarrow_s \varphi(x,A)] \vee [x \in A \Leftrightarrow_w \varphi(x,A)]\} \quad (2.15)$$

and similarly we rewrite the defining axioms in general case for the classical sets in the
following form

$$\forall x[x \in a \Leftrightarrow_s \varphi(x,a) \wedge (x \in u)]. \quad (2.16)$$

Abbreviation 2.2.We write instead (2.15):

$$\forall x\{[x \in A \Leftrightarrow_{s,w} \varphi(x,A)]\} \quad (2.17)$$

Definition 2.1. (1) Let A be a nonclassical set defined by formula (2.17).
Assum that: (i) for some y statement $\varphi(y)$ and statement $\varphi(y) \Rightarrow y \in A$ holds and
(ii) $\varphi(y), \varphi(y) \Rightarrow y \in A \nvdash_{RMP} y \in A$, $y \in A, y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$.
Then we say that y is a weak member of non-classical set A and abbreviate $y \in_w A$.
Abbreviation 2.3. Let A be a nonclassical set defined by formula (6.1) or by formula (6.2). We abbreviate $x \in_{s,w} A$ if the following statement $x \in_s A \vee x \in_w A$ holds, i.e.

$$x \in_{s,w} A \leftrightarrow_{def} (x \in_s A \vee x \in_w A). \quad (2.18)$$

Definition 2.2.(1) Two nonclassical sets A, B are defined to be equal and we write $A = B$ if $\forall x[x \in_{s,w} A \Leftrightarrow_s x \in_{s,w} B]$. (2) A is a subset of B, and we often write $A \subset_{s,v} B$, if $\forall x[x \in_{s,w} A \Rightarrow_s x \in_{s,w} B]$. (3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u\forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x)]$ and $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x,A)]$.
Remark 2.5.$\mathbf{CL.Set}(A)$) asserts that the set A is a classical set. For any classical set u,
it follows from the defining axiom for the classical set $\{x|x \in_s u \wedge \varphi(x)\}$ that $\mathbf{CL.Set}(\{x|x \in_s u \wedge \varphi(x)\})$.
We shall identify $\{x|x \in_s u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset_s A, u \subseteqq_s A$, etc.
Abbreviation 2.4.Let $\varphi(t)$ be a formula of $\mathbf{NC}^{\#}_{\infty^{\#}}$.
(i) $\forall x\varphi(x)$ and $\forall^{\mathbf{CL}}x\varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(ii) $\exists x\varphi(x)$ and $\exists^{\mathbf{CL}}x\varphi(x)$ abbreviates $\forall x(\mathbf{CL.Set}(x) \Rightarrow \varphi(x))$
(iii) $\forall X\varphi(X)$ and $\forall^{\mathbf{NCL}}X\varphi(X)$ abbreviates $\forall X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
(iv) $\exists X\varphi(X)$ and $\exists^{\mathbf{NCL}}X\varphi(X)$ abbreviates $\exists X(\mathbf{NCL.Set}(X) \Rightarrow \varphi(X))$
Remark 2.6.If A is a nonclassical set, we write $\exists x \in A\, \varphi(x,A)$ for $\exists x[x \in A \wedge \varphi(x,A)]$ and $\forall x \in A\varphi(x,A)$ for $\forall x[x \in A \Rightarrow \varphi(x,A)]$.
We define now the following sets:
1.$\{u_1, u_2, \ldots, u_n\} = \{x|x = u_1 \vee x = u_2 \vee \ldots \vee x = u_n\}$. 2. $\{A_1, A_2, \ldots, A_n\} =$ $= \{x|x = A_1 \vee x = A_2 \vee \ldots \vee x = A_n\}$. 3.$\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.
4.$\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5.$A \cup B = \{x|x \in A \vee x \in B\}$.
5.$A \cap B = \{x|x \in A \wedge x \in B\}$. 6.$A - B = \{x|x \in A \wedge x \notin B\}$. 7.$u^+ = u \cup \{u\}$.
8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10.$\mathbf{V} = \{x|: x = x\}$.
11.$\varnothing = \{x|x \neq x\}$.
The system $\mathbf{NC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:

Extensionality1: $\forall u \forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A \forall B[\forall x(x \in A \Leftrightarrow_{s,w} x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCL.Set}(\mathbf{V})$
Empty Set: $\mathbf{CL.Set}(\varnothing)$
Pairing1: $\forall u \forall v\ \mathbf{Cl.Set}(\{u,v\})$
Pairing2: $\forall A \forall B\ \mathbf{NCl.Set}(\{A,B\})$
Union1: $\forall u\ \mathbf{CL.Set}(\cup u)$
Union2: $\forall A\ \mathbf{NCL.Set}(\cup A)$
Powerset1: $\forall u\ \mathbf{CL.Set}(\mathbf{P}(u))$
Powerset2: $\forall A\ \mathbf{NCl.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^{+} \in a)]$
Separation1$\forall u_1 \forall u_2, \ldots \forall u_n \forall a \exists \mathbf{Cl.Set}(\{x \in_s a | \varphi(x,u_1,u_2,\ldots,u_n)\})$
Separation2$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{NCl.Set}(\{x \in_{s,w} A | \varphi(x,A;u_1,u_2,\ldots,u_n)\})$
Comprehension1$\forall u_1 \forall u_2, \ldots \forall u_n \exists A \forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x;u_1,u_2,\ldots,u_n)]$
Comprehension 2 $\forall u_1 \forall u_2, \ldots \forall u_n \exists A \forall x[x \in_{s,w} A \Leftrightarrow_{s,w} \varphi(x,A;u_1,u_2,\ldots,u_n)]$
Comprehension 3 $\forall u_1 \forall u_2, \ldots \forall u_n \exists a \forall x[x \in_s a \Leftrightarrow_s (a \subset u_1) \wedge \varphi(x,a;u_1,u_2,\ldots,u_n)]$
In particular:
Comprehension 3$'$ $\forall u \exists a \forall x[x \in_s a \Leftrightarrow_s (a \subset u) \wedge \varphi(x,a;u)]$
Hyperinfinity: see subsection 2.1.
Remark 2.**7**.Note that the axiom of hyper infinity follows from the schemata Comprehension 3.
Definition 2.**3**. The ordered pair of two sets u,v is defined as usual by

$$\langle u,v \rangle = \{\{u\},\{u,v\}\}. \tag{2.19}$$

Definition 2.**4**. We define the Cartesian product of two nonclassical sets A and B as usual by

$$A \times_{s,w} B = \{\langle x,y \rangle | x \in_{s,w} A \wedge y \in_{s,w} B\} \tag{2.20}$$

Definition 2.**5**. A binary relation between two nonclassical sets A,B is a subset $R \subseteq_{s,w} A \times_{s,w} B.$ We also write $aR_{s,w}b$ for $< a,b >\in_{s,w} R.$ The doman $\mathbf{dom}(R)$ and the range $\mathbf{ran}(R)$ of R are defined by

$$\mathbf{dom}(R) = \{x | \exists y(xR_{s,w}y)\}, \mathbf{ran}(R) = \{y : \exists x(xR_{s,w}y)\}. \tag{2.21}$$

Definition 2.**6**.A relation $F_{s,w}$ is a function, or map, written $\mathbf{Fun}(F_{s,w})$, if for each $a \in_{s,w} \mathbf{dom}(F)$ there is a unique b for which $aF_{s,w}b.$ This unique b is written $F(a)$ or $Fa.$ We write $F_{s,w} : A \to B$ for the assertion that $F_{s,w}$ s a function with $\mathbf{dom}(F_{s,w}) = A$ and $\mathbf{ran}(F_{s,w}) = B.$ In this case we write $a \mapsto F_{s,w}(a)$ for $F_{s,w}a.$
Definition 2.**7**.The identity map $\mathbf{1}_A$ on A is the map $A \to A$ given by $a \mapsto a$. If $X \subseteq_{s,w} A,$ the
map $x \mapsto x : X \to A$ is called the insertion map of X into $A.$
Definition 2.**8**.If $F_{s,w} : A \to B$ and $X \subseteq_{s,w} A,$ the restriction $F_{s,w}|X$ of $F_{s,w}$ to X is the map $X \to A$ given by $x \mapsto F_{s,w}(x)$. If $Y \subseteq_{s,w} B,$ the inverse image of Y under $F_{s,w}$ is the set

$$F_{s,w}^{-1}[Y] = \{x \in_{s,w} A : F_{s,w}(x) \in_{s,w} Y\}. \tag{2.22}$$

Given two functions $F_{s,w} : A \to B, G_{s,w} : B \to C,$ we define the composite function

$G_{s,w} \circ F_{s,w} : A \to C$ to be the function $a \mapsto G_{s,w}(F_{s,w}(a)).$ If $F_{s,w} : A \to A,$ we write $F_{s,w}^2$ for $F_{s,w} \circ F_{s,w}, F_{s,w}^3$ for $F_{s,w} \circ F_{s,w} \circ F_{s,w}$ etc.

Definition 2.9.A function $F_{s,w} : A \to B$ is said to be monic if for all
$x,y \in_{s,w} A, F_{s,w}(x) = F_{s,w}(y)$ implies $x = y$, epi if for any $b \in_{s,w} B$ there is $a \in_{s,w} A$ for which $b = F_{s,w}(a)$, and bijective, or a bijection, if it is both monic and epi. It is easily shown that
$F_{s,w}$ is bijective if and only if $F_{s,w}$ has an inverse, that is, a map $G_{s,w} : B \to A$ such that $F_{s,w} \circ G_{s,w} = \mathbf{1}_B$ and $G_{s,w} \circ F_{s,w} = \mathbf{1}_A$.

Definition 2.10.Two sets X and Y are said to be equipollent, and we write $X \approx_{s,w} Y$, if there is a bijection between them.

Definition 2.11.Suppose we are given two sets I,A and an epi map $F_{s,w} : I \to A$. Then $A = \{F_{s,w}(i) | i \in I\}$ and so, if, for each $i \in_{s,w} I$, we write a_i for $F_{s,w}(i)$, then A can be presented in the form of an indexed set $\{a_i : i \in_{s,w} I\}$. If A is presented as an indexed set of sets $\{X_i | i \in_{s,w} I\}$, then we write $\bigcup_{i\in I} X_i$ and $\bigcap_{i\in I} X_i$ for $\cup A$ and $\cap A$, respectively.

Definition 2.12.The projection maps $\pi_1 : A \times_{s,w} B \to A$ and $\pi_2 : A \times_{s,w} B \to B$ are defined to be the maps $< a,b > \mapsto$ a and $< a,b > \mapsto b$ respectively.

Definition 2.13.For sets A,B, the exponential B^A is defined to be the set of all functions
from A to B.

2.1.Axiom of nonregularity and axiom of hyperinfinity

Axiom of nonregularity

Remind that a non-empty set u is called regular iff $\forall x[x \neq \varnothing \to (\exists y \in x)(x \cap y = \varnothing)]$.

Let's investigate what it says: suppose there were a non-empty x such that $(\forall y \in x)(x \cap y \neq \varnothing)$. For any $z_1 \in x$ we would be able to get $z_2 \in z_1 \cap x$. Since $z_2 \in x$ we would be able to get $z_3 \in z_2 \cap x$. The process continues forever:
$\ldots \in z_{n+1} \in z_n \ldots \in z_4 \in z_3 \in z_2 \in z_1 \in x$.Thus if we don't wish to rule out such an infinite regress we forced accept the following statement:

$$\exists x[x \neq \varnothing \to (\forall y \in x)(x \cap y \neq \varnothing)]. \quad (2.23)$$

Axiom of hyperinfinity.

Definition 2.14.(i) A non-empty transitive non regular set u is a well formed non regular
set iff:

(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \quad (2.24)$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u_{n+1}^{+}, \quad (2.25)$$

where $a^+ = a \cup \{a\}$.

(ii) we define a function $a^{+[k]}$inductively by $a^{+[k+1]} = (a^{+[k]})^+$

Definition 2.15. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any
$n \in \mathbb{N}$

$$w \in u_n. \quad (2.26)$$

Definition 2.16. We say that an well formed non regular set u is infinite (or hyperfinite)

hypernatural nuber iff:
(I) For any member $w \in u$ one and only one of the following conditions are satified:
(i) $w \in \mathbb{N}$ or
(ii) $w = u_n$ for some $n \in \mathbb{N}$ or
(iii) $w \prec u$.
(II) Let ${}_{\prec}u$ be a set ${}_{\prec}u = \{z | z \prec u\}$, then by relation $(\cdot \prec \cdot)$ a set ${}_{\prec}u$ is densely ordered with no first element.
(III) $\mathbb{N} \subset u$.
Definition 2.17. Assume $u \in \mathbb{N}^{\#}$, then u is infinite (hypernatural) number if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$.
Axiom of hyperinfinity
There exists unique set $\mathbb{N}^{\#}$ such that:
(i) $\mathbb{N} \subset \mathbb{N}^{\#}$
(ii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number v such that $v \prec u$
(iii) if $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$ then there exists infinite (hypernatural) number w such that $u \prec w$
(v) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\cdot \prec \cdot)$ with no first and no last element.

3.Hypernaturals $\mathbb{N}^{\#}$.

In this section nonstandard arithmetic $\mathbf{A}^{\#}$ related to hypernaturals $\mathbb{N}^{\#}$ is considered axiomatically.

Axioms of the nonstandard arithmetic $\mathbf{A}^{\#}$ are:

Axiom of hyperinfinity
There exists unique set $\mathbb{N}^{\#}$ such that:
(i) $\mathbb{N} \subset \mathbb{N}^{\#}$
(ii) if u is infinite (hypernatural) number then there exists infinite (hypernatural) number v such that $v \prec u$
(iii) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that $u \prec w$
(iv) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\cdot \prec \cdot)$ with no first and no last element.
Axioms of infite ω-induction

(i)

$$\forall S (S \subset \mathbb{N}) \left\{ \left[\bigwedge_{n \in \omega} (n \in S \Rightarrow_s n^+ \in S) \right] \Rightarrow_s S = \mathbb{N} \right\}. \tag{3.1}$$

(ii) Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$, then

$$\left[\bigwedge_{n \in \omega} (F(n) \Rightarrow_s F(n^+)) \right] \Rightarrow_s \forall n (n \in \omega) F(n). \tag{3.2}$$

Definition 3.1.(i) Let β be a hypernatural such that $\beta \in \mathbb{N}^{\#} \backslash \mathbb{N}$. Let $[0, \beta] \subset \mathbb{N}^{\#}$ be a set such that $\forall x [x \in [0, \beta] \Leftrightarrow 0 \leq x \leq \beta]$ and let $[0, \beta)$ be a set $[0, \beta) = [0, \beta] \backslash \{\beta\}$.
(ii) Let $\beta \in \mathbb{N}^{\#} \backslash \mathbb{N}$ and let $\beta_\infty \subset \mathbb{N}^{\#}$be a set such that

$$\forall x \{x \in \beta_\infty \Leftrightarrow \exists k (k \geq 0) [0 \leq x \leq \beta^{+[k]}]\}. \tag{3.3}$$

Definition 3.2.Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is restricted on a classical set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff the following condition is satisfied

$$\forall\alpha[\alpha \in \mathbb{N}^{\#}\backslash S \Rightarrow_s \neg F(\alpha)]. \tag{3.4}$$

Definition 3.3.Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is strictly restricted on a set S such that $S \subsetneqq_s \mathbb{N}^{\#}$ iff there is no proper subset $S' \subset S$ such that a wff $F(x)$ is restricted on a set S'.

Example 3.1.(i)Let $\mathbf{fin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{fin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}$. Obviously wff $\mathbf{fin}(\alpha)$ is strictly restricted on a set $\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N}^{\#}\backslash\mathbb{N} \Rightarrow_s \neg\mathbf{fin}(\alpha)]$.
Let $\mathbf{hfin}(\alpha), \alpha \in \mathbb{N}^{\#}$be a wff formula such that $\mathbf{hfin}(\alpha) \Leftrightarrow_s \alpha \in \mathbb{N}^{\#}\backslash\mathbb{N}$ since $\forall\alpha[\alpha \in \mathbb{N} \Rightarrow_s \neg\mathbf{hfin}(\alpha)]$.

Definition 3.4. Let $F(x)$ be a wff of $\mathbf{NC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is unrestricted if wff $F(x)$ is not restricted on any set S such that $S \subsetneqq \mathbb{N}^{\#}$.

Axiom of hyperfinite induction 1

$$\forall S(S \subseteq_s [0,\beta])\forall\beta(\beta \in_s \mathbb{N}^{\#}) \searrow$$
$$\left\{\forall\alpha(\alpha \in_s [0,\beta))\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^{+} \in_s S)\right] \Rightarrow_s S = [0,\beta]\right\}. \tag{3.5}$$

Axiom of hyperfinite induction 1$'$

$$\forall S(S \subseteq_s [0,\beta_{\infty}])\forall\beta(\beta \in \mathbb{N}^{\#}) \searrow$$
$$\left\{\forall\alpha(\alpha \in [0,\beta_{\infty}])\left[\bigwedge_{0\leq\alpha<\beta_{\infty}}(\alpha \in S \Rightarrow \alpha^{+} \in S)\right] \Rightarrow S = [0,\beta_{\infty}]\right\}. \tag{3.5}$$

Axiom of hyper infinite induction 1

$$\forall S(S \subset_s \mathbb{N}^{\#})\left\{\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in_s S \Rightarrow \alpha^{+} \in_s S)\right] \Rightarrow_s S =_s \mathbb{N}^{\#}\right\}. \tag{3.6}$$

Definition 3.5.A set $S \subset_s \mathbb{N}^{\#}$is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha\in\mathbb{N}^{\#}}(\alpha \in_s S \Rightarrow_s \alpha^{+} \in_s S). \tag{3.7}$$

Obviously a set $\mathbb{N}^{\#}$ is a hyper inductive. Thus axiom of hyper infinite induction 1 asserts that a set $\mathbb{N}^{\#}$ this is the smallest hyper inductive set.

Axioms of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta \in [0,\beta])\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta])F(\alpha). \tag{3.8}$$

Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta_{\infty}]$ then

$$\left[\forall\beta(\beta \in [0,\beta_{\infty}])\left[\bigwedge_{0\leq\alpha<\beta_{\infty}}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\alpha(\alpha \in [0,\beta_{\infty}])F(\alpha). \tag{3.9}$$

Axiom of hyper infinite induction 2

Let $F(x)$ be anrestricted wff of the set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow_s F(\alpha^{+}))\right]\right] \Rightarrow_s \forall\beta(\beta \in \mathbb{N}^{\#})F(\beta). \tag{3.10}$$

4.Hyper inductive definitions in general.

A function $f : \mathbb{N}^\# \to A$ whose domain is the set $\mathbb{N}^\#$ is colled an hyper infinite sequence and denoted by $\{f_n\}_{n\in\mathbb{N}^\#}$ or by $\{f(n)\}_{n\in\mathbb{N}^\#}$The set of all hyperinfinite sequences whose terms belong to A is clearly $A^{\mathbb{N}^\#}$; the set of all hyperfinite sequences of $n \in \mathbb{N}^\#\backslash\mathbb{N}$ terms in A is A^n. The set of all hyperfinite sequences with terms in A can be defined as

$$\left\{R \subset \mathbb{N}^\# \times A : (R \text{ is a function}) \wedge \bigvee_{n\in\mathbb{N}^\#}(Dom(R) = n)\right\}, \qquad (4.1)$$

where $Dom(R)$ is domain of R. This definition implies the existence of the set of all hyper finite finite sequences with terms in A. The simplest case is the hyper inductive definition of a hyper infinite sequence $\{\varphi(n)\}_{n\in\mathbb{N}^\#}$ (with terms belonging to a certain set Z) satisfying the following conditions:

(a)

$$\varphi(0) = z, \varphi(n^+) = e(\varphi(n), n), \qquad (4.2)$$

where $z \in Z$ and e is a function mapping $Z \times \mathbb{N}^\#$ into Z.

More generally, we consider a mapping f of the cartesian product $Z \times \mathbb{N}^\# \times A$ into Z and seek a function $\varphi \in Z^{\mathbb{N}^\# \times A}$ satisfying the conditions :

(b)

$$\varphi(0,a) = g(a), \varphi(n^+,a) = f(\varphi(n,a), n, a), \qquad (4.3)$$

where $g \in Z^A$. This is a definition by hyper infinite induction with parameter a ranging over the set A. Schemes (a) and (b) correspond to induction "from n to $n^+ = n+1$",i.e. $\varphi(n^+)$ or $\varphi(n^+,a)$ depends upon $\varphi(n)$ or $\varphi(n,a)$ respectively. More generally, $\varphi(n^+)$ may depend upon all values $\varphi(m)$ where $m \leq n$ (i.e. $m \in n^+$). In the case of induction with parameter, $\varphi(n^+,a)$ may depend upon all values $\varphi(m,a)$, where $m \leq n$; or even upon all values $\varphi(m,a)$, where $m \leq n^+$ and $b \in A$. In this way we obtain the following schemes of

definitions by hyper infinite induction:

(c) $\varphi(0) = z, \varphi(n^+) = h(\varphi|n^+, n)$,

(d) $\varphi(0,a) = g(a), \quad \varphi(n^+,a) = H(\varphi|(n^+ \times A), n, a)$.

In the scheme (c), $z \in Z$ and $h \in Z^{C\times\mathbb{N}^\#}$, where C is the set of hyperfinite sequences whose terms belong to Z; in the scheme (d), $g \in Z^A$ and $H \in Z^{T\times\mathbb{N}^\#\times A}$, where T is the set of functions whose domains are included in $\mathbb{N}^\# \times A$ and whose values belong to Z.

It is clear that the scheme (d) is the most general of all the schemes considered above.

By choice of functions one obtains from (d) any of the schemes (a)-(d). For example, taking the function defined by $H(c,n,a) = f(c(n,a),n,a)$ for $a \in A, n \in \mathbb{N}^\#, c \in Z^{\mathbb{N}^\# \times A}$ as H in (d), one obtain (b). We shall now show that, conversely, the scheme (d) can be obtained from (a). Let g and H be functions belonging to Z^A and $Z^{T\times\mathbb{N}^\#\times A}$ respectively, and let φ be a function satisfying (d). We shall show that the sequence $\Psi = \{\Psi_n\}_{n\in\mathbb{N}^\#}$ with $\Psi_n = \varphi|(n^+,A)$ can be defined by (a).Obviously, $\Psi_n \in T$ for every $n \in \mathbb{N}^\#$. The first term of the sequence Ψ is equal to $\varphi|(0^+,A)$, i.e. to the set: $z^* = \{\langle\langle 0,a\rangle, g(a)\rangle | a \in A\}$. The relation between Ψ_n, and Ψ_{n^+} is given by the formula:$\Psi_{n^+} = \Psi_n \cup \varphi|(\{n^+\} \times A)$, where the second component is

$$\{\langle\langle n^+,a\rangle, \varphi(n^+,a)\rangle | a \in A\} = \{\langle n^+,a\rangle, H(\Psi_n,n,a) | a \in A\}. \qquad (4.4)$$

Thus we see that the sequence Ψ can be defined by (a) if we substitute T for Z, z^* for z

and let $e(c,n) = c \cup \{\langle n^+, a\rangle, H(c,n,a) | a \in A\}$ for $c \in T$.
Now we shall prove the existence and uniqueness of the function satisfying (a).
This theorem shows that we are entitled to use definitions by induction of the type (a).
According to the remark made above, this will imply the existence of functions satisfying
the formulas (b), (c), and (d). Since the uniqueness of such functions can be proved in
the same manner as for (a), we shall use in the sequel definitions by induction of any of
the types (a)-(d).

Theorem 3.1. If Z is any set $z \in Z$ and $e \in Z^{Z \times \mathbb{N}^{\#}}$, then there exists exactly one hyper infinite sequence φ satisfying formulas (a).

Proof. Uniqueness. Suppose that $\{\varphi_1(n)\}_{n\in\mathbb{N}^{\#}}$ and $\{\varphi_2(n)\}_{n\in\mathbb{N}^{\#}}$ satisfy (a) and let

$$K = \{n | n \in \mathbb{N}^{\#} \wedge \varphi_1(n) = \varphi_2(n)\} \tag{4.5}$$

Then (a) implies that K is hyperinductive. Hence $\mathbb{N}^{\#} = K$ and therefore $\varphi_1(n) \equiv \varphi_2(n)$.
Existence. Let $\Phi(z,n,t)$ be the formula $e(z,n) = t$ and let $\Psi(w,z,F_n)$ be the following formula:

$$(F_n \text{ is a function}) \wedge (Dom(F) = n^+) \wedge (F(0) = z) \wedge \bigwedge_{m\in n} \Phi(F_n(m), m, F_n(m^+)). \tag{4.6}$$

In other words, F is a function defined on the set of numbers $\leq n \in \mathbb{N}^{\#}$ such that $F(0) = z$ and $F(m^+) = e(F(m),m)$ for all $m < n \in \mathbb{N}^{\#}$.

Assumption 3.1.We assume now (but without loss of generality) that predicate $\Psi(w,z,F_n)$ is unrestricted on variable $n \in \mathbb{N}^{\#}$,see Definition 3.3.
We prove by hyper infinite induction that there exists exactly one function F_n such that $\Psi(n,z,F_n)$.
The proof of uniqueness of this function is similar to that given in the Theorem 3.1.
The existence of F_n can be proved as follows: for $n = 0$ it suffices to take $\{\langle 0,z\rangle\}$ as F_n; if $n \in \mathbb{N}^{\#}$ and F_n satisfies $\Psi(n,z,F_n)$, then $F_{n^+} = F_n \cup \{\langle n^+, e(F_n(n),n)\rangle\}$
satisfies the condition $\Psi(n^+, z, F_{n^+})$.
Now, we take as φ the set of pairs $\langle n,s\rangle$ such that $n \in \mathbb{N}^{\#}, s \in Z$ and

$$\exists F[\Psi(n,z,F) \wedge (s = F(n))]. \tag{4.7}$$

Since F is the unique function satisfying $\Psi(n,z,F)$, it follows that φ is a function.
For $n = 0$ we have $\varphi(0) = F_0(0) = z$; if $n \in \mathbb{N}^{\#}$, then $\varphi(n^+) = F_{n^+}(n^+) = e(F_n(n),n)$ by the definition of F_n; hence we obtain $\varphi(n^+) = e(\varphi(0),n)$. Theorem 3.2 is thus proved.

Remark 3.1.Note that Assumption 3.1 is not necessarily,see Appendix B.
We frequently define not one but several functions (with the same range Z) by a simultaneous induction:

$$\varphi(0) = z, \varphi(n^+) = f(\varphi(n),\psi(n),n), \psi(0) = t, \psi(n^+) = g(\varphi(n),\psi(n),n) \tag{4.8}$$

where $z,t \in Z$ and $f,g \in Z^{Z\times Z\times\mathbb{N}^{\#}}$.
This kind of definition can be reduced to the previous one. It suffices to notice that the hyper infinite sequence $\vartheta_n = \langle\varphi(n),\psi(n)\rangle$ satisfies the formulas:

$$\vartheta_0 = \langle z,t\rangle, \vartheta_{n^+} = e(\vartheta_n, n), \tag{4.9}$$

where we set

$$e(u,n) = \langle f(K(u),L(u),n), g(K(u),F(w),n)\rangle, \tag{4.10}$$

and K, L denote functions such that $K(\langle x,y\rangle)$ and $L(\langle x,y\rangle) = y$ respectively. Thus the function ϑ is defined by hyper infinite induction by means of (a). We now define φ and ψ by

$$\varphi(n) = K(\vartheta_n), \psi(n) = L(\vartheta_n). \qquad (4.11)$$

Remark 3.2.We assume now that predicate $\Psi(w,z,F_n)$ is restricted on variable $n \in \mathbb{N}^{\#}$, on a set $[0,\beta] \cup \widehat{\omega} \subset \mathbb{N}^{\#}$, see Definition 3.2, then there exists exactly one hyperfinite sequence φ satisfying formulas (a). Note that is a case if and only if $f, g \in Z^{Z\times Z\times[0,\beta]\cup\widehat{\omega}}$.

5.Fundamental examples of the hyper inductive definitions.

1.Addition operation of hypernatural numbers

The function $+(m,n) \triangleq m+n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$ is defined by

$m+0 = m, m+n^{+} = (m+n)^{+}$.

This definition is obtained from (b) by seting $Z = A = \mathbb{N}^{\#}, g(a) = a, f(p,n,a) = p^{+}$.

This function satisfies all properties of addition such as: for all $m,n,k \in \mathbb{N}^{\#}$

(i) $m+0 = m$ (ii) $m+n = n+m$ (iii) $m+(n+k) = (m+n)+k$.

2.Multiplicattion operation of gypernatural numbers

The function $\times(m,n) \triangleq m\times n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$ is defined by

$m\times 1 = 1, m\times n^{+} = m\times n + m$.

(i) $m\times 1 = 1$ (ii) $m\times n = n\times m$ (iii) $m\times(n\times k) = (m\times n)\times k$.

4.Distributivity with respect to multiplication over addition.

$m\times(n+k) = m\times n + m\times k$.

5. Let $Z = A = X^X, g(a) = I_X, f(u,n,a) = u\circ a$ in (b). Then (b) takes on the following form

$$\varphi(0,a) = I_X, \varphi(n^{+},a) = \varphi(n,a)\circ a. \qquad (5.1)$$

The function $\varphi(n,a)$ is denoted by a^n and is colled n-th iteration of the function a :

$$a^0(x) = x, a^{n^{+}}(x) = a^n(a(x)), x \in X, a \in X^X, n \in \mathbb{N}^{\#}. \qquad (5.2)$$

6.Let $A = (\mathbb{N}^{\#})^{\mathbb{N}^{\#}}, g(a) = a_0, f(u,n,a) = u + a_{n^{+}}$.Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^{+},a) = \varphi(n,a) + a_{n^{+}} \qquad (5.3)$$

The function is defined by the Eqs.(5.3) is denoted by

$$\sum_{i=0}^{n} a_i \qquad (5.4)$$

7.Let $A = (\mathbb{N}^{\#})^{\mathbb{N}^{\#}}, g(a) = a_0, f(u,n,a) = u\times a_{n^{+}}$.Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^{+},a) = \varphi(n,a)\times a_{n^{+}} \qquad (5.5)$$

The function is defined by the Eqs.(5.5) is denoted by

$$\prod_{i=0}^{n} a_i \qquad (5.6)$$

Theorem 5.1. The following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^{\#}$:

(1) using distributivity

$$b\times\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b\times a_i \qquad (5.7)$$

(2) using commutativity and associativity

$$\sum_{i=0}^{n} a_i \pm \sum_{i=0}^{n} b_i = \sum_{i=0}^{n}(a_i \pm b_i) \quad (5.8)$$

(3) splitting a sum, using associativity

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{j} a_i + \sum_{i=j+1}^{n} a_i \quad (5.9)$$

(4) using commutativity and associativity, again

$$\sum_{i=k_0}^{k_1} \sum_{j=l_0}^{l_1} a_{ij} = \sum_{j=l_0}^{l_1} \sum_{i=k_0}^{k_1} a_{ij} \quad (5.10)$$

(5) using distributivity

$$\left(\sum_{i=0}^{n} a_i\right) \times \left(\sum_{j=0}^{n} b_j\right) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_i \times b_j \quad (5.11)$$

(6)

$$\left(\prod_{i=0}^{n} a_i\right) \times \left(\prod_{i=0}^{n} b_i\right) = \prod_{i=0}^{n} a_i \times b_i \quad (5.12)$$

(7)

$$\left(\prod_{i=0}^{n} a_i\right)^m = \prod_{i=0}^{n} a_i^m \quad (5.13)$$

Proof. Imediately by hyper infinite induction principle.

6. Nonstandard arithmetic $\mathbf{A}^{\#}$.

Addition Operation of Hypernatural Numbers $\mathbb{N}^{\#}$

There is a unique binary operation $+(\bullet,\bullet) : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$

such that

(1) $+(m,0) = m$ for all $m \in \mathbb{N}^{\#}$

(2) $+(m,n+1) = +(m,n) + 1$ for all $m,n \in \mathbb{N}^{\#}$.

This definition satisfies all properties of addition such as

(i) $m + 0 = a$; (ii) $m + n = n + m$; (iii) $m + (n + k) = (m + n) + k$

Multiplication Operation of Hypernatural Numbers $\mathbb{N}^{\#}$

There is a unique binary operation $\times(m,n) \triangleq m \times n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$

such that

$m \times 1 = 1, m \times (n + 1) = m \times n + m$.

(i) $m \times 1 = 1$ (ii) $m \times n = n \times m$ (iii) $m \times (n \times k) = (m \times n) \times k$.

Inequalities

The usual total order binary relation $\leq$ on hypernatural numbers $\mathbb{N}^{\#}$ defined as follows, assuming 0 is a hypernatural number:

For all $a,b \in \mathbb{N}^{\#}, a \leq b$ if and only if there exists some $c \in \mathbb{N}^{\#}$ such that $a + c = b$.

This relation is stable under addition and multiplication: for $a,b,c \in \mathbb{N}^{\#}$, if $a \leq b$, then: $a + c \leq b + c$, and $a \times c \leq b \times c$.

Proposition 6.1. (a) For any natural or hypernatural number $k \in \mathbb{N}^{\#}$,

$$\vdash \bigvee_{0 \leq m \leq k} (x = m) \Leftrightarrow_s x \leq k. \quad (6.1)$$

(a′) For any natural or hypernatural number $\kappa \in \mathbb{N}^{\#}$ and any wff $B(x)$ unbounded on variable x

$$\vdash \bigwedge_{0\leq m\leq k} B(m) \Leftrightarrow_s \forall x(x \leq k \Rightarrow_s B(x)), \tag{6.2}$$

i.e.

$$\vdash \forall k(k \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq m\leq k} B(m)\right] \Leftrightarrow_s \forall k(k \in \mathbb{N}^{\#})\forall x(x \leq k \Rightarrow_s B(x)). \tag{6.3}$$

(b) For any natural or hypernatural number $k \in \mathbb{N}^{\#}$ such that $k > 0$,

$$\vdash \bigvee_{1\leq m\leq k} (x = m - 1) \Leftrightarrow_s x < k. \tag{6.4}$$

(b′) For any natural or hypernatural number $k \in \mathbb{N}^{\#}$ such that $k > 0$ and any wff $B(x)$ unbounded on variable x

$$\vdash \bigwedge_{0\leq m\leq k-1} B(m) \Leftrightarrow_s \forall x(x < k \Rightarrow_s B(x)). \tag{6.5}$$

(c) For any wff $B(x)$ strictly restricted on a set $[0,y)$ and any wff $E(x)$ strictly restricted on a set $\mathbb{N}^{\#}\backslash[0,y)$

$$\vdash (\forall x(x < y \Rightarrow_s B(x))) \wedge (\forall x(x \geq y \Rightarrow_s E(x))) \Rightarrow_s \forall x(B(x) \vee E(x)). \tag{6.6}$$

Proof. (a) We prove $\bigvee_{0\leq m\leq k} (x = m) \Leftrightarrow_s x \leq k$ by hyperfinite induction in the metalanguage on k. The case for $k = 0, \vdash x = 0 \Leftrightarrow_s x \leq 0,$ is obvious from the definitions. Assume as inductive hypothesis that

$$\bigvee_{0\leq m\leq k} (x = m) \Leftrightarrow_s x \leq k. \tag{6.7}$$

Now assume that

$$\left[\bigvee_{0\leq m\leq k} (x = m)\right] \vee (x = k + 1). \tag{6.8}$$

But $\vdash x = k + 1 \Rightarrow_s \ x \leq k + 1$ and, by the inductive hypothesis,

$$\bigvee_{0\leq m\leq k} (x = m). \tag{6.9}$$

Also $\vdash x \leq k \Rightarrow_s x < k + 1$. Thus, $x \leq k + 1$. So,

$$\vdash \bigvee_{0\leq m\leq k+1} (x = m) \Rightarrow_s x \leq k + 1. \tag{6.10}$$

Conversely, assume $x \leq k + 1$. Then $x = k + 1 \vee x < k + 1$. If $x = k + 1$, then

$$\bigvee_{0\leq m\leq k+1} (x = m). \tag{6.11}$$

If $x < k + 1$, then we have $x \leq k$. By the inductive hypothesis,

$$\bigvee_{0\leq m\leq k} (x = m) \tag{6.12}$$

and,therefore,

$$\bigvee_{0\leq m\leq k+1} (x = m). \tag{6.13}$$

Thus in either case,

$$\bigvee_{0 \le m \le k+1} (x = m). \qquad (6.14)$$

This proves

$$\vdash x \le k+1 \Rightarrow_s \bigvee_{0 \le m \le k+1} (x = m). \qquad (6.15)$$

From the inductive hypothesis, we have derived

$$\bigvee_{0 \le m \le k+1} (x = m) \Leftrightarrow_s x \le k+1 \qquad (6.16)$$

and this completes the proof to part (a).Parts (a′), (b), and (b′) follow easily from part (a). Part (c) follows almost immediately from the statement $t \neq r \Rightarrow_s (t < r) \vee (r < t)$, using obvious tautologies.

Proposition 6.2. (a) For any natural or hypernatural number $k \in \mathbb{N}^{\#}$ and any wff $B(x)$ unbounded on variable x

$$\vdash \forall k (k \in \mathbb{N}^{\#}) \forall m (0 \le m \le k) B(m) \Leftrightarrow_s \forall k (k \in \mathbb{N}^{\#}) \forall x (x \le k \Rightarrow_s B(x)). \qquad (6.17)$$

Proof. It follows from (a′) and definition of hyper infinite conjunction $\bigwedge_{0 \le m \le k} B(m)$, see Appendix A.

Proposition 6.3. (a) Axiom of hyperfinite induction (3.5) can be expressed by usual set theoretical language in the following form

$$\forall S (\subset_s [0,\beta]) \forall \beta (\beta \in \mathbb{N}^{\#}) \{ \forall \alpha (0 \le \alpha < \beta) [(\alpha \in S \Rightarrow \alpha^{+} \in S)] \Rightarrow_s S = [0,\beta] \}. \quad (6.18)$$

(b) Axiom of hyper infinite induction (3.6) can be expressed by usual set theoretical language in the following form

$$\forall S (S \subset_s \mathbb{N}^{\#}) \{ \forall \beta (\beta \in \mathbb{N}^{\#}) \forall \alpha (0 \le \alpha < \beta) [(\alpha \in S \Rightarrow_s \alpha^{+} \in S)] \Rightarrow_s S = \mathbb{N}^{\#} \}. \qquad (6.19)$$

Proof. (a)-(b) It follows from Proposition 6.3 and definition of hyper infinite conjunction, see Appendix A.

There are several stronger forms of the hyper infinite induction principles that we can prove at this point.

Theorem 6.1.(Complete hyperfinite induction 1) (i) Let $S \subset [0,\beta]$, then

$$\forall x (x \in [0,\beta]) [\forall z (z < x \Rightarrow_s z \in_s S) \Rightarrow x \in_s S] \Rightarrow_s S = [0,\beta] \qquad (6.20)$$

(ii) Let $S \subset [0,\beta_\infty]$, then

$$\forall x (x \in [0,\beta_\infty]) [\forall z (z < x \Rightarrow_s z \in_s S) \Rightarrow x \in_s S] \Rightarrow_s S = [0,\beta_\infty] \qquad (6.21)$$

Theorem 6.2.(Complete hyperfinite induction 2) (i) Let $B(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta] \subset \mathbb{N}^{\#} \backslash \mathbb{N}$, then

$$\forall x (x \in [0,\beta]) [\forall z (z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)] \Rightarrow_s \forall x (x \in [0,\beta]) B(x) \qquad (6.22)$$

(ii) Let $F(x)$ be a wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ strictly restricted on a set $[0,\beta_\infty]$, then

$$\forall x (x \in [0,\beta_\infty]) [\forall z (z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)] \Rightarrow_s \forall x (x \in [0,\beta_\infty]) B(x) \qquad (6.23)$$

Theorem 6.3.(Complete hyper infinite induction 2) Let $B(x)$ be anrestricted wff of the set theory $\mathbf{NC}^{\#}_{\infty^{\#}}$ then

$$\forall x (x \in \mathbb{N}^{\#}) [\forall z (z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)] \Rightarrow_s \forall x (x \in \mathbb{N}^{\#}) B(x) \qquad (6.24)$$

In ordinary languageI consider a property $B(x)$ such that, for any x, if $B(x)$ holds for all hypernatural numbers less than x, then $B(x)$ holds for x also. Then $B(x)$ holds for all natural and hypernatural numbers $x \in \mathbb{N}^{\#}$.

Proof.Let $E(x)$ be a wff $\forall z(z \leq x \Rightarrow_s B(z))$.

(i) 1.Assume that $\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)]$, then

2.$[\forall z(z < 0 \Rightarrow_s B(z)) \Rightarrow_s B(0)]$ it follows from 1.

3. $z \nless 0$, then

4. $\forall z(z < 0 \Rightarrow_s B(z))$ it follows from 1,

5. $B(0)$ it follows from 2,4 by MP

6. $\forall z(z \leq 0 \Rightarrow_s B(z))$ i.e.,$E(0)$ holds it follows from Proposition 6.1(a$'$)

7.$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)] \vdash E(0)$ it follows from 1,6 by MP

(ii) 1.Assume that: $\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)]$.

2.Assume that: $E(x) \equiv \forall z(z \leq x \Rightarrow_s B(z))$, then

3.$\forall z(z < x^{+} \Rightarrow_s B(z))$ it follows from 2 since $z \leq x \Rightarrow z < x^{+}$.

4.$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x^{+} \Rightarrow_s B(z)) \Rightarrow_s B(x^{+})]$ it follows from 1 by rule A4:if t is free for x in $B(x)$, then $\forall x B(x) \vdash B(t)$.

5. $B(x^{+})$ it follows from 3,4 by MP rule.

6. $z \leq x^{+} \Rightarrow_s z < x^{+} \vee z = x^{+}$ it follows from definitions.

7. $z < x^{+} \Rightarrow_s B(z)$ it follows from 3 by particularization rule (PR), see Appendix A.

8. $z = x^{+} \Rightarrow_s B(z)$ it follows from 5.

9. $E(x^{+}) \equiv \forall z(z \leq x^{+} \Rightarrow_s B(z))$ it follows from 6,7,8,rule Gen.

10.$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)] \vdash \forall x(x \in \mathbb{N}^{\#})[E(x) \Rightarrow_s E(x^{+})]$ it follows from 1,9 by generalized deduction theorem,rule Gen.

Now by (i), (ii) and the induction axiom, we obtain $D \vdash \forall x(x \in \mathbb{N}^{\#})E(x)$ that is $D \vdash \forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z))]$, where
$D \equiv \forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow_s B(z)) \Rightarrow_s B(x)]$.

Hence, by rule A4 twice, $D \vdash x \leq x \Rightarrow_s B(x)$. But $\vdash x \leq x$. So,$D \vdash B(x)$, and, by Gen and the generalized deduction theorem (see Appendix A),
$D \vdash \forall x(x \in \mathbb{N}^{\#})B(x)$.

Remind that.

Theorem.(Complete ω-induction)

$$\forall S(S \subset \mathbb{N})\forall x[\forall z(z < x \Rightarrow_s z \in_s S) \Rightarrow x \in_s S] \Rightarrow_s S = \mathbb{N}. \tag{6.25}$$

Theorem.$(\mathbb{N},<)$ is a well-ordered set.

Proof. We will prove by reductio ad absurdum using complete ω-induction (6.25).
Let X be a nonempty subset of $\mathbb{N}$. Suppose X does not have a $<$-least element.
Then consider the set $\Delta = \mathbb{N}\backslash X$.

Case 1. $\mathbb{N}\backslash X = \varnothing$. Then $X = \mathbb{N}$ and so 0 is a $<$-least element. Contradiction.

Case 2. $\mathbb{N}\backslash X \neq \varnothing$. There exists an $n \in \mathbb{N}\backslash X$ such that for all $k < n, k \in \mathbb{N}\backslash X$.
Note tat such n necessarily exists because $0 \in \mathbb{N}\backslash X$, else $0 \in X$ and would be a $<$-least element of X.

Since we have supposed that $\Delta = \mathbb{N}\backslash X$ does not have a $<$-least element, thus $n \notin X$. Thus we see that for all $k < n, k \in \Delta \Rightarrow_s n \in \mathbb{N}\backslash X$,i.e. a set Δ has a property

$$\forall n[\forall k(k < n \Rightarrow_s k \in_s \Delta) \Rightarrow n \in_s \Delta] \tag{6.26}$$

Using complete ω-induction (6.25) we can conclude that $n \in \mathbb{N}\backslash X$ for all $n \in \mathbb{N}$.

Thus $\mathbb{N}\backslash X = \mathbb{N}$.But $\mathbb{N}\backslash X = \mathbb{N}$ implies $X = \varnothing$. This is a contradiction to X being a nonempty subset of $\mathbb{N}$.

Remark 6.1.Let X be a nonempty subset of $\mathbb{N}^{\#}$. Suppose X does not have a $<$ -least element. Then consider the set $\Delta^{\#} = \mathbb{N}^{\#}\backslash X \neq \varnothing$. There exists an $n \in \mathbb{N}^{\#}\backslash X$ such that for all $k < n, k \in \mathbb{N}^{\#}\backslash X$. Note tat such n necessarily exists because $0 \in \mathbb{N}^{\#}\backslash X$, else $0 \in X$ and would be a $<$ -least element of X. Since we have supposed that $\Delta^{\#} = \mathbb{N}^{\#}\backslash X$ does not have a least element, thus $n \notin X$.

Therefore we see that for all $k < n, k \in \Delta^{\#} \Rightarrow_s n \in \mathbb{N}^{\#}\backslash X$,i.e. a set $\Delta^{\#}$ has a property

$$\forall n[\forall k(k < n \Rightarrow_s k \in_s \Delta^{\#}) \Rightarrow n \in_s \Delta^{\#}]. \qquad (6.27)$$

But in contrast with (6.26) we can not conclude from (6.27) that $n \in \mathbb{N}^{\#}\backslash X$ for all $n \in \mathbb{N}^{\#}$. For example let X be a set $\mathbb{N}^{\#}\backslash\mathbb{N}$. Thus $\Delta^{\#} = \mathbb{N}$ and (6.27) is satisfied but $\mathbb{N} \neq \mathbb{N}^{\#}$. Obviously $\mathbb{N}^{\#}\backslash\mathbb{N}$ does not have a $<$ -least element.

Definition 6.1.A sequence $\{u_n\}_{n\in\mathbb{Z}}, u_n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is a blok corresponding to gyperfinite number $u = u_0 \in \mathbb{N}^{\#}\backslash\mathbb{N}$ iff there is gyperfinite number u such that $\ldots\in u_{-(n+1)} \in u_{-n}\ldots\in u_{-4} \in u_{-3} \in u_{-2} \in u_{-1} \in u$ and the following conditions are satisfied

$$\ldots\in u_{-(n+1)} \in u_{-n}\ldots\in u_{-4} \in u_{-3} \in u_{-2} \in u_{-1} \in u \in u_1 \in u_2 \in\ldots\in u_n \in u_{n+1} \in\ldots \quad (6.28)$$

where for any $n \in \mathbb{N} : u_{-(n+1)} \in u_{-n}$, where $u_{-n} = u^{+}_{-(n+1)}$.

Thus beginning with an infinite integer $u \in \mathbb{N}^{\#}\backslash\mathbb{N}$ we obtain a block (8.20) of infinite integers.However, given a “block,” there is another block consisting of even larger infinite integers. For example, there is the integer $u + u$, where $u + k < u + u$ for each $k \in \mathbb{N}$. And $v = u + u$ is itself part of the block:

$$\ldots< v - 3 < v - 2 < v - 1 < v < v + 1 < v - 2 <\ldots \qquad (6.29)$$

Of course, $v < v + u < v + v$, and so forth. There are even infinite integers $u \times u$ and u^u, and so forth.Proceeding in the opposite direction, if $u \in \mathbb{N}^{\#}\backslash\mathbb{N}$, either u or $u + 1$ is of the form $v + v$. Here v must be infinite. So there is no first block, since $v < u$. In fact, the ordering of the blocks is dense. For let the block containing v precede the one containing u, that is,

$$v - 2 < v - 1 < v < v + 1 <\ldots<\ldots< u - 2 < u - 1 < u < u + 1 <\ldots \qquad (6.30)$$

Either $u + v$ or $u + v + 1$ can be written $z + z$ where $v + k < z < u - l$ for all $k, l \in \mathbb{N}$.

Remark 6.2. Note that $\mathbb{N}^{\#}$ consists of $\mathbb{N}$ as an initial segment followed by an ordered set of blocks. These blocks are densely ordered with no first or last element. Each block is itself order-isomorphic to the integers

$$\ldots-3, -2, -1, 0, 1, 2, 3, \ldots \qquad (6.31)$$

Although $\mathbb{N}^{\#}\backslash\mathbb{N}$ is a nonempty subset of $\mathbb{N}^{\#}$, as we have just seen it has no least element and likewise for any block.

7.Hyperrationals $\mathbb{Q}^{\#}$.

Now that we have the hypernatural numbers $\mathbb{N}^{\#}$, defining hyperintegers and hyperrational numbers is well within reach [2].

Definition 7.1. Let $Z^{\#\prime} = \mathbb{N}^{\#} \times \mathbb{N}^{\#}$. We can define an equivalence relation $\approx$ on $Z^{\#\prime}$ by $(a,b) \approx (c,d)$ if and only if $a + d = b + c$. Then we denote the set of all hyperintegers by $\mathbb{Z}^{\#} = Z^{\#\prime}/\approx$ (The set of all equivalence classes of $Z^{\#\prime}$ modulo $\approx$).

Definition 7.2. Let $Q^{\#\prime} = \mathbb{Z}^{\#} \times (\mathbb{Z}^{\#} - \{0\}) \ = \{(a,b) \in \mathbb{Z}^{\#} \times \mathbb{Z}^{\#} | b \neq 0\}$. We can define an equivalence relation $\approx$ on $Q^{\#\prime}$ by $(a,b) \approx (c,d)$ if and only if $a \times d = b \times c$. Then we denote the set of all hyperrational numbers by $\mathbb{Q}^{\#} = Q'/\approx$ (The set of all equivalence classes of Q' modulo $\approx$).

Definition 7.3. A linearly ordered set $(P,<)$ is called dense if for any $a,b \in P$ such that $a < b$, there exists $z \in P$ such that $a < z < b$.

Lemma 7.1. $(\mathbb{Q}^{\#},<)$ is dense.

Proof. Let $x = (a,b), y = (c,d) \in \mathbb{Q}^{\#}$ be such that $x < y$. Consider $z = (ad + bc, 2bd) \in \mathbb{Q}^{\#}$. It is easily shown that $x < z < y$.

8.External Cauchy hyperreals $\mathbb{R}^{\#}_{\mathbf{c}}$ via Cauchy completion.

Definition 8.1. A hyper infinite sequence of hyperrational numbers (or for the sake of brevity simply hyperrational sequence) is a function from the hypernatural numbers $\mathbb{N}^{\#}$ into the hyperrational numbers $\mathbb{Q}^{\#}$. We usually denote such a function by $n \mapsto a_n$, or by $a : n \to a_n$, so the terms in the sequence are written $\{a_1,a_2,a_3,\ldots,a_n\ldots\}$. To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{\infty^{\#}}$, or $\{a_n\}_{n\in\mathbb{N}^{\#}}$, or for the sake of brevity simply $\{a_n\}$.

Definition 8.2. Let $\{a_n\}$ be a hyperrational sequence. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that, after N (i.e.for all $n > N$), $|a_n| \leq \varepsilon$. We often denote this symbolically by $a_n \to_{\#} 0$.
We can also, at this point, define what it means for a hyperrational sequence #-tends to any given number $q \in \mathbb{Q}^{\#}$: $\{a_n\}$ #-tends to q if the hyperrational sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_{\#} 0$.

Definition 8.3. Let $\{a_n\}$ be a hyper infinite hyperrational sequence. We call $\{a_n\}$ a Cauchy hyperrational sequence if the difference between its terms #-tends to 0.
To be precise: given any hyperrational number $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon$.

Theorem 8.1. If $\{a_n\}$ is a #-convergent hyperrational sequence (that is, $a_n \to_{\#} q$ for some hyperrational number $q \in \mathbb{Q}^{\#}$), then $\{a_n\}$ is a Cauchy hyperrational sequence.

Proof.We know that $a_n \to_{\#} q$. Here is a ubiquitous trick: instead of using ε in the definition, start with an arbitrary small $\varepsilon > 0, \varepsilon \approx 0$ and then choose $N \in \mathbb{N}^{\#}/\mathbb{N}$ so that $|a_n - q| < \varepsilon/2$ when $n > N$. Then if $m,n > N$, we have
$|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
This shows that $\{a_n\}$ is a Cauchy hyper infinite sequence.

Theorem 8.2. If $\{a_n\}$ is a Cauchy hyperrational sequence, then it is bounded or hyper bounded; that is, there is some $M \in \mathbb{Q}^{\#}$ finite or hyperfinite such that $|a_n| \leq M$ for all $n \in \mathbb{N}^{\#}$.

Proof.Since $\{a_n\}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in \mathbb{N}^{\#}/\mathbb{N}$ such that $|a_m - a_n| < 1$ whenever $m,n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as $a_{N+1} - 1 < a_n < a_{N+1} + 1$. This means that |an| is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list: $\{|a_0|,|a_1|,\ldots,|a_N|,|a_{N+1} - 1|,|a_{N+1} + 1|\}$. Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then (as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$. Hence, M is a bound for the sequence.

Definition 8.4. Let S be a set . A relation $x \sim y$ among pairs of elements of S is said to be an equivalence relation if the following three properties hold:

Reflexivity: for any $s \in S, s{\sim}s$.

Symmetry: for any $s,t \in S$, if $s{\sim}t$ then $t{\sim}s$.

Transitivity: for any $s,t,r \in S$, if $s{\sim}t$ and $t{\sim}r$, then $s{\sim}r$.

Theorem 8.3. Let S be a set, with an equivalence relation $(\bullet\sim\bullet)$ on pairs of elements. For $s \in S$, denote by $\mathbf{cl}[s]$ the set of all elements in S that are related to s. Then for any $s,t \in S$, either $\mathbf{cl}[s] = \mathbf{cl}[t]$ or $\mathbf{cl}[s]$ and $\mathbf{cl}[t]$ are disjoint.

The hyperreal numbers $\mathbb{R}^{\#}_{\mathbf{c}}$ will be constructed as equivalence classes of Cauchy hyperrational sequences. Let $\mathcal{F}_{\mathbb{Q}^{\#}}$ denote the set of all Cauchy hyperrational sequences of hyperrational numbers. We define the equivalence relation on $\mathcal{F}_{\mathbb{Q}^{\#}}$.

Definition 8.5. Let $\{a_n\}$ and $\{b_n\}$ be in $\mathcal{F}_{\mathbb{Q}^{\#}}$. Say they are #-equivalent if $a_n - b_n \to_{\#} 0$ i.e., if and only if the hyperrational sequence $\{a_n - b_n\}$ tends to 0.

Theorem 8.4.Definition 8.4 yields an equivalence relation on $\mathcal{F}_{\mathbb{Q}^{\#}}$.

Proof. We need to show that this relation is reflexive, symmetric, and transitive.

Reflexive: $a_n - a_n = 0$, and the sequence all of whose terms are 0 clearly #-converges to 0. So $\{a_n\}$ is related to $\{a_n\}$.

Symmetric: Suppose $\{a_n\}$ is related to $\{b_n\}$, so $a_n - b_n \to_{\#} 0$.

But $b_n - a_n = -(a_n - b_n)$, and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 8.2, it follows that $b_n - a_n \to_{\#} 0$ as well. Hence, $\{b_n\}$ is related to $\{a_n\}$.

Transitive: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 3.1. Suppose $\{a_n\}$ is related to $\{b_n\}$, and $\{b_n\}$ is related to $\{c_n\}$. This means that $a_n - b_n \to_{\#} 0$ and $b_n - c_n \to_{\#} 0$. To be fully precise, let us fix $\varepsilon > 0, \varepsilon \approx 0$; then there exists an $N \in \mathbb{N}^{\#}$ such that for all $n > N, |a_n - b_n| < \varepsilon/2$; also, there exists an M such that for all $n > M$, $|b_n - c_n| < \varepsilon/2$. Well, then, as long as n is bigger than both N and M, we have that $|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

So, choosing L equal to the max of N, M, we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |a_n - c_n| < \varepsilon$. This means that $a_n - c_n \to_{\#} 0$ – i.e. $\{a_n\}$ is related to $\{c_n\}$.

Definition 8.6. The hyperreal numbers $\mathbb{R}^{\#}_{c}$ are the equivalence classes $\mathbf{cl}[\{a_n\}]$ of Cauchy sequences of hyperrational numbers, as per Definition 3.5. That is, each such equivalence class is a hyperreal number.

Definition 8.7. Given any hyperrational number $q \in \mathbb{Q}^{\#}$, define a hyperreal number $q^{\#}$ to be the equivalence class of the sequence $q^{\#} = (q,q,q,q,\ldots)$ consisting entirely of q. So we view $\mathbb{Q}^{\#}$ as being inside $\mathbb{R}^{\#}_{c}$ by thinking of each hyperrational number $q \in \mathbb{Q}^{\#}$ as its associated equivalence class $q^{\#}$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 8.8. Let $s,t \in \mathbb{R}^{\#}_{c}$, so there are Cauchy sequences $\{a_n\},\{b_n\}$ of hyperrational numbers with $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$.

(a) Define $s + t$ to be the equivalence class of the sequence $\{a_n + b_n\}$.

(b) Define $s \times t$ to be the equivalence class of the sequence $\{a_n \times b_n\}$.

Theorem 8.5.The operations $+, \times$ in Definition 8.8 (a),(b) are well-defined.

Proof. Suppose that $\mathbf{cl}[\{a_n\}] = \mathbf{cl}[\{c_n\}]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$. Thus means that

$a_n - c_n \to_{\#} 0$ and $b_n - d_n \to_{\#} 0$. Then $(a_n + b_n) - (c_n + d_n) = (a_n - c_n) + (b_n - d_n)$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to 0, and so $\mathbf{cl}[\{a_n + b_n\}] = \mathbf{cl}[\{c_n + d_n\}]$.

Multiplication is a little trickier; this is where we will use Theorem 8.2. We will also use another ubiquitous technique: adding 0 in the form of $s - s$. Again, suppose that $\mathbf{cl}[(a_n) = \mathbf{cl}[(c_n)]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$; we wish to show that $\mathbf{cl}[\{a_n \times b_n\}] = \mathbf{cl}[\{c_n \times d_n\}]$, or, in other words, that $a_n \times b_n - c_n \cdot d_n \to_{\#} 0$. Well, we add and subtract one of the other cross terms, say

$$b_n \times c_n : a_n \times b_n - c_n \times d_n = a_n \times b_n + (b_n \times c_n - b_n \times c_n) - c_n \times d_n =$$
$$= (a_n \times b_n - b_n \times c_n) + (b_n \times c_n - c_n \times d_n) = b_n \times (a_n - c_n) + c_n \times (b_n - d_n).$$

Hence, we have $|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n|$. Now, from Theorem 8.2, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all $n \in \mathbb{N}^{\#}$. Taking some number K which is bigger than both, we have

$$|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n| \leq K(|a_n - c_n| + |b_n - d_n|).$$

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2K$), we see that $a_n \times b_n - c_n \times d_n \to_{\#} 0$.

Theorem 8.6. Given any hyperreal number $s \neq 0$, there is a hyperreal number t such that $s \times t = 1$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\{0,0,0,0,\ldots\}$. In other words, $s = \mathbf{cl}[\{a_n\}]$ where $a_n - 0$ does not #-converge to 0. From this, we are to deduce the existence of a hyperreal number $t = \mathbf{cl}[\{b_n\}]$ such that $s \times t = \mathbf{cl}[\{a_n \times b_n\}]$ is the same equivalence class as $\mathbf{cl}[\{1,1,1,1,\ldots\}]$. Doing so is actually an easy consequence of the fact that nonzero rational numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{a_n\}$ does not tend to 0), there's no reason any number of the terms in $\{a_n\}$ can't equal 0. However, it turns out that eventually, $a_n \neq 0$.

That is,

Lemma 8.1. If $\{a_n\}$ is a Cauchy hyper infinite sequence which does not #-tend to 0, then there is an $N \in \mathbb{N}^{\#}/\mathbb{N}$ such that, for $n > N, a_n \neq 0$.

We will now use Lemma 8.1 to complete the proof of Theorem 8.7.

Let N be such that $a_n \neq 0$ for $n > N$. Define a hyper infinite sequence b_n of hyperrational numbers as follows:

for $n \leq N, b_n = 0$, and for $n > N, b_n = 1/a_n$; $\{b_n\} = (0,0,\ldots,0,1/a_{N+1},1/a_{N+2},\ldots)$.

This makes sense since, for $n > N$, an is a nonzero hyperrational number, so $1/a_n$ exists. Then $a_n \cdot b_n$ is equal to $a_n \cdot 0 = 0$ for $n \leq N$, and equals $a_n \cdot b_n = a_n \cdot 1/a_n = 1$ for $n > N$. Well, then, if we look at the hyper infinite sequence $(1,1,1,1,\ldots)$, we have $(1,1,1,1,\ldots) - (a_n \cdot b_n)$ is the hyper infinite sequence which is $1 - 0 = 1$ for $n \leq N$ and equals $1 - 1 = 0$ for $n > N$. Since this sequence is eventually equal to 0, it #-converges to 0, and so $\mathbf{cl}[\{a_n \cdot b_n\}] = \mathbf{cl}[(1,1,1,1,\ldots)] = 1 \in \mathbb{R}_c^{\#}$. This shows that $t = \mathbf{cl}[\{b_n\}]$ is a multiplicative inverse to $s = \mathbf{cl}[\{a_n\}]$.

Definition 8.9. Let $s \in \mathbb{R}_c^{\#}$. Say that s is positive if $s \neq 0,$ and if $s = \mathbf{cl}[\{a_n\}]$ for some Cauchy sequence of hyperrational numbers such that for some $N \in \mathbb{N}^{\#}, a_n > 0$ for all $n > N.$ Given two hyperreal numbers $s,\ t,$ say that $s > t$ if $s - t$ is positive.

Theorem 8.7. Let s, t be hyperreal numbers such that $s > t,$ and let $r \in \mathbb{R}_c^{\#}$. Then $s + r > t + r.$

Proof. Let $s = \mathbf{cl}[\{a_n\}], t = \mathbf{cl}[\{b_n\}],$ and $r = \mathbf{cl}[\{c_n\}].$ Since $s > t$ i.e., $s - t > 0,$ we know that there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N,\ a_n - b_n > 0.$ So $a_n > b_n$ for $n > N.$ Now, adding c_n to both sides of this inequality (as we know we can do for hyperrational numbers), we have $a_n + c_n > b_n + c_n$ for $n > N,$ or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N.$ Note also that $(a_n + c_n) - (b_n + c_n) = a_n - b_n$ does not #-converge to $0,$ by the assumption that $s - t > 0.$ Thus, by Definition 8.8, this means that $s + r = \mathbf{cl}[\{a_n + c_n\}] > \mathbf{cl}[\{b_n + c_n\}] = t + r.$

Theorem 8.8. (Generalized Archimedean property)Let $s, t > 0$ be hyperreal numbers. Then there is $m \in \mathbb{N}^{\#}$ such that $m \times s > t.$

Proof. Let $s, t > 0$ be hyperreal numbers. We need to find a hypernatural number m so that
$m \times s > t.$ First, recall that, by m in this context, we mean $\mathbf{cl}[\{m, m, m, m, \ldots\}].$ So, letting $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}],$what we need to show is that there exists m with
$\mathbf{cl}[\{m, m, m, m, \ldots\}] \times \mathbf{cl}[\{a_1, a_2, a_3, a_4, \ldots\}] = \mathbf{cl}[\{m \times a_1, m \times a_2, m \times a_3, m \times a_4, \ldots\}] >$
$> \mathbf{cl}[\{b_1, b_2, b_3, b_4, \ldots\}].$

Now, to say that $\mathbf{cl}[\{m \times a_n\}] > \mathbf{cl}[\{b_n\}],$ or $\mathbf{cl}[\{m \times a_n - b_n\}]$ is positive, is, by Definition 8.9, just to say that there is $N \in \mathbb{N}^{\#}$ such that $m \times a_n - b_n > 0$ for all $n > N,$ while $m \times a_n - b_n \nrightarrow_{\#} 0.$ To be precise, the first statement is:

There exist $m, N \in \mathbb{N}^{\#}$ so that $m \times a_n > b_n$ for all $n > N.$

To produce a contradiction, we assume this is not the case; assume that

(#) for every m and $N,$ there exists an $n > N$ so that $m \times a_n \leq b_n.$

Now, since $\{b_n\}$ is a Cauchy sequence, by Theorem 3.2 it is hyperbounded – there is a
hyperrational number $M \in \mathbb{Q}^{\#}$ such that $b_n \leq M$ for all $n.$ Now, by the properties for the hyperrational numbers $\mathbb{Q}^{\#}$, given any hyperrational number $\varepsilon > 0, \varepsilon \approx 0,$ there is an $m \in \mathbb{N}^{\#}$ such that $M/m < \varepsilon/2.$ Fix such an $m.$ Then if $m \times a_n \leq b_n,$ we have $a_n \leq b_n/m \leq M/m < \varepsilon/2.$

Now, $\{a_n\}$ is a Cauchy sequence, and so there exists N so that for
$n, k > N, |a_n - a_k| < \varepsilon/2.$

By Asumption (#), we also have an $n > N$ such that $m \times a_n \leq b_n,$ which means that $a_n < \varepsilon/2.$ But then for every $k > N,$ we have that $a_k - a_n < \varepsilon/2$, so $a_k < a_n + \varepsilon/2 < \quad \varepsilon/2 + \varepsilon/2 = \varepsilon.$ Hence, $a_k < \varepsilon$ for all $k > N.$ This proves that $a_k \to_{\#} 0,$ which by Definition 8.9 contradicts the fact that $\mathbf{cl}[\{a_n\}] = s > 0.$

Thus, there is indeed some $m \in N$ so that $m \times a_n - b_n > 0$ for all sufficiently infinite large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. To conclude the proof, we must also show that $m \times a_n - b_n \nrightarrow 0.$ Actually, it is possible that $m \times a_n - b_n \to 0$ (for example if $\{a_n\} = \{1, 1, 1, \ldots\}$ and $\{b_n\} = \{m, m, m, \ldots\}$). But that's okay: then we can simply choose a larger $m.$ That is: let m be a hypernatural number constructed as above, so that $m \times a_n - b_n > 0$ for all sufficiently large $\in \mathbb{N}^{\#} \backslash \mathbb{N}$. If it happens to be true that $m \times a_n - b_n \nrightarrow 0,$ then the proof is complete.

If, on the other hand, it turned out that $m \times a_n - b_n \to 0$, then take instead the integer $m+1$. Since $s = \mathbf{cl}[\{a_n\}] > 0$, we have a $n > 0$ for all infinite large n, so $(m+1) \times a_n - b_n = m \times a_n - b_n + a_n > a_n > 0$ for all infinite large n, so $m+1$ works just as well as m did in this regard; and since $m \times a_n - b_n \to 0$, we have $(m+1) \times a_n - b_n = (m \times a_n - b_n) + a_n \nrightarrow 0$ since $s = \mathbf{cl}[\{a_n\}] > 0$ (so $a_n \nrightarrow 0$).

It will be handy to have one more Theorem about how the hyperrationals $\mathbb{Q}^\#$ and hyperreals $\mathbb{R}^\#_c$ compare before we proceed. This theorem is known as the density of $\mathbb{Q}^\#$ in $\mathbb{R}^\#_c$, and it follows almost immediately from the construction of the $\mathbb{R}^\#_c$ from $\mathbb{Q}^\#$.

Theorem 8.9. Given any hyperreal number $r \in \mathbb{R}^\#_c$, and any hyperrational number $\varepsilon > 0$, $\varepsilon \approx 0$, there is a hyperrational number $q \in \mathbb{Q}^\#$ such that $|r - q| < \varepsilon$.

Proof. The hyperreal number r is represented by a Cauchy hyperrational sequence $\{a_n\}$.

Since this sequence is Cauchy, given $\varepsilon > 0, \varepsilon \approx 0$, there is $N \in \mathbb{N}^\#$ so that for all $m, n > N$, $|a_n - a_m| < \varepsilon$. Picking some fixed $l > N$, we can take the hyperrational number q given by $q = \mathbf{cl}[\{a_l, a_l, a_l, \ldots\}]$. Then we have $r - q = \mathbf{cl}[\{a_n - a_l\}_{n \in \mathbb{N}^\#}]$, and $q - r = \mathbf{cl}[\{a_l - a_n\}_{n \in \mathbb{N}^\#}]$.

Now, since $l > N$, we see that for $n > N, a_n - a_l < \varepsilon$ and $a_l - a_n < \varepsilon$, which means by Definition 8.9 that $r - q < \varepsilon$ and $q - r < \varepsilon$; hence, $|r - q| < \varepsilon$.

Definition 8.10. Let $S \subsetneqq \mathbb{R}^\#_c$ be a non-empty set of hyperreal numbers.

A hyperreal number $x \in \mathbb{R}^\#_c$ is called an upper bound for S if $x \geq s$ for all $s \in S$.

A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper bound for S and $x \leq y$ for every upper bound y of S.

Remark 8.1. The order $\leq$ given by Definition 8.9 obviously is $\leq$-incomplete.

Definition 8.11. Let $S \subsetneqq \mathbb{R}^\#_c$ be a nonempty subset of $\mathbb{R}^\#_c$. We we will say that:

(1) S is $\leq$-admissible above if the following conditions are satisfied:

(i) S bounded above;

(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.

(2) S is $\leq$-admissible belov if the following condition are satisfied:

(i) S bounded belov;

(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

Theorem 8.10. (i) Any $\leq$-admissible above subset $S \subset \mathbb{R}^\#_c$ has the least upper bound property.(ii) Any $\leq$-admissible below subset $S \subset \mathbb{R}^\#_c$ has the greatest lower bound property.

Proof. Let $S \subset \mathbb{R}^\#_c$ be a nonempty subset, and let M be an upper bound for S. We are going to construct two sequences of hyperreal numbers, $\{u_n\}$ and $\{l_n\}$. First, since S is nonempty, there is some element $s_0 \in S$. Now, we go through the following hyperinductive procedure to produce numbers $u_0, u_1, u_2, \ldots, u_n, \ldots$ and $l_1, l_2, l_3, \ldots, l_n, \ldots$

(i) Set $u_0 = M$ and $l_0 = s_0$.

(ii) Suppose that we have already defined u_n and l_n. Consider the number $m_n = (u_n + l_n)/2$, the average between u_n and l_n.

(1) If m_n is an upper bound for $S,$ define $u_{n+1} = m_n$ and $l_{n+1} = l_n.$

(2) If m_n is not an upper bound for $S,$ define $u_{n+1} = u_n$ and $l_{n+1} = m_n.$

Since $s_0 < M,$ it is easy to prove by hyper infinite induction that $\{u_n\}$ is a non-increasing sequence: $u_{n+1} \leq u_n, n \in \mathbb{N}^\#$and $\{l_n\}$ is a non-decreasing sequence $l_{n+1} \geq l_n, n \in \mathbb{N}^\#.$ This gives us the following lemma.

Lemma 8.2. $\{u_n\}$ and $\{l_n\}$ are Cauchy sequences of hyperreal numbers.

Proof. Note that each $l_n \leq M$ for all $n \in \mathbb{N}^\#$. Since $\{l_n\}$ is non-decreasing, it follows that $\{l_n\}$ is Cauchy. For $\{u_n\}$, we have $u_n \geq s_0$ for all $n \in \mathbb{N}^\#$, and so $-u_n \leq -s_0.$ Since $\{u_n\}$ is non-increasing, $\{-u_n\}$ is non-decreasing, and so as above, $\{-u_n\}$ is Cauchy. It is easy to verify that, therefore, $\{u_n\}$ is Cauchy.

The following Lemma shows that $\{u_n\}$ does tend to a hyperreal number.

Lemma 8.3. There is a hyperreal number u such that $u_n \to_\# u.$

Proof. Fix a term u_n in the sequence $\{u_n\}$. By Theorem 8.9, there is a hyperrational number q_n such that $|u_n - q_n| < 1/n.$ Consider the sequence $\{q_1, q_2, q_3, \ldots, q_n, \ldots\}$ of hyperrational numbers. We will show this hypersequence is Cauchy. Fix $\varepsilon > 0, \varepsilon \approx 0.$ By the Theorem 8.8, we can choose $N \in \mathbb{N}^\#$ so that $1/N < \varepsilon/3.$ We know, since $\{u_n\}$ is Cauchy, that there is an $M \in \mathbb{N}^\#$ such that for $n,m > M, |u_n - u_m| < \varepsilon/3.$ Then, so long as $n,m > max\{N,M\},$ we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq$$
$$\leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\{q_n\}$ is a Cauchy sequence of hyperrational numbers, and so it represents a hyperreal number $u = \mathbf{cl}[\{q_n\}].$We must show that $u_n - u \to_\# 0,$ but this is practically built into the definition of $u.$ To be precise, letting q_n^* be the hyperreal number $\mathbf{cl}[\{q_n, q_n, q_n, \ldots\}],$ we see immediately that $q_n^* - u \to_\# 0$ (this is precisely equivalent to the statement that $\{q_n\}$ is Cauchy). But $u_n - q_n^* < 1/n$ by construction; it is easily verify that the assertion that if a sequence $q_n^* \to_\# u$ and $u_n - q_n^* \to_\# 0,$ then $u_n \to_\# u.$So $\{u_n\}$, a non-increasing sequence of upper bounds for $S,$ tends to a hyperreal number $u.$ As you've guessed, u is the least upper bound of our set $S.$ To prove this, we need one more lemma.

Lemma 8.4. $l_n \to_\# u.$

Proof. First, note in the first case above, we have that

$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}.$$

In the second case, we also have

$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}.$$

Now, this means that $u_1 - l_1 = \frac{1}{2}(M - s),$ and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(L - s),$ and in general by hyperinfinite induction, $u_n - l_n = 2^{-n}(L - s).$ Since $L > s$ so $L - s > 0,$ and since $2^{-n} < 1/n,$ by the Theorem 8.8, we have for any $\varepsilon > 0$ that $2^{-n}(L - s) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^\#/\mathbb{N}$. Thus, $u_n - l_n < 2^{-n}(L - s) < \varepsilon$ as well, and so $u_n - l_n \to_{\#} 0$. Again, it is easily verify that, since $u_n \to_{\#} u$, we have $l_n \to_{\#} u$ as well.

Proof of Theorem 8.10. First, we show that u is an upper bound. Well, suppose it is not, so that $u < s$ for some $s \in S$. Then $\varepsilon = s - u$ is > 0, and since $u_n \to u$ and is non-increasing, there must be an n so that $u_n - u < \varepsilon$, meaning that $u_n < u + \varepsilon = u + (s - u) = s$. Since u_n is an upper bound for S, however, this is a contradiction. Hence, u is an upper bound for S.

Now, we also know that, for each n, l_n is not an upper bound, meaning that for each n, there is an $s_n \in S$ so that $l_n \leq s_n$. Lemma 8.4 tells us that $l_n \to_{\#} u$, and since the sequence $\{l_n\}$ is non-decreasing, this means that for each $\varepsilon > 0$, there is an $N \in \mathbb{N}^{\#}/\mathbb{N}$ so that for $n > N, l_n > u - \varepsilon$. Hence, for $n > N, s_n \geq l_n > u - \varepsilon$ as well. In particular, for each $\varepsilon > 0$, there is an element $s \in S$ such that $s > u - \varepsilon$. This means that no number smaller than u can be an upper bound for S. Hence, u is the least upper bound for S.

Remark 8.2.Note that assumption in Theorem 8.10 that S is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$ is necessarily, othervice Theorem 8.10 is not holds. For example let $\Delta = \{\varepsilon | \varepsilon \geq 0 \wedge \varepsilon \approx 0\}$. Obviously a set Δ is not $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$. It is clear that Theorem 8.10 is not holds for a set Δ.

Theorem 8.11.(**Generalized Nested Intervals Theorem**)

Let $\{I_n\}_{n \in \mathbb{N}^{\#}} = \{[a_n, b_n]\}_{n \in \mathbb{N}^{\#}}, [a_n, b_n] \subset \mathbb{R}_c^{\#}$ be a hyper infinite sequence of closed intervals satisfying each of the following conditions:

(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$,

(ii) $b_n - a_n \to_{\#} 0$ as $n \to \infty^{\#}$.

Then $\bigcap_{n=1}^{\infty^{\#}} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^{\#}$. Moreover both sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.

Proof.Note that: (a) the set $A = \{a_n | n \in \mathbb{N}^{\#}\}$ is bounded or hyperbouded above by b_1 and (b) the set $A = \{a_n | n \in \mathbb{N}^{\#}\}$ is $\leq$-admissible above subset of $\mathbb{R}_c^{\#}$.

By Theorem 8.10 there exists $\sup A$. Let $\xi = \sup A$.

Since I_n are nested,for any positive hyperintegers m and n we have $a_m \leq a_{m+n} \leq b_{m+n} \leq b_n$, so that $\xi \leq b_n$ for each $n \in \mathbb{N}^{\#}$. Since we obviously have $a_n \leq \xi$ for each $n \in \mathbb{N}^{\#}$, we have $a_n \leq \xi \leq b_n$ for all $n \in \mathbb{N}^{\#}$, which implies $\xi \in \bigcap_{n=1}^{\infty^{\#}} I_n$. Finally, if $\xi, \eta \in \bigcap_{n=1}^{\infty^{\#}} I_n$, with $\xi \leq \eta$, then we get $0 \leq \eta - \xi \leq b_n - a_n$, for all $n \in \mathbb{N}^{\#}$, so that $0 \leq \eta - \xi \leq \inf_{n \in \mathbb{N}^{\#}} |b_n - a_n| = 0$.

Theorem 8.12.(**Generalized Squeeze Theorem**)

Let $\{a_n\}, \{c_n\}$ be two hyper infinite sequences #-converging to L, and $\{b_n\}$ a hyper infinite sequence. If $\forall n \geq K, K \in \mathbb{N}^{\#}$ we have $a_n \leq b_n \leq c_n$, then $\{b_n\}$ also #-converges to L.

Proof. Choose an $\varepsilon > 0, \varepsilon \approx 0$. By definition of the #-limit,there is an $N_1 \in \mathbb{N}^{\#}$ such that for all $n > N_1$ we have $|a_n - L| < \varepsilon$, in other words $L - \varepsilon < a_n < L + \varepsilon$.Similarly, there is an $N_2 \in \mathbb{N}^{\#}$ such that for all $n > N_2$ we have $L - \varepsilon < c_n < L + \varepsilon$. Denote $N = \max(N_1, N_2, K)$. Then for $n > N, L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, in other words $|b_n - L| < \varepsilon$. Since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary, by definition of the #-limit this says that #-$\lim_{n \to \infty^{\#}} b_n = L$.

Theorem 8.13.(**Corollary of the Generalized Squeeze Theorem**).

If #-$\lim_{n \to \infty^{\#}} |a_n| = 0$ then #-$\lim_{n \to \infty^{\#}} a_n = 0$.

Proof.We know that $-|a_n| \leq a_n \leq |a_n|$.We want to apply the Generalized Squeeze Theorem.We are given that #-$\lim_{n \to \infty^{\#}} |a_n| = 0$.This also implies that

$\#\text{-}\lim_{n\to\infty^{\#}}(-|a_n|) = 0.$ So by the Generalized Squeeze Theorem, $\#\text{-}\lim_{n\to\infty^{\#}} a_n = 0.$

Theorem 8.14. (**Generalized Bolzano-Weierstrass Theorem**)
Every hyperbounded hyperinfinite sequence has a #-convergent hyper infinite subsequence.

Proof. Let $\{w_n\}_{n\in\mathbb{N}^{\#}}$ be a hyperbounded hyper infinite sequence. Then, there exists an interval $[a_1,b_1]$ such that $a_1 \leq w_n \leq b_1$ for all $n \in \mathbb{N}^{\#}$.
Either $\left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains hyper infinitely many terms of $\{w_n\}$. That is, there exists hyperinfinitely many n in $\mathbb{N}^{\#}$ such that a_n is in $\left[a_1, \frac{a_1+b_1}{2}\right]$ or there exists hyper infinitely many n in $\mathbb{N}^{\#}$ such that a_n is in $\left[\frac{a_1+b_1}{2}, b_1\right]$. If $\left[a_1, \frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_2,b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$. Otherwise, let $[a_2,b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$. Either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$. If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_3,b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$. Otherwise, let $[a_3,b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$. By hyper infinite induction, we can continue this construction and obtain hyper infinite sequence of intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}^{\#}}$ such that:
(i) for each $n \in \mathbb{N}^{\#}, [a_n,b_n]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$,
(ii) for each $n \in \mathbb{N}^{\#}, [a_{n+1},b_{n+1}] \subseteq [a_n,b_n]$ and
(iii) for each $n \in \mathbb{N}^{\#}, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
Then generalized nested intervals theorem implies that the intersection of all of the intervals $[a_n,b_n]$ is a single point w. We will now construct a hyper infinite subsequence of $\{w_n\}_{n\in\mathbb{N}^{\#}}$ which will #-converge to w.
Since $[a_1,b_1]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_1 \in \mathbb{N}^{\#}$ such that w_{k_1} is in $[a_1,b_1]$. Since $[a_2,b_2]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_2 \in \mathbb{N}^{\#}, k_2 > k_1,$ such that w_{k_2} is in $[a_2,b_2]$. Since $[a_3,b_3]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_3 \in \mathbb{N}^{\#}, k_3 > k_2,$ such that w_{k_3} is in $[a_3,b_3]$. Continuing this process by hyper infinite induction, we obtain hyper infinite sequence $\{w_{k_n}\}_{n\in\mathbb{N}^{\#}}$ such that $w_{k_n} \in [a_n,b_n]$ for each $n \in \mathbb{N}^{\#}$. The sequence $\{w_{k_n}\}_{n\in\mathbb{N}^{\#}}$ is a subsequence of $\{w_n\}_{n\in\mathbb{N}^{\#}}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^{\#}$. Since $a_n \to_{\#} w,$ and $a_n \leq w_n \leq b_n$ for each $n \in \mathbb{N}^{\#}$, the squeeze theorem implies that $w_{k_n} \to_{\#} w.$

Definition 8.12. Let $\{a_n\}$ be a $\mathbb{R}_c^{\#}$-valued hyper infinite sequence i.e., $a_n \in \mathbb{R}_c^{\#}, n \in \mathbb{N}^{\#}$. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N \in \mathbb{N}^{\#}\backslash\mathbb{N},$ $N = N(\varepsilon)$ such that, for all $n > N$, $|a_n| \leq \varepsilon.$ We often denote this symbolically by $a_n \to_{\#} 0.$
We can also, at this point, define what it means for a hyperreal sequence #-tends to a given number $q \in \mathbb{R}_c^{\#}$: $\{a_n\}$ #-tends to q if the hyperreal sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_{\#} 0.$

Definition 3.13. Let $\{a_n\}, n \in \mathbb{N}^{\#}$ be a hyperreal sequence. We call $\{a_n\}$ a Cauchy hyperreal sequence if the difference between its terms #-tends to $0.$ To be precise: given any hyperreal number $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon.$

Theorem 8.15. If $\{a_n\}$ is a #-convergent hyperreal sequence (that is, $a_n \to_{\#} b$ for some hyperreal number $b \in \mathbb{R}_c^{\#}$), then $\{a_n\}$ is a Cauchy hyperreal sequence.

Theorem 8.16. If $\{a_n\}$ is a Cauchy hyperreal sequence, then it is bounded or hyper bounded; that is, there is some $M \in \mathbb{R}_c^\#$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^\#$.

Theorem 8.17. Any Cauchy hyperreal sequence $\{a_n\}$ has a #-limit in $\mathbb{R}_c^\#$ i.e.,there exists

$b \in \mathbb{R}_c^\#$ such that $a_n \to_\# b$.

Proof.By Definition 8.13 given $\varepsilon > 0, \varepsilon \approx 0,$ there is a hypernatural number $N = N(\varepsilon)$ such that for any $n, n' > N$,

$$|a_n - a_{n'}| < \varepsilon. \quad (8.1)$$

From (8.1) for any $n, n' > N$ we get

$$a_{n'} - \varepsilon < a_n < a_n + \varepsilon. \quad (8.2)$$

The generalized Bolzano-Weierstrass theorem implies there is a #-convergent hyper infinite subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to_\# b$ for some hyperreal number $b \in \mathbb{R}_c^\#$. Let us show that the sequence $\{a_n\}$ also #-convergent to this $b \in \mathbb{R}_c^\#$. We can choose $k \in \mathbb{N}^\#$ so large that $n_k > N$ and

$$|a_{n_k} - b| < \varepsilon. \quad (8.3)$$

We choose now in (8.1) $n' = n_k$ and therefore

$$|a_n - a_{n_k}| < \varepsilon. \quad (8.4)$$

From (8.3) and (8.4) for any $n > N$ we get

$$|(a_{n_k} - b) + (a_n - a_{n_k})| = |a_n - b| < 2\varepsilon. \quad (8.5)$$

Thus $a_n \to_\# b$ as well. .

9.The Extended Hyperreal Number System $\hat{\mathbb{R}}_c^\#$

Definition 9.1.(a) A set $S \subset \mathbb{N}^\#$ is hyperfinite if $card(S) = card(\{x | 0 \leq x \leq n\})$, where $n \in \mathbb{N}^\# \backslash \mathbb{N}$. (b) A set $S \subseteq \mathbb{N}^\#$ is hyper infinite if $card(S) = card(\mathbb{N}^\#)$

Notation 9.1. If F is an arbitrary collection of sets, then $\cup\{S | S \in F\}$is the set of all elements that are members of at least one of the sets in F , and $\cap\{S | S \in F\}$ is the set of all elements that are members of every set in F. The union and intersection of finitely or hyper finitely many sets $S_k, 0 \leq k \leq n \in \mathbb{N}^\#$ are also written as $\cup_{k=0}^n S_k$ and $\cap_{k=0}^n S_k$. The union and intersection of an hyperinfinite sequence $S_k, k \in \mathbb{N}^\#$ of sets are written as $\cup_{k=0}^{\infty^\#} S$ or $\cup_{n \in \mathbb{N}^\#} S$ and $\cap_{k=0}^{\infty^\#} S$ or $\cap_{n \in \mathbb{N}^\#} S$ correspondingly.

A nonempty set S of hyperreal numbers $\mathbb{R}_c^\#$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to the hyperreal number system two points, $+\infty^\#$ (which we also write more simply as $\infty^\#$) and $-\infty^\#$,and to define the order relationships between them and any hyperreal number $x \in \mathbb{R}_c^\#$ by $-\infty^\# < x < \infty^\#$.

We call $-\infty^\#$ and $\infty^\#$ points at hyperinfinity. If S is a nonempty set of hyperreals, we write $\sup S = \infty^\#$ to indicate that S is hyper unbounded above, and $\inf S = -\infty^\#$ to indicate that S is hyper unbounded below. .

#-Open and #-Closed Sets on $\hat{\mathbb{R}}_c^\#$.

Definition 9.2.If a and b are in the extended hyperreals and $a < b$, then the #-open interval (a,b) is defined by $(a,b) \triangleq \{x | a < x < b\}$. :

The #-open intervals $(a, \infty^\#)$ and $(-\infty^\#, b)$ are semi-hyper infinite if a and b are finite or hyperfinite, and $(-\infty^\#, \infty^\#)$ is the entire hyperreal line.

If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] \triangleq \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#},a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. Semi-hyper infinite #-closed intervals are sets of the form $[a,\infty) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$, where a is finite or hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$, respectively.

Definition 9.3. If $x_0 \in \mathbb{R}_c^{\#}$ is a hyperreal number and $\varepsilon > 0, \varepsilon \approx 0$ then the #-open interval

$(x_0 - \varepsilon, x_0 + \varepsilon)$ is an #-neighborhood of x_0. If a set $S \subset \mathbb{R}_c^{\#}$ contains an #-neighborhood of x_0, then S is a #-neighborhood of x_0, and x_0 is an #-interior point of S. The set of #-interior points of S is the #-interior of S, denoted by $\#\text{-}Int(S)$.

(i) If every point of S is an #-interior point (that is, $S = \#\text{-}Int(S)$), then S is #-open.

(ii) A set S is #-closed if $S^c = \mathbb{R}_c^{\#} \backslash S$ is #-open.

Example 9.1. An open interval (a,b) is an #-open set, because if $x_0 \in (a,b)$ and $\varepsilon \leq \min\{x_0 - a; b - x_0\}$, then $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,b)$.

Remark 9.1. The entire hyperline $\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#},\infty^{\#})$ is #-open, and therefore $\varnothing$ is #-closed.

However, $\varnothing$ is also #-open, for to deny this is to say that $\varnothing$ contains a point that is not an #-interior point, which is absurd because $\varnothing$ contains no points. Since $\varnothing$ is #-open, $\hat{\mathbb{R}}_c^{\#}$ is #-closed. Thus, $\hat{\mathbb{R}}_c^{\#}$ and $\varnothing$ are both #-open and #-closed.

Remark 9.2. They are not the only subsets of $\hat{\mathbb{R}}_c^{\#}$ with this property mentioned above.

Definition 9.4. A deleted #-neighborhood of a point x_0 is a set that contains every point of some #-neighborhood of x_0 except for x_0 itself. For example, $S = \{x|0 < |x - x_0| < \varepsilon\}$, where $\varepsilon \approx 0$, is a deleted #-neighborhood of x_0. We also say that it is a deleted ε-#-neighborhood of x_0.

Theorem 9.1. (a) The union of #-open sets is #-open:

(b) The #-intersection of #-closed sets is #-closed:

These statements apply to arbitrary collections, hyperfinite or hyperinfinite, of #-open and #-closed sets.

Proof (a) Let L be a collection of #-open sets and $S = \cup\{G|G \in L\}$.

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in L, and since G_0 is #-open, it contains some ε-#-neighborhood of x_0. Since $G_0 \subset S$, this ε-#–neighborhood is in S, which is consequently a #-neighborhood of x_0. Thus, S is a #-neighborhood of each of its points, and therefore #-open, by definition.

(b) Let F be a collection of #-closed sets and $T = \cap\{H|H \in F\}$. Then $T^c = \cup\{H^c|H \in F\}$ and, since each H^c is #-open, T^c is #-open, from (a). Therefore, T is #-closed, by definition.

Example 9.2. If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] = \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#}a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. The set $[a,b) = \{x|a \leq x < b\}$ is a half-#-closed or half-#-open interval if $-\infty^{\#} < a < b < \infty^{\#}$, as is $(a,b] = \{x|a < x \leq b\}$ however, neither of these sets is #-open or #-closed. Semi-infinite #-closed intervals are sets of the form $[a,\infty^{\#}) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$, where a is hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$, respectively.

Definition 9.5. Let S be a subset of $\hat{\mathbb{R}}_c^{\#} = (-\infty^{\#},\infty^{\#})$. Then

(a) x_0 is a #-limit point of S if every deleted #-neighborhood of x_0 contains a point of S.

(b) x_0 is a boundary point of S if every #-neighborhood of x_0 contains at least one point in S and one not in S. The set of #-boundary points of S is the #-boundary of S, denoted by #-∂S. The #-closure of S, denoted by #-$\overline{S}$, is $S \cup$ #-∂S.

(c) x_0 is an #-isolated point of S if $x_0 \in S$ and there is a #-neighborhood of x_0 that contains no other point of S.

(d) x_0 is #-exterior to S if x_0 is in the #-interior of S^c. The collection of such points is the #-exterior of S.

Theorem 9.2. A set S is #-closed if and only if no point of S^c is a #-limit point of S.

Proof. Suppose that S is #-closed and $x_0 \in S^c$. Since S^c is #-open, there is a #-neighborhood of x_0 that is contained in S^c and therefore contains no points of S. Hence, x_0 cannot be a #-limit point of S. For the converse, if no point of S^c is a #-limit point of S then every point in S^c must have a #-neighborhood contained in S^c. Therefore, S^c is #-open and S is #-closed.

Corollary 9.1.A set S is #-closed if and only if it contains all its #-limit points.
If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.

Proposition 9.1. If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S. .

#-Open Coverings

Definition 9.6.A collection H of #-open sets of $\mathbb{R}_c^\#$ is an #-open covering of a set S if every point in S is contained in a set H belonging to H; that is, if $S \subset \cup\{F | F \in H\}$.

Definition 9.7.A set $S \subset \mathbb{R}_c^\#$ is called #-compact (or hyper compact) if each of its #-open covers has a finite or hyperfinite subcover. .

Theorem 9.3.(**Generalized Heine–Borel Theorem**) If H is an #-open covering of a #-closed and hyper bounded subset S of the hyperreal line $\mathbb{R}_c^\#$ (or of the $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^\#$) then S has an #-open covering $\widetilde{H}$ consisting of hyper finite many #-open sets belonging to H.

Proof. If a set S in $\mathbb{R}_c^{\#n}$ is hyper bounded, then it can be enclosed within an n-box $T_0 = [-a,a]^n$ where $a > 0$. By the property above, it is enough to show that T_0 is #-compact.

Assume, by way of contradiction, that T_0 is not #-compact. Then there exists an hyper infinite open cover $C_{\infty^\#}$ of T_0 that does not admit any hyperfinite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into $2n$ sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the $2n$ sections of T_0 must require an hyper infinite subcover of $C_{\infty^\#}$, otherwise $C_{\infty^\#}$ itself would have a hyperfinite subcover, by uniting together the hyperfinite covers of the sections. Call this section T_1.Likewise, the sides of T_1 can be bisected, yielding $2n$ sections of T_1, at least one of which must require an hyper infinite subcover of $C_{\infty^\#}$. Continuing in like manner yields a decreasing hyper infinite sequence of nested n-boxes: $T_0 \supset T_1 \supset T_2 \supset \ldots \supset T_k \supset \ldots, k \in \mathbb{N}^\#$, where the side length of T_k is $(2a)/2^k$, which #-converges to 0 as k tends to hyper infinity, $k \to \infty^\#$. Let us define a hyper infinite sequence $\{x_k\}_{k \in \mathbb{N}^\#}$ such that each $x_k : x_k \in T_k$. This hyper infinite sequence is Cauchy, so it must #-converge to some #-limit L. Since each $T_k$is #-closed, and for each k the sequence $\{x_k\}_{k \in \mathbb{N}^\#}$ is eventually always inside T_k, we see that $L \in T_k$ for each $k \in \mathbb{N}^\#$.

Since $C_{\infty^\#}$ covers $T_0,$ then it has some member $U \in C_{\infty^\#}$ such that $L \in U.$ Since U is open, there is an n-ball $B(L) \subseteq U.$ For large enough $k,$ one has $T_k \subseteq B(L) \subseteq U,$ but then the infinite number of members of $C_{\infty^\#}$ needed to cover T_k can be replaced by just one: $U,$ a contradiction.Thus, T_0 is #-compact. Since S is #-closed and a subset of the #-compact set $T_0,$ then S is also #-compact.

As an application of the Generalized Heine–Borel theorem, we give a short proof of the

Generalized Bolzano–Weierstrass Theorem.

Theorem 9.4.(Generalized Bolzano–Weierstrass Theorem) Every hyper bounded hyper infinite set $S \subset \mathbb{R}_c^\#$ has at least one #-limit point.

Proof. We will show that a hyper bounded nonempty set without a #-limit point can contain only finite or a hyper finite number of points. If S has no #-limit points, then S is #-closed (Theorem 9.) and every point $x \in S$ has an w-#-open neighborhood N_x that contains no point of S other than $x.$The collection $H = \{N_x | x \in S\}$ is an w-#-open covering for $S.$ Since S is also hyper bounded, Theorem 9.3 implies that S can be covered by finite or a hyper finite collection of sets from H, say $N_{x_1},\ldots,N_{x_n}, n \in \mathbb{N}^\#.$ Since these sets contain only $x_1,\ldots,x_n$ from $S,$ it follows that $S = \{x_k\}_{1 \leq k \leq n}, n \in \mathbb{N}^\#.$

10.External non-Archimedean field ${}^*\mathbb{R}_c^\#.$

10.1.External non-Archimedean field ${}^*\mathbb{R}_c^\#$ via Cauchy completion of internal non-Archimedean field ${}^*\mathbb{R}.$

Definition 10.1. A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a : \mathbb{N}^\# \to {}^*\mathbb{R}$from hypernatural numbers $\mathbb{N}^\#$ into the hyperreal numbers ${}^*\mathbb{R}.$

We usually denote such a function by $n \mapsto a_n,$or by $a : n \to a_n,$so the terms in the sequence are written $\{a_1,a_2,a_3,\ldots,a_n\ldots\}.$ To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{\infty^\#}$,or $\{a_n\}_{n \in \mathbb{N}^\#}$, or for the sake of brevity simply $\{a_n\}.$

Definition 10.2. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0,$there is a hypernatural number $N \in \mathbb{N}^\# \backslash \mathbb{N},$ $N = N(\varepsilon)$ such that, after N (i.e.for all $n > N$), $|a_n| \leq \varepsilon.$ We denote this symbolically by $a_n \to_\# 0.$

We can also, at this point, define what it means for a hyper infinite ${}^*\mathbb{R}$-valued sequence #-tends to any given number $q \in {}^*\mathbb{R}$: $\{a_n\}$ #-tends to q if the hyper infinite sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_\# 0.$

Definition 10.3. Let $\{a_n\}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence. We call $\{a_n\}$ a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence if the difference between its terms #-tends to $0.$ To be precise: given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0,$there is a hypernatural number $N = N(\varepsilon)$ such that for any $m,n > N, |a_n - a_m| < \varepsilon.$

Theorem 10.1.If $\{a_n\}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence (that is, $a_n \to_\# q$ for some hyperreal number $q \in {}^*\mathbb{R}$), then $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.

Proof. We know that $a_n \to_\# q.$ Here is a ubiquitous trick: instead of using ε in the definition Definition 10.3, start with an arbitrary infinite small $\varepsilon > 0, \varepsilon \approx 0$ and then choose $N \in \mathbb{N}^\# \backslash \mathbb{N}$ so that $|a_n - q| < \varepsilon/2$ when $n > N.$ Then if $m,n > N,$ we have $|a_n - a_m| = |(a_n - q) - (a_m - q)| \leq |a_n - q| + |a_m - q| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$This shows that $\{a_n\}_{n \in \mathbb{N}^\#}$ is a Cauchy sequence.

Theorem 10.2. If $\{a_n\}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is bounded or hyper bounded; that is, there is some finite or hyperfinite $M \in {}^*\mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^\#$.
Proof. Since $\{a_n\}$ is Cauchy, setting $\varepsilon = 1$ we know that there is some $N \in \mathbb{N}^\#$ such that $|a_m - a_n| < 1$ whenever $m, n > N$. Thus, $|a_{N+1} - a_n| < 1$ for $n > N$. We can rewrite this as $a_{N+1} - 1 < a_n < a_{N+1} + 1$. This means that $|a_n|$ is less than the maximum of $|a_{N+1} - 1|$ and $|a_{N+1} + 1|$. So, set M equal to the maximum number in the following list: $\{|a_0|, |a_1|, \ldots, |a_N|, |a_{N+1} - 1|, |a_{N+1} + 1|\}$. Then for any term a_n, if $n \leq N$, then $|a_n|$ appears in the list and so $|a_n| \leq M$; if $n > N$, then
(as shown above) $|a_n|$ is less than at least one of the last two entries in the list, and so $|a_n| \leq M$. Hence, $M \in {}^*\mathbb{R}$ is a bound for the sequence $\{a_n\}$.
Definition 10.4. Let S be a set. A relation $x \sim y$ among pairs of elements of S is said to be an equivalence relation if the following three properties hold:
Reflexivity: for any $s \in S, s \sim s$.
Symmetry: for any $s, t \in S$, if $s \sim t$ then $t \sim s$.
Transitivity: for any $s, t, r \in S$, if $s \sim t$ and $t \sim r$, then $s \sim r$.
Theorem 10.3. Let S be a set, with an equivalence relation $(\bullet \sim \bullet)$ on pairs of elements. For $s \in S$, denote by $\mathbf{cl}[s]$ the set of all elements in S that are related to s. Then for any $s, t \in S$, either $\mathbf{cl}[s] = \mathbf{cl}[t]$ or $\mathbf{cl}[s]$ and $\mathbf{cl}[t]$ are disjoint.
The hyperreal numbers ${}^*\mathbb{R}_c^\#$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences. Let $\mathcal{F}_{{}^*\mathbb{R}}$ denote the set of all Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.
Definition 10.5. Let $\{a_n\}$ and $\{b_n\}$ be in $\mathcal{F}_{{}^*\mathbb{R}}$. Say they are #-equivalent if $a_n - b_n \to_\# 0$ i.e., if and only if the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n - b_n\}$ tends to 0.
Theorem 10.4. Definition 10.5 yields an equivalence relation on $\mathcal{F}_{{}^*\mathbb{R}}$.
Proof. We need to show that this relation is reflexive, symmetric, and transitive.
Reflexive: $a_n - a_n = 0$, and the hyper infinite sequence all of whose terms are 0 clearly #-converges to 0. So $\{a_n\}$ is related to $\{a_n\}$.
Symmetric: Suppose $\{a_n\}$ is related to $\{b_n\}$, so $a_n - b_n \to_\# 0$.
But $b_n - a_n = -(a_n - b_n)$, and since only the absolute value $|a_n - b_n| = |b_n - a_n|$ comes into play in Definition 10.2, it follows that $b_n - a_n \to_\# 0$ as well. Hence, $\{b_n\}$ is related to $\{a_n\}$.
Transitive: Here we will use the $\varepsilon/2$ trick we applied to prove Theorem 10.1. Suppose $\{a_n\}$ is related to $\{b_n\}$, and $\{b_n\}$ is related to $\{c_n\}$. This means that $a_n - b_n \to_\# 0$ and $b_n - c_n \to_\# 0$. To be fully precise, let us fix $\varepsilon > 0, \varepsilon \approx 0$; then there exists an $N \in \mathbb{N}^\#$ such that for all $n > N, |a_n - b_n| < \varepsilon/2$; also, there exists an M such that for all $n > M$, $|b_n - c_n| < \varepsilon/2$. Well, then, as long as n is bigger than both N and M, we have that $|a_n - c_n| = |(a_n - b_n) + (b_n - c_n)| \leq |a_n - b_n| + |b_n - c_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So, choosing L equal to the max of N, M, we see that given $\varepsilon > 0$ we can always choose L so that for $n > L, |a_n - c_n| < \varepsilon$. This means that $a_n - c_n \to_\# 0$ – i.e. $\{a_n\}$ is related to $\{c_n\}$.
Definition 10.6. The external hyperreal numbers ${}^*\mathbb{R}_c^\#$ are the equivalence classes $\mathbf{cl}[\{a_n\}]$ of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers, as per Definition 10.5. That is, each such equivalence class is an external hyperreal number.

Definition 10.7. Given any hyperreal number $q \in {}^*\mathbb{R}$, define a hyperreal number $q^\#$ to be the equivalence class of the hyper infinite ${}^*\mathbb{R}$-valued sequence $q^\# = (q,q,q,q,\dots)$ consisting entirely of q. So we view ${}^*\mathbb{R}$ as being inside ${}^*\mathbb{R}_c^\#$ by thinking of each hyperreal number q as its associated equivalence class $q^\#$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

Definition 10.8. Let $s,t \in {}^*\mathbb{R}_c^\#$, so there are Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences $\{a_n\},\{b_n\}$ of hyperreal numbers with $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$.

(a) Define $s+t$ to be the equivalence class of the sequence $\{a_n + b_n\}$.

(b) Define $s \times t$ to be the equivalence class of the sequence $\{a_n \times b_n\}$.

Theorem 10.5.The operations $+,\times$ in Definition 10.8 (a),(b) are well-defined.

Proof. Suppose that $\mathbf{cl}[\{a_n\}] = \mathbf{cl}[\{c_n\}]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$. Thus means that $a_n - c_n \to_\# 0$ and $b_n - d_n \to_\# 0$. Then $(a_n+b_n)-(c_n+d_n) = (a_n-c_n)+(b_n-d_n)$. Now, using the familiar $\varepsilon/2$ trick, you can construct a proof that this tends to 0, and so $\mathbf{cl}[\{a_n+b_n\}] = \mathbf{cl}[\{c_n+d_n\}]$.

Multiplication is a little trickier; this is where we will use Theorem 10.3. We will also use another ubiquitous technique: adding 0 in the form of $s-s$. Again, suppose that $\mathbf{cl}[(a_n)] = \mathbf{cl}[(c_n)]$ and $\mathbf{cl}[\{b_n\}] = \mathbf{cl}[\{d_n\}]$; we wish to show that $\mathbf{cl}[\{a_n \times b_n\}] = \mathbf{cl}[\{c_n \times d_n\}]$, or, in other words, that $a_n \times b_n - c_n \cdot d_n \to_\# 0$. Well, we add and subtract one of the other cross terms, say $b_n \times c_n$: $a_n \times b_n - c_n \times d_n = a_n \times b_n + (b_n \times c_n - b_n \times c_n) - c_n \times d_n =$

$= (a_n \times b_n - b_n \times c_n) + (b_n \times c_n - c_n \times d_n) = b_n \times (a_n - c_n) + c_n \times (b_n - d_n)$.

Hence, we have $|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n|$. Now, from Theorem 10.2, there are numbers M and L such that $|b_n| \leq M$ and $|c_n| \leq L$ for all $n \in \mathbb{N}^\#$. Taking some number K which is bigger than both, we have

$|a_n \times b_n - c_n \times d_n| \leq |b_n| \times |a_n - c_n| + |c_n| \times |b_n - d_n| \leq K(|a_n - c_n| + |b_n - d_n|)$.

Now, noting that both $a_n - c_n$ and $b_n - d_n$ tend to 0 and using the $\varepsilon/2$ trick (actually, this time we'll want to use $\varepsilon/2K$), we see that $a_n \times b_n - c_n \times d_n \to_\# 0$.

Theorem 10.6. Given any hyperreal number $s \in {}^*\mathbb{R}_c^\#$, $s \neq 0$, there is a hyperreal number $t \in {}^*\mathbb{R}_c^\#$ such that $s \times t = 1$.

Proof. First we must properly understand what the theorem says. The premise is that s is nonzero, which means that s is not in the equivalence class of $\{0,0,0,0,\dots\}$. In other words, $s = \mathbf{cl}[\{a_n\}]$ where $a_n - 0$ does not #-converge to 0. From this, we are to deduce the existence of a hyperreal number $t = \mathbf{cl}[\{b_n\}]$ such that $s \times t = \mathbf{cl}[\{a_n \times b_n\}]$ is the same equivalence class as $\mathbf{cl}[\{1,1,1,1,\dots\}]$. Doing so is actually an easy consequence of the fact that nonzero hyperreal numbers have multiplicative inverses, but there is a subtle difficulty. Just because s is nonzero (i.e. $\{a_n\}$ does not tend to 0), there's no reason any number of the terms in $\{a_n\}$ can't equal 0. However, it turns out that eventually, $a_n \neq 0$. That is:

Lemma 10.1. If $\{a_n\}$ is a Cauchy sequence which does not #-tend to 0, then there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N, a_n \neq 0$.

Definition 10.9. Let $s \in {}^*\mathbb{R}_c^{\#}$. Say that s is positive if $s \neq 0$, and if $s = \mathbf{cl}[\{a_n\}]$ for some Cauchy sequence of hyperreal numbers such that for some $N \in \mathbb{N}^{\#}, a_n > 0$ for all $n > N$. Given two hyperreal numbers s, t, say that $s > t$ if $s - t$ is positive.

Theorem 10.7. Let $s,t \in {}^*\mathbb{R}_c^{\#}$ be hyperreal numbers such that $s > t$, and let $r \in {}^*\mathbb{R}_c^{\#}$. Then $s + r > t + r$.

Proof. Let $s = \mathbf{cl}[\{a_n\}], t = \mathbf{cl}[\{b_n\}]$, and $r = \mathbf{cl}[\{c_n\}]$. Since $s > t$ i.e., $s - t > 0$, we know that there is an $N \in \mathbb{N}^{\#}$ such that, for $n > N$, $a_n - b_n > 0$. So $a_n > b_n$ for $n > N$. Now, adding c_n to both sides of this inequality (as we know we can do for hyperreal numbers ${}^*\mathbb{R}$), we have $a_n + c_n > b_n + c_n$ for $n > N$, or $(a_n + c_n) - (b_n + c_n) > 0$ for $n > N$. Note also that $(a_n + c_n) - (b_n + c_n) = a_n - b_n$ does not #-converge to 0, by the assumption that $s - t > 0$. Thus, by Definition 10.8, this means that $s + r = \mathbf{cl}[\{a_n + c_n\}] > \mathbf{cl}[\{b_n + c_n\}] = t + r$.

Theorem 10.8. Let $s,t \in {}^*\mathbb{R}_c^{\#}$ $s,t > 0$ be hyperreal numbers.Then there is $m \in \mathbb{N}^{\#}$ such that $m \times s > t$.

Proof. Let $s,t > 0$ be hyperreal numbers. We need to find a natural number m so that $m \times s > t$. First, recall that, by m in this context, we mean $\mathbf{cl}[\{m,m,m,m,\dots\}]$. So, letting $s = \mathbf{cl}[\{a_n\}]$ and $t = \mathbf{cl}[\{b_n\}]$,what we need to show is that there exists m with $\mathbf{cl}[\{m,m,m,m,\dots\}] \times \mathbf{cl}[\{a_1,a_2,a_3,a_4,\dots\}] = \mathbf{cl}[\{m \times a_1, m \times a_2, m \times a_3, m \times a_4,\dots\}] >$
$> \mathbf{cl}[\{b_1,b_2,b_3,b_4,\dots\}]$.

Now, to say that $\mathbf{cl}[\{m \times a_n\}] > \mathbf{cl}[\{b_n\}]$, or $\mathbf{cl}[\{m \times a_n - b_n\}]$ is positive, is, by Definition 10.9, just to say that there is $N \in \mathbb{N}^{\#}$ such that $m \times a_n - b_n > 0$ for all $n > N$, while $m \times a_n - b_n \nrightarrow_{\#} 0$. To be precise, the first statement is:

There exist $m,N \in \mathbb{N}^{\#}$ so that $m \times a_n > b_n$ for all $n > N$.

To produce a contradiction, we assume this is not the case; assume that

(#) for every m and N, there exists an $n > N$ so that $m \times a_n \leq b_n$.

Now, since $\{b_n\}$ is a Cauchy sequence, by Theorem 10.2 it is hyperbounded - there is a hyperreal number $M \in {}^*\mathbb{R}$ such that $b_n \leq M$ for all $n \in \mathbb{N}^{\#}$. Now, by the properties for the hyperreal numbers ${}^*\mathbb{R}$, given any hyperreal number such that $\varepsilon > 0, \varepsilon \approx 0$, there is an $m \in \mathbb{N}^{\#}$ such that $M/m < \varepsilon/2$. Fix such an m. Then if $m \times a_n \leq b_n$, we have $a_n \leq b_n/m \leq M/m < \varepsilon/2$.

Now, $\{a_n\}$ is a Cauchy sequence, and so there exists N so that for $n,k > N, |a_n - a_k| < \varepsilon/2$.

By Asumption (#), we also have an $n > N$ such that $m \times a_n \leq b_n$, which means that $a_n < \varepsilon/2$. But then for every $k > N$, we have that $a_k - a_n < \varepsilon/2$, so $a_k < a_n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $a_k < \varepsilon$ for all $k > N$. This proves that $a_k \to_{\#} 0$, which by Definition 10.9 contradicts the fact that $\mathbf{cl}[\{a_n\}] = s > 0$.

Thus, there is indeed some $m \in N$ so that $m \times a_n - b_n > 0$ for all sufficiently infinite large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. To conclude the proof, we must also show that $m \times a_n - b_n \nrightarrow 0$. Actually, it is possible that $m \times a_n - b_n \to 0$ (for example if $\{a_n\} = \{1,1,1,\dots\}$ and $\{b_n\} = \{m,m,m,\dots\}$). But that's okay: then we can simply choose a larger m. That is: let m be a hypernatural number constructed as above, so that $m \times a_n - b_n > 0$ for all sufficiently large $\in \mathbb{N}^{\#} \backslash \mathbb{N}$. If it happens to be true that $m \times a_n - b_n \nrightarrow 0$, then the proof is complete.

If, on the other hand, it turned out that $m \times a_n - b_n \to 0$, then take instead the integer $m + 1$.Since $s = \mathbf{cl}[\{a_n\}] > 0$, we have a n > 0 for all infinite large n, so

$(m+1) \times a_n - b_n = m \times a_n - b_n + a_n > a_n > 0$ for all infinite large n, so $m+1$ works just as well as m did in this regard; and since $m \times a_n - b_n \to 0,$ we have $(m+1) \times a_n - b_n = (m \times a_n - b_n) + a_n \nrightarrow 0$ since $s = \mathbf{cl}[\{a_n\}] > 0$ (so $a_n \nrightarrow 0$).

It will be handy to have one more Theorem about how the hyperreals ${}^*\mathbb{R}$ and hyperreals ${}^*\mathbb{R}_c^\#$ compare before we proceed. This theorem is known as the density of ${}^*\mathbb{R}$ in ${}^*\mathbb{R}_c^\#$, and it follows almost immediately from the construction of the ${}^*\mathbb{R}_c^\#$ from ${}^*\mathbb{R}$.

Theorem 10.9. Given any hyperreal number $r \in {}^*\mathbb{R}_c^\#$, and any hyperreal number $\varepsilon > 0,\ \varepsilon \approx 0,$ there is a hyperreal number $q \in {}^*\mathbb{R}$ such that $|r - q| < \varepsilon$.

Proof. The hyperreal number r is represented by a Cauchy ${}^*\mathbb{R}$-valued sequence $\{a_n\}$. Since this sequence is Cauchy, given $\varepsilon > 0, \varepsilon \approx 0,$ there is $N \in \mathbb{N}^\#$ so that for all $m,n > N,$
$|a_n - a_m| < \varepsilon.$ Picking some fixed $l > N,$ we can take the hyperreal number q given by $q = \mathbf{cl}[\{a_l, a_l, a_l, \ldots\}]$. Then we have $r - q = \mathbf{cl}[\{a_n - a_l\}_{n\in\mathbb{N}^\#}]$, and $q - r = \mathbf{cl}[\{a_l - a_n\}_{n\in\mathbb{N}^\#}]$.

Now, since $l > N,$ we see that for $n > N, a_n - a_l < \varepsilon$ and $a_l - a_n < \varepsilon,$ which means by Definition 10.9 that $r - q < \varepsilon$ and $q - r < \varepsilon;$ hence, $|r - q| < \varepsilon.$

Definition 10.10.Let $S \subsetneqq {}^*\mathbb{R}_c^\#$ be a non-empty set of hyperreal numbers.

A hyperreal number $x \in {}^*\mathbb{R}_c^\#$ is called an upper bound for S if $x \geq s$ for all $s \in S.$

A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper
bound for S and $x \leq y$ for every upper bound y of $S.$

Remark 10.1.The order $\leq$ given by Definition 10.9 obviously is $\leq$ -incomplete.

Definition 10.11. Let $S \subsetneqq {}^*\mathbb{R}_c^\#$ be a nonempty subset of ${}^*\mathbb{R}_c^\#$. We we will say that:

(1) S is $\leq$ -admissible above if the following conditions are satisfied:

(i) S bounded or hyperbounded above;

(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0.$

(2) S is $\leq$-admissible belov if the following condition are satisfied:

(i) S bounded belov;

(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0.$

Theorem 10.10. (i) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^\#$ has the least upper bound property.(ii) Any $\leq$-admissible below subset $S \subset {}^*\mathbb{R}_c^\#$ has the greatest lower bound property.

Proof. Let $S \subset {}^*\mathbb{R}_c^\#$ be a nonempty subset, and let M be an upper bound for $S.$ We are going to construct two sequences of hyperreal numbers, $\{u_n\}$ and $\{l_n\}$. First, since S is nonempty, there is some element $s_0 \in S.$ Now, we go through the following hyperinductive procedure to produce numbers $u_0, u_1, u_2, \ldots, u_n, \ldots$ and $l_1, l_2, l_3, \ldots, l_n, \ldots$

(i) Set $u_0 = M$ and $l_0 = s.$

(ii) Suppose that we have already defined u_n and $l_n.$ Consider the number $m_n = (u_n + l_n)/2,$ the average between u_n and $l_n.$

(1) If m_n is an upper bound for $S,$ define $u_{n+1} = m_n$ and $l_{n+1} = l_n.$

(2) If m_n is not an upper bound for $S,$ define $u_{n+1} = u_n$ and $l_{n+1} = l_n.$

Remark 10.1.Since $s < M,$ it is easy to prove by hyper infinite induction that

(i) $\{u_n\}$ is a non-increasing sequence: $u_{n+1} \leq u_n, n \in \mathbb{N}^\#$and $\{l_n\}$ is a non-decreasing sequence $l_{n+1} \geq l_n, n \in \mathbb{N}^\#$, (ii) u_n is an upper bound for S for all $n \in \mathbb{N}^\#$ and l_n is never an upper bound for S for any $n \in \mathbb{N}^\#$.

This gives us the following lemma.

Lemma 10.2. $\{u_n\}$ and $\{l_n\}$ are Cauchy ${}^*\mathbb{R}$-valued sequences of hyperreal numbers.

Proof. Note that each $l_n \leq M$ for all $n \in \mathbb{N}^\#$. Since $\{l_n\}$ is non-decreasing, it follows that $\{l_n\}$ is Cauchy. For $\{u_n\}$, we have $u_n \geq s_0$ for all $n \in \mathbb{N}^\#$, and so $-u_n \leq -s_0$. Since $\{u_n\}$ is non-increasing, $\{-u_n\}$ is non-decreasing, and so as above, $\{-u_n\}$ is Cauchy. It is easy to verify that, therefore, $\{u_n\}$ is Cauchy.

The following Lemma shows that $\{u_n\}$ does #-tend to a hyperreal number $u \in {}^*\mathbb{R}_c^\#$.

Lemma 10.3. There is a hyperreal number $u \in {}^*\mathbb{R}_c^\#$ such that $u_n \to_\# u$.

Proof. Fix a term u_n in the sequence $\{u_n\}$. By Theorem 10.9, there is a hyperreal number $q_n \in {}^*\mathbb{R}, n \in \mathbb{N}^\#$ such that $|u_n - q_n| < 1/n$. Consider the sequence $\{q_1, q_2, q_3, \ldots, q_n, \ldots\}$ of hyperreal numbers. We will show this sequence is Cauchy. Fix $\varepsilon > 0, \varepsilon \approx 0$. By the Theorem 10.8, we can choose $N \in \mathbb{N}^\#$ so that $1/N < \varepsilon/3$. We know, since $\{u_n\}$ is Cauchy, that there is an $M \in \mathbb{N}^\#$ such that for $n, m > M$, $|u_n - u_m| < \varepsilon/3$. Then, so long as $n, m > max\{N, M\}$, we have

$$|q_n - q_m| = |(q_n - u_n) + (u_n - u_m) + (u_m - q_m)| \leq$$
$$\leq |q_n - u_n| + |u_n - u_m| + |u_m - q_m| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Thus, $\{q_n\}$ is a Cauchy sequence of internal hyperreal numbers, and so it represents the external hyperreal number $u = \mathbf{cl}[\{q_n\}]$. We must show that $u_n - u \to_\# 0$, but this is practically built into the definition of u. To be precise, letting q_n^* be the hyperreal number
$\mathbf{cl}[\{q_n, q_n, q_n, \ldots\}]$, we see immediately that $q_n^* - u \to_\# 0$ (this is precisely equivalent to the statement that $\{q_n\}$ is Cauchy). But $u_n - q_n^* < 1/n$ by construction; it is easily verify that the assertion that if a sequence $q_n^* \to_\# u$ and $u_n - q_n^* \to_\# 0$, then $u_n \to_\# u$. So $\{u_n\}$, a non-increasing sequence of upper bounds for S, tends to a hyperreal
number u. As you've guessed, u is the least upper bound of our set S. To prove this, we
need one more lemma.

Lemma 10.4. $l_n \to_\# u$.

Proof. First, note in the first case above, we have that
$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}.$$
In the second case, we also have
$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}.$$
Now, this means that $u_1 - l_1 = \frac{1}{2}(M - s)$, and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(L - s)$, and in general by hyperinfinite induction, $u_n - l_n = 2^{-n}(L - s)$. Since $L > s$ so $L - s > 0$, and since $2^{-n} < 1/n$, by the Theorem 10.8, we have for any $\varepsilon > 0$ that $2^{-n}(L - s) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^\#$. Thus, $u_n - l_n < 2^{-n}(L - s) < \varepsilon$ as well, and so $u_n - l_n \to_\# 0$. Again, it is easily verify that, since $u_n \to_\# u$, we have $l_n \to_\# u$ as well.

Remark 10.2.Note that assumption in Theorem 10.10 that S is $\leq$-admissible above subset of $\mathbb{R}_c^\#$ is necessarily, othervice Theorem 10.10 is not holds.

Theorem 10.11.(**Generalized Nested Intervals Theorem**)

Let $\{I_n\}_{n\in\mathbb{N}^\#} = \{[a_n,b_n]\}_{n\in\mathbb{N}^\#}, [a_n,b_n] \subset \mathbb{R}_c^\#$ be a hyper infinite sequence of closed intervals satisfying each of the following conditions:
(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots,$
(ii) $b_n - a_n \to_\# 0$ as $n \to \infty^\#$.
Then $\bigcap_{n=1}^{\infty^\#} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^\#$. Moreover both sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.
Proof.Note that: (a) the set $A = \{a_n | n \in \mathbb{N}^\#\}$ is hyperbouded above by $b_1$and
(b) the set $A = \{a_n | n \in \mathbb{N}^\#\}$ is $\leq$-admissible above subset of $\mathbb{R}_c^\#$.
By Theorem 10.10 there exists $\sup A$. Let $\xi = \sup A$.
Since I_n are nested,for any positive hyperintegers m and n we have
$a_m \leq a_{m+n} \leq b_{m+n} \leq b_n$,so that $\xi \leq b_n$ for each $n \in \mathbb{N}^\#$.Since we obviously have $a_n \leq \xi$ for each $n \in \mathbb{N}^\#$,we have $a_n \leq \xi \leq b_n$ for all $n \in \mathbb{N}^\#$,which implies $\xi \in \bigcap_{n=1}^{\infty^\#} I_n$.Finally, if $\xi,\eta \in \bigcap_{n=1}^{\infty^\#} I_n$, with $\xi \leq \eta$, then we get $0 \leq \eta - \xi \leq b_n - a_n$,for all $n \in \mathbb{N}^\#$,so that
$0 \leq \eta - \xi \leq \inf_{n\in\mathbb{N}^\#} |b_n - a_n| = 0$.
Theorem 10.12.(**Generalized Squeeze Theorem**)
Let $\{a_n\},\{c_n\}$ be two hyper infinite sequences #-converging to L,and $\{b_n\}$ a hyper infinite sequence. If $\forall n \geq K, K \in \mathbb{N}^\#$ we have $a_n \leq b_n \leq c_n$, then $\{b_n\}$ also #-converges to L.
Proof. Choose an $\varepsilon > 0, \varepsilon \approx 0$. By definition of the #-limit,there is an $N_1 \in \mathbb{N}^\#$ such that for all $n > N_1$ we have $|a_n - L| < \varepsilon$, in other words $L - \varepsilon < a_n < L + \varepsilon$.Similarly, there is an $N_2 \in \mathbb{N}^\#$ such that for all $n > N_2$ we have $L - \varepsilon < c_n < L + \varepsilon$. Denote $N = \max(N_1, N_2, K)$. Then for $n > N, L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, in other words $|b_n - L| < \varepsilon$.Since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary, by definition of the #-limit this says that #-$\lim_{n\to\infty^\#} b_n = L$.
Theorem 10.13.(**Corollary of the Generalized Squeeze Theorem**).
If #-$\lim_{n\to\infty^\#} |a_n| = 0$ then #-$\lim_{n\to\infty^\#} a_n = 0$.
Proof.We know that $-|a_n| \leq a_n \leq |a_n|$.We want to apply the Generalized Squeeze Theorem.We are given that #-$\lim_{n\to\infty^\#} |a_n| = 0$.This also implies that #-$\lim_{n\to\infty^\#}(-|a_n|) = 0$.So by the Generalized Squeeze Theorem, #-$\lim_{n\to\infty^\#} a_n = 0$.
Theorem 10.14. (**Generalized Bolzano-Weierstrass Theorem**)
Every hyperbounded hyper infinite ${}^*\mathbb{R}_c^\#$-valued sequence has a #-convergent hyper infinite subsequence.
Proof. Let $\{w_n\}_{n\in\mathbb{N}^\#}$ be a hyperbounded hyper infinite sequence. Then, there exists an interval $[a_1,b_1]$ such that $a_1 \leq w_n \leq b_1$ for all $n \in \mathbb{N}^\#$.
Either $\left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains hyper infinitely many terms of $\{w_n\}$. That is, there exists hyper infinitely many n in $\mathbb{N}^\#$ such that a_n is in $\left[a_1, \frac{a_1+b_1}{2}\right]$ or there exists hyper infinitely many n in $\mathbb{N}^\#$ such that a_n is in $\left[\frac{a_1+b_1}{2}, b_1\right]$. If $\left[a_1, \frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_2,b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$. Otherwise, let $[a_2,b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$.
Either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$. If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{w_n\}$, let $[a_3,b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$. Otherwise, let $[a_3,b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$. By hyper infinite induction, we can continue this construction and obtain hyper infinite sequence of intervals $\{[a_n,b_n]\}_{n\in\mathbb{N}^\#}$ such that:
(i) for each $n \in \mathbb{N}^\#, [a_n,b_n]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^\#}$,

(ii) for each $n \in \mathbb{N}^{\#}, [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and
(iii) for each $n \in \mathbb{N}^{\#}, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
Then generalized nested intervals theorem implies that the intersection of all of the intervals $[a_n, b_n]$ is a single point w. We will now construct a hyper infinite subsequence of $\{w_n\}_{n\in\mathbb{N}^{\#}}$ which will #-converge to w.
Since $[a_1, b_1]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_1 \in \mathbb{N}^{\#}$ such that w_{k_1} is in $[a_1, b_1]$. Since $[a_2, b_2]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_2 \in \mathbb{N}^{\#}, k_2 > k_1$, such that w_{k_2} is in $[a_2, b_2]$. Since $[a_3, b_3]$ contains hyper infinitely many terms of $\{w_n\}_{n\in\mathbb{N}^{\#}}$, there exists $k_3 \in \mathbb{N}^{\#}, k_3 > k_2$, such that w_{k_3} is in $[a_3, b_3]$. Continuing this process by hyper infinite induction, we obtain hyper infinite sequence $\{w_{k_n}\}_{n\in\mathbb{N}^{\#}}$ such that $w_{k_n} \in [a_n, b_n]$ for each $n \in \mathbb{N}^{\#}$. The sequence $\{w_{k_n}\}_{n\in\mathbb{N}^{\#}}$ is a subsequence of $\{w_n\}_{n\in\mathbb{N}^{\#}}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^{\#}$. Since $a_n \to_{\#} w$, and $a_n \leq w_n \leq b_n$ for each $n \in \mathbb{N}^{\#}$, the squeeze theorem implies that $w_{k_n} \to_{\#} w$.

Definition 10.12. Let $\{a_n\}$ be a hyperreal sequence i.e., $a_n \in {}^*\mathbb{R}_c^{\#}, n \in \mathbb{N}^{\#}$. Say that $\{a_n\}$ #-tends to 0 if, given any $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$, $N = N(\varepsilon)$ such that, for all $n > N$, $|a_n| \leq \varepsilon$. We often denote this symbolically by $a_n \to_{\#} 0$. We can also, at this point, define what it means for a hyperreal sequence #-tends to a given number $q \in {}^*\mathbb{R}_c^{\#}$: $\{a_n\}$ #-tends to q if the hyperreal sequence $\{a_n - q\}$ #-tends to 0 i.e., $a_n - q \to_{\#} 0$.

Definition 10.13. Let $\{a_n\}, n \in \mathbb{N}^{\#}$ be a hyperreal sequence. We call $\{a_n\}$ a Cauchy hyperreal sequence if the difference between its terms #-tends to 0. To be precise: given any hyperreal number $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N, |a_n - a_m| < \varepsilon$.

Theorem 10.15. If $\{a_n\}$ is a #-convergent hyperreal sequence (that is, $a_n \to_{\#} b$ for some hyperreal number $b \in \mathbb{R}_c^{\#}$), then $\{a_n\}$ is a Cauchy hyperreal sequence.

Theorem 10.16. If $\{a_n\}$ is a Cauchy hyperreal sequence, then it is hyper bounded; that is, there is some $M \in \mathbb{R}_c^{\#}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}^{\#}$.

Theorem 10.17. Any Cauchy hyperreal sequence $\{a_n\}$ has a #-limit in ${}^*\mathbb{R}_c^{\#}$ i.e., there exists $b \in {}^*\mathbb{R}_c^{\#}$ such that $a_n \to_{\#} b$.

Proof. By Definition 10.13 given $\varepsilon > 0, \varepsilon \approx 0$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $n, n' > N$,

$$|a_n - a_{n'}| < \varepsilon. \tag{10.1}$$

From (10.1) for any $n, n' > N$ we get

$$a_{n'} - \varepsilon < a_n < a_n + \varepsilon. \tag{10.2}$$

The generalized Bolzano-Weierstrass theorem implies there is a #-convergent hyper infinite subsequence $\{a_{n_k}\} \subset \{a_n\}$ such that $a_{n_k} \to_{\#} b$ for some hyperreal number $b \in {}^*\mathbb{R}_c^{\#}$. Let us show that the sequence $\{a_n\}$ also #-convergent to this $b \in {}^*\mathbb{R}_c^{\#}$.
We can choose $k \in \mathbb{N}^{\#}$ so large that $n_k > N$ and

$$|a_{n_k} - b| < \varepsilon. \tag{10.3}$$

We choose now in (10.1) $n' = n_k$ and therefore

$$|a_n - a_{n_k}| < \varepsilon. \tag{10.4}$$

From (10.3) and (10.4) for any $n > N$ we get

$$|(a_{n_k} - b) + (a_n - a_{n_k})| = |a_n - b| < 2\varepsilon. \quad (10.5)$$

Thus $a_n \to_{\#} b$ as well.

Remark 10.3.Note that there exist canonical natural embedings

$$\mathbb{R} \hookrightarrow {}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}^{\#}_c. \quad (10.6)$$

.

10.2.The Extended Hyperreal Number System ${}^*\hat{\mathbb{R}}^{\#}_c$

Definition 10.14.(a) A set $S \subset \mathbb{N}^{\#}$ is hyperfinite if $card(S) = card(\{x|0 \leq x \leq n\})$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. (b) A set $S \subseteq \mathbb{N}^{\#}$ is hyper infinite if $card(S) = card(\mathbb{N}^{\#})$.

Notation 10.2. If F is an arbitrary collection of subsets of ${}^*\mathbb{R}^{\#}_c$, then $\cup\{S|S \in F\}$is the set of all elements that are members of at least one of the sets in F , and $\cap\{S|S \in F\}$ is the set of all elements that are members of every set in F. The union and intersection of finitely or hyperfinitely many sets $S_k, 0 \leq k \leq n \in \mathbb{N}^{\#}$ are also written as $\bigcup_{k=0}^{n} S_k$ and $\bigcap_{k=0}^{n} S_k$. The union and intersection of an hyperinfinite sequence $S_k, k \in \mathbb{N}^{\#}$ of sets are written as $\bigcup_{k=0}^{\infty^{\#}} S$ or $\bigcup_{n \in \mathbb{N}^{\#}} S$ and $\bigcap_{k=0}^{\infty^{\#}} S$ or $\bigcap_{n \in \mathbb{N}^{\#}} S$ correspondingly.

A nonempty set S of hyperreal numbers ${}^*\mathbb{R}^{\#}_c$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to the hyperreal number system two points, $+\infty^{\#}$ (which we also write more simply as $\infty^{\#}$) and $-\infty^{\#}$,and to define the order relationships between them and any hyperreal number $x \in {}^*\mathbb{R}^{\#}_c$ by $-\infty^{\#} < x < \infty^{\#}$.

We call $-\infty^{\#}$ and $\infty^{\#}$ points at hyperinfinity. If S is a nonempty set of hyperreals, we write $\sup S = \infty^{\#}$ to indicate that S is unbounded above, and $\inf S = -\infty^{\#}$ to indicate that S is unbounded below.

#-Open and #-Closed Sets on ${}^*\hat{\mathbb{R}}^{\#}_c$.

Definition 10.15.If a and b are in the extended hyperreals and $a < b$, then the open interval (a,b) is defined by $(a,b) \triangleq \{x|a < x < b\}$. :

The open intervals $(a,+\infty^{\#})$ and $(-\infty^{\#},b)$ are semi-hyperinfinite if a and b are finite or hyperfinite, and $(-\infty^{\#},\infty^{\#})$ is the entire hyperreal line.

If $-\infty^{\#} < a < b < \infty^{\#}$, the set $[a,b] \triangleq \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^{\#},a)$ and $(b,\infty^{\#})$. We say that $[a,b]$ is a #-closed interval. Semi-hyper infinite #-closed intervals are sets of the form $[a,\infty) = \{x|a \leq x\}$ and $(-\infty^{\#},a] = \{x|x \leq a\}$,where a is finite or hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^{\#},a)$ and $(a,\infty^{\#})$, respectively.

Definition 10.16.If $x_0 \in \mathbb{R}^{\#}_c$ is a hyperreal number and $\varepsilon > 0, \varepsilon \approx 0$ then the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ is an #-neighborhood of x_0. If a set $S \subset {}^*\mathbb{R}^{\#}_c$ contains an #-neighborhood of x_0, then S is a #-neighborhood of x_0, and x_0 is an #-interior point of S.

The set of #-interior points of S is the #-interior of S, denoted by #-$Int(S)$.

(i) If every point of S is an #-interior point (that is, $S = $ #-$Int(S)$), then S is #-open.

(ii) A set S is #-closed if $S^c = {}^*\mathbb{R}^{\#}_c\backslash S$ is #-open.

Example 10.1. An open interval (a,b) is an #-open set, because if $x_0 \in (a,b)$ and

$\varepsilon \leq \min\{x_0 - a; b - x_0\}$, then $(x_0 - \varepsilon, x_0 + \varepsilon) \subset (a,b)$

Remark 10.4.The entire hyperline ${}^*\hat{\mathbb{R}}_c^\# = (-\infty^\#, \infty^\#)$ is #-open, and therefore $\varnothing$ is #-closed.

However, $\varnothing$ is also #-open, for to deny this is to say that $\varnothing$ contains a point that is not an #-interior point, which is absurd because $\varnothing$ contains no points. Since $\varnothing$ is #-open, ${}^*\hat{\mathbb{R}}_c^\#$ is #-closed. Thus, ${}^*\hat{\mathbb{R}}_c^\#$ and $\varnothing$ are both #-open and #-closed.

Remark 10.5.They are not the only subsets of ${}^*\hat{\mathbb{R}}_c^\#$ with this property.

Definition 10.17.A deleted #-neighborhood of a point x_0 is a set that contains every point of some #-neighborhood of x_0 except for x_0 itself. For example, $S = \{x|0 < |x - x_0| < \varepsilon\}$, where $\varepsilon \approx 0$, is a deleted #-neighborhood of x_0. We also say that it is a deleted ε-#-neighborhood of x_0.

Theorem 10.18.(a) The union of #-open sets is #-open:

(b) The #-intersection of #-closed sets is #-closed:

These statements apply to arbitrary collections, hyperfinite or hyperinfinite, of #-open and #-closed sets.

Proof (a) Let L be a collection of #-open sets and $S = \cup\{G|G \in L\}$.

If $x_0 \in S$, then $x_0 \in G_0$ for some G_0 in L, and since G_0 is #-open, it contains some ε-#-neighborhood of x_0. Since $G_0 \subset S$, this ε-#–neighborhood is in S, which is consequently a #-neighborhood of x_0. Thus, S is a #-neighborhood of each of its points, and therefore #-open, by definition.

(b) Let F be a collection of #-closed sets and $T = \cap\{H|H \in F\}$. Then $T^c = \cup\{H^c|H \in F\}$ and, since each H^c is #-open, T^c is #-open, from (a). Therefore, T is #-closed, by definition.

Example 10.2. If $-\infty^\# < a < b < \infty^\#$, the set $[a,b] = \{x|a \leq x \leq b\}$ is #-closed, since its complement is the union of the #-open sets $(-\infty^\# a)$ and $(b, \infty^\#)$. We say that $[a,b]$ is a #-closed interval. The set $[a,b) = \{x|a \leq x < b\}$ is a half-#-closed or half-#-open interval if $-\infty^\# < a < b < \infty^\#$, as is $(a,b] = \{x|a < x \leq b\}$ however, neither of these sets is #-open or #-closed. Semi-infinite #-closed intervals are sets of the form $[a, \infty^\#) = \{x|a \leq x\}$ and $(-\infty^\#, a] = \{x|x \leq a\}$,where a is hyperfinite. They are #-closed sets, since their complements are the #-open intervals $(-\infty^\#, a)$ and $(a, \infty^\#)$,respectively.

Definition 10.18. Let S be a subset of $\hat{\mathbb{R}}_c^\# = (-\infty^\#, \infty^\#)$. Then

(a) x_0 is a #-limit point of S if every deleted #-neighborhood of x_0 contains a point of S.

(b) x_0 is a boundary point of S if every #-neighborhood of x_0 contains at least one point in S and one not in S. The set of #-boundary points of S is the #-boundary of S, denoted by #-∂S. The #-closure of S, denoted by #-$\bar{S}$, is $S \cup$ #-∂S.

(c) x_0 is an #-isolated point of S if $x_0 \in S$ and there is a #-neighborhood of x_0 that contains no other point of S.

(d) x_0 is #-exterior to S if x_0 is in the #-interior of S^c. The collection of such points is the #-exterior of S.

Theorem 10.19. A set S is #-closed if and only if no point of S^c is a #-limit point of S.

Proof. Suppose that S is #-closed and $x_0 \in S^c$. Since S^c is #-open, there is a #-neighborhood of x_0 that is contained in S^c and therefore contains no points of S.

Hence, x_0 cannot be a #-limit point of S. For the converse, if no point of S^c is a #-limit point of S then every point in S^c must have a #-neighborhood contained in S^c. Therefore, S^c is #-open and S is #-closed.

Corollary 10.1.A set S is #-closed if and only if it contains all its #-limit points.

If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S.

Proposition 10.1. If S is #-closed and hyper bounded, then $\inf(S)$ and $\sup(S)$ are both in S. .

#-Open Coverings

Definition 10.19.A collection H of #-open sets of $\mathbb{R}_c^{\#}$ is an #-open covering of a set S if every point in S is contained in a set H belonging to H; that is, if $S \subset \cup\{F|F \in H\}$.

Definition 10.20.A set $S \subset \mathbb{R}_c^{\#}$ is called #-compact (or hyper compact) if each of its #-open covers has a hyperfinite subcover. .

Theorem 10.20.(**Generalized Heine–Borel Theorem**) If H is an #-open covering of a #-closed and hyper bounded subset S of the hyperreal line $\mathbb{R}_c^{\#}$ (or of the $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$) then S has an #-open

covering $\widetilde{H}$ consisting of hyper finite many #-open sets belonging to H.

Proof. If a set S in $\mathbb{R}_c^{\#n}$ is hyper bounded, then it can be enclosed within an n-box $T_0 = [-a,a]^n$ where $a > 0$. By the property above, it is enough to show that T_0 is #-compact.

Assume, by way of contradiction, that T_0 is not #-compact. Then there exists an hyper infinite open cover $C_{\infty^{\#}}$ of T_0 that does not admit any hyperfinite subcover. Through bisection of each of the sides of T_0, the box T_0 can be broken up into $2n$ sub n-boxes, each of which has diameter equal to half the diameter of T_0. Then at least one of the $2n$ sections of T_0 must require an hyper infinite subcover of $C_{\infty^{\#}}$, otherwise $C_{\infty^{\#}}$ itself would have a hyperfinite subcover, by uniting together the hyperfinite covers of the sections. Call this section T_1. Likewise, the sides of T_1 can be bisected, yielding $2n$ sections of T_1, at least one of which must require an hyper infinite subcover of $C_{\infty^{\#}}$. Continuing in like manner yields a decreasing hyper infinite sequence of nested n-boxes: $T_0 \supset T_1 \supset T_2 \supset\ldots\supset T_k \supset\ldots, k \in \mathbb{N}^{\#}$, where the side length of T_k is $(2a)/2^k$, which #-converges to 0 as k tends to hyper infinity, $k \to \infty^{\#}$. Let us define a hyper infinite sequence $\{x_k\}_{k\in\mathbb{N}^{\#}}$ such that each $x_k : x_k \in T_k$. This hyper infinite sequence is Cauchy, so it must #-converge to some #-limit L. Since each $T_k$is #-closed, and for each k the sequence $\{x_k\}_{k\in\mathbb{N}^{\#}}$ is eventually always inside T_k, we see that $L \in T_k$ for each $k \in \mathbb{N}^{\#}$. Since $C_{\infty^{\#}}$ covers T_0, then it has some member $U \in C_{\infty^{\#}}$ such that $L \in U$. Since U is open, there is an n-ball $B(L) \subseteq U$. For large enough k, one has $T_k \subseteq B(L) \subseteq U$, but then the infinite number of members of $C_{\infty^{\#}}$ needed to cover T_k can be replaced by just one: U, a contradiction.Thus, T_0 is #-compact. Since S is #-closed and a subset of the #-compact set T_0, then S is also #-compact.

As an application of the Generalized Heine–Borel theorem, we give a short proof of the

Generalized Bolzano–Weierstrass Theorem.

Theorem 10.21.(Generalized Bolzano–Weierstrass Theorem) Every hyper bounded hyper infinite set $S \subset \mathbb{R}_c^{\#}$ has at least one #-limit point.

Proof. We will show that a hyper bounded nonempty set without a #-limit point can contain only finite or a hyper finite number of points. If S has no #-limit points, then S is

#-closed (Theorem 9.) and every point $x \in S$ has an w-#-open neighborhood N_x that contains no point of S other than x. The collection $H = \{N_x | x \in S\}$ is an w-#-open covering for S. Since S is also hyper bounded, Theorem 9.3 implies that S can be covered by finite or a hyper finite collection of sets from H, say $N_{x_1},\ldots,N_{x_n}, n \in \mathbb{N}^{\#}$. Since these sets contain only $x_1,\ldots,x_n$ from S, it follows that $S = \{x_k\}_{1 \leq k \leq n}, n \in \mathbb{N}^{\#}$.

10.3.External Cauchy hyperreals $\mathbb{R}_c^{\#}$ and ${}^*\mathbb{R}_c^{\#}$ axiomatically.

A model for the Cauchy hyperreal number system consists of a set $\mathbb{R}_c^{\#}$, two distinct elements 0 and 1 of $\mathbb{R}_c^{\#}$, two binary operations $+$ and $\times$ on $\mathbb{R}^{\#}$ (called addition and multiplication, respectively), and a binary relation $\leq$ on $\mathbb{R}^{\#}$, satisfying the following properties.

Axioms:

I.$(\mathbb{R}_c^{\#},+,\times)$ forms a field i.e.,

(i) For all $x,y,$ and z in $\mathbb{R}^{\#}$, $x + (y + z) = (x + y) + z$ and $x \times (y \times z) = (x \times y) \times z$.
(associativity of addition and multiplication)

(ii) For all x and y in $\mathbb{R}^{\#}$, $x + y = y + x$ and $x \times y = y \times x$.
(commutativity of addition and multiplication)

(iii)For all $x,y,$ and z in $\mathbb{R}^{\#}$,$x \times (y + z) = (x \times y) + (x \times z)$.
(distributivity of multiplication over addition)

(iv)For all x in $\mathbb{R}^{\#}$, $x + 0 = x$.
(existence of additive identity)

0 is not equal to 1, and for all x in $\mathbb{R}^{\#}$, $x \times 1 = x$.
(existence of multiplicative identity)

(v) For every x in $\mathbb{R}^{\#}$, there exists an element $-x$ in $\mathbb{R}^{\#}$, such that $x + (-x) = 0$.
(existence of additive inverses)

(vi)For every $x \neq 0$ in $\mathbb{R}^{\#}$, there exists an element $x - 1$ in $\mathbb{R}^{\#}$, such that $x \times\ x - 1 = 1$.
(existence of multiplicative inverses)

II.$(\mathbb{R}^{\#},\leq)$ forms a totally ordered set. In other words,

(i) For all x in $\mathbb{R}^{\#}$, $x \leq x$. (reflexivity)

(ii) For all x and y in $\mathbb{R}^{\#}$, if $x \leq y$ and $y \leq x$, then $x = y$. (antisymmetry)

(iii)For all $x,y,$ and z in $\mathbb{R}^{\#}$, if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

(iv)For all x and y in $\mathbb{R}^{\#}$, $x \leq y$ or $y \leq x$. (totality)

The field operations $+$ and $\times$ on $\mathbb{R}^{\#}$ are compatible with the order $\leq$. In other words,

(v)For all x,y and z in $\mathbb{R}^{\#}$, if $x \leq y$, then $x + z \leq y + z$. (preservation of order under addition)

(vi) For all x and y in $\mathbb{R}^{\#}$, if $0 \leq x$ and $0 \leq y$, then $0 \leq x \times\ y$ (preservation of order under multiplication)

III.Non-Archimedean property

$\mathbb{Q}^{\#} \subset \mathbb{R}^{\#}$ i.e.,$\mathbb{R}^{\#}$ is non-Archimedean ordered field.

Remark 10.1.Here a hyperrational is by definition a ratio of two hyperintegers. Consider
the ring $\mathbb{Q}^{\#}_{\mathbf{fin}}$ of all limited (i.e. finite) elements in $\mathbb{Q}^{\#}$. Then $\mathbb{Q}^{\#}_{\mathbf{fin}}$ has a unique maximal ideal $\mathbf{I}^{\#}_{\approx}$, the infinitesimals or infinitesimal numbers are quantities that are closer to zero
than any real number from the field $\mathbb{R}$, but are not zero.The quotient ring $\mathbb{Q}^{\#}_{\mathbf{fin}}/\mathbf{I}^{\#}_{\approx}$ gives

the
field $\mathbb{R}$ of real numbers.
Definition 10.1. An element $x \in \mathbb{R}^{\#}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q},\ r > 0$.
As we shall see in a moment in bivalent case,
Theorem 10.1.Every finite $x \in \mathbb{R}^{\#}$ is infinitely close to some (unique) $r \in \mathbb{R}$ in the sense that $|x - r|$ is either 0 or positively infinitesimal in $\mathbb{R}^{\#}$. This unique r is called the standard
part of x and is denoted by $st(x)$.
Proof. Let $x \in \mathbb{R}^{\#}$ be finite. Let $D_1,$ be the set of $r \in \mathbb{R}$ such that $r < x$ and D_2 the set of
$r' \in \mathbb{R}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}$, hence determines a
unique $r_0 \in \mathbb{R}$. A simple argument shows that $|x - r_0|$ is infinitesimal, i.e., $st(x) = r_0$.
Notation 10.1.We usually write $x \approx 0$ iff $x \in \mathbf{I}^{\#}_{\approx}$.
Definition 10.2. A hypersequence of hyperreal numbers is any function $a : \mathbb{N}^{\#} \to \mathbb{R}^{\#}$.
Often hypersequences such as these are called hyperreal hypersequences,
hypersequences of hyperreal numbers or hypersequences in $\mathbb{R}^{\#}$ to make it clear that the
elements of the sequence are hyperreal numbers. Analogous definitions can be given for
sequences of hypernatural numbers, hyperintegers, etc.
Notation 10.2.However, we usually write a_n for the image of $n \in \mathbb{N}^{\#}$ under $a,$ rather than
$a(n)$.The values a_n are often called the elements of the hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$.
Definition 10.3. We call $x \in \mathbb{R}^{\#}$ the limit of the hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$ if the following condition holds: for each hyperreal number $\varepsilon \in \mathbb{R}^{\#}$ such that $\varepsilon \approx 0, \varepsilon > 0,$ there exists a hypernatural number $N \in \mathbb{N}^{\#}$ such that, for every hypernatural number $n \geq N,$ we have $|x_n - x| < \varepsilon.$
Definition 10.4.The hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$ is said to #-converge to the #-limit $x,$
written $x_n \to x, n \to \infty^{\#}$ or $\lim_{n \to \infty^{\#}}(x_n) = x.$ Symbolically, this reads:

$$\forall \varepsilon[(\varepsilon \approx 0) \wedge (\varepsilon > 0)][\exists N \in \mathbb{N}^{\#}(\forall n \in \mathbb{N}^{\#}(n \geq N \Longrightarrow |x_n - x| < \varepsilon))]. \qquad (10.1)$$

If a hypersequence $(x_n)_{n \in \mathbb{N}^{\#}}$ converges to some limit, then it is convergent; otherwise it is #-divergent. A hypersequence that has zero as a #-limit is sometimes called a null hypersequence.
Limits of hypersequences behave well with respect to the usual arithmetic operations.
If $a_n \to a, n \to \infty^{\#}$ and $b_n \to b, n \to \infty^{\#}$, then $a_n + b_n \to a + b, n \to \infty^{\#}$ and
$a_n \times b_n \to a \times b, n \to \infty^{\#}$ if neither b_n or any b_n is zero, $a_n \times b_n \to a \times b, n \to \infty^{\#}$.
The following properties of limits of real hypersequences provided, in each equation below, that the limits on the right exist.
The limit of a hypersequence is unique.
1.$\#\text{-}\lim_{n \to \infty^{\#}}(a_n \pm b_n) = \#\text{-}\lim_{n \to \infty^{\#}} a_n \pm \#\text{-}\lim_{n \to \infty^{\#}} b_n$
2.$\#\text{-}\lim_{n \to \infty^{\#}}(c \times a_n) = c \times \#\text{-}\lim_{n \to \infty^{\#}} a_n$
3.$\#\text{-}\lim_{n \to \infty^{\#}}(a_n \times b_n) = (\#\text{-}\lim_{n \to \infty^{\#}}) \times (\#\text{-}\lim_{n \to \infty^{\#}} b_n)$
4.$\#\text{-}\lim_{n \to \infty^{\#}}(a_n / b_n) = \#\text{-}\lim_{n \to \infty^{\#}} a_n / \#\text{-}\lim_{n \to \infty^{\#}} b_n$ provided $\#\text{-}\lim_{n \to \infty^{\#}} b_n \neq 0$
5.$\#\text{-}\lim_{n \to \infty^{\#}} a_n^p = [\#\text{-}\lim_{n \to \infty^{\#}} a_n]^p$

6. If $a_n \leq b_n$ where n greater than some N, then $\#\text{-}\lim_{n\to\infty^\#} a_n \leq \#\text{-}\lim_{n\to\infty^\#} b_n$

7. (Squeeze theorem) If $a_n \leq c_n \leq b_n$, and $\#\text{-}\lim_{n\to\infty^\#} a_n = \#\text{-}\lim_{n\to\infty^\#} b_n = L$, then $\#\text{-}\lim_{n\to\infty^\#} c_n = L$.

Definition 10.5.A hyper infinite sequence (x_n) is said to tend to hyperinfinity, written $x_n \to \infty^\#$ or $\#\text{-}\lim_{n\to\infty^\#} x_n = \infty^\#$, if for every $K \in \mathbb{R}^\#$, there is an $N \in \mathbb{N}^\#$ such that for every $n \geq N$; that is, the hypersequence terms are eventually larger than any fixed K. Similarly, $x_n \to -\infty^\#$ if for every $K \in \mathbb{R}^\#$, there is an $N \in \mathbb{N}^\#$ such that for every $n \geq N$, $x_n < K$. If a hypersequence tends to infinity or minus infinity, then it is divergent. However, a divergent hypersequence need not tend to plus or minus hyperinfinity

Definition 10.6.A hypersequence $(x_n)_{n\in\mathbb{N}^\#}$of hyperreal numbers is called a Cauchy hypersequence if for every positive hyperreal number ε, there is a positive hyperinteger $N \in \mathbb{N}^\#$ such that for all hypernatural numbers $m,n > N : |x_m - x_n| < \varepsilon$,where the vertical bars denote the absolute value. In a similar way one can define Cauchy hypersequences of hyperrational numbers,etc. Cauchy formulated such a condition by requiring $|x_m - x_n| \approx 0$ i.e., to be infinite small for every pair of infinite large $m,n \in \mathbb{N}^\#$.

Definition 10.7.Let $\mathbb{R}_c^\#$ be the set of Cauchy hypersequences of hyperrational numbers.

That is, hypersequences $(x_n)_{n\in\mathbb{N}^\#}$ of hyperrational numbers such that for every hyperrational $\varepsilon > 0$, there exists an hyperinteger $N \in \mathbb{N}^\#\backslash\mathbb{N}$ such that for all hypernatural numbers $m,n > N, |x_m - x_n| < \varepsilon$. Here the vertical bars as usial denote the absolute value.

Definition 10.8. A standard procedure to force all Cauchy hypersequences in a metric space to converge is adding new points to the metric space in a process called completion. $\mathbb{R}_c^\#$ is defined as the completion of $\mathbb{Q}^\#$ with respect to the metric $|x - y|$, as will be detailed below.

Definition 10.9. Cauchy hypersequences $(x_n)_{n\in\mathbb{N}^\#}$ and $(y_n)_{n\in\mathbb{N}^\#}$ can be added and multiplied as follows:

$$(x_n)_{n\in\mathbb{N}^\#} + (y_n)_{n\in\mathbb{N}^\#} = (x_n + y_n)_{n\in\mathbb{N}^\#}, \tag{10.2}$$

and

$$(x_n)_{n\in\mathbb{N}^\#} \times (y_n)_{n\in\mathbb{N}^\#} = (x_n \times y_n)_{n\in\mathbb{N}^\#}. \tag{10.3}$$

Definition 10.10. Two Cauchy hypersequences are called equivalent if and only if the difference between them tends to zero. This defines an equivalence relation that is compatible with the operations (10.2)-(10.3) defined above, and the set $\mathbb{R}_c^\#$ of all equivalence classes $\mathbf{cl}[(x_n)_{n\in\mathbb{N}^\#}]$ can be shown to satisfy all axioms of the hyperreal numbers.

We can embed $\mathbb{Q}^\#$ into $\mathbb{R}_c^\#$ by identifying the rational number $r \in \mathbb{Q}^\#$ with the equivalence class of the hypersequence $(r_n)_{n\in\mathbb{N}^\#}$ with $r_n = r$ for all $n \in \mathbb{N}^\#$.

Remark 10.2.Comparison between hyperreal numbers is obtained by defining the following comparison between Cauchy hypersequences:

$$(x_n)_{n\in\mathbb{N}^\#} \geq (y_n)_{n\in\mathbb{N}^\#} \tag{10.4}$$

if and only if x is equivalent to y or there exists an hyperinteger $N \in \mathbb{N}^\#$ such that

$x_n \geq y_n$
for all $n > N$.
Remark 10.3.By construction, every hyperreal number $x \in \mathbb{R}_c^{\#}$ is represented by a Cauchy
hypersequence of hyperrational numbers. This representation is far from unique; every
hyperrational hypersequence that converges to x is a representation of x. This reflects the observation that one can often use different hypersequences to approximate the same hyperreal number.The equation 0.999... $=$ 1 states that the hypersequences (0, 0.9, 0.99, 0.999,...) and (1, 1, 1, 1,...) are equivalent, i.e., their difference #-converges
to 0.
IV.The field $\mathbb{R}^{\#}$ is complete in the following sense:
Definition 10.11.Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a non-empty set of hyperreal numbers.
A hyperreal number $x \in \mathbb{R}_c^{\#}$ is called an upper bound for S if $x \geq s$ for all $s \in S$.
A hyperreal number x is the least upper bound (or supremum $\sup S$) for S if x is an upper
bound for S and $x \leq y$ for every upper bound y of S.
Remark 10.4.The order $\leq$ given by Eq.(3.4) obviously is $\leq$-incomplete.
Definition 10.12. Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a nonempty subset of $\mathbb{R}_c^{\#}$.We we will say that:
(1) S is $\leq$-admissible above if the following conditions are satisfied:
(i) S bounded above;
(ii) let $A(S)$ be a set $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.
(2) S is $\leq$-admissible belov if the following condition are satisfied:
(i) S bounded belov;
(ii) let $L(S)$ be a set $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there exst $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.
Theorem 10.2.(i) Every $\leq$-admissible above subset $S \subsetneqq \mathbb{R}_c^{\#}$ has a supremum $\sup S$.
(ii) Every $\leq$-admissible belov subset $S \subsetneqq \mathbb{R}_c^{\#}$ has infinum $\inf S$.
Proof.Let $S \subsetneqq \mathbb{R}_c^{\#}$ be a nonempty subset of $\mathbb{R}_c^{\#}$, and let $M \in \mathbb{Q}^{\#}$ be an hyperrational upper bound for S. We are going to construct two hypersequences of hyperrational numbers, $(u_n)_{n \in \mathbb{N}^{\#}}$and $(l_n)_{n \in \mathbb{N}^{\#}}$. First, since S is nonempty, there is some element $s_0 \in S$.
We can choose a hyperrational number $L \in \mathbb{Q}^{\#}$ such that $L < s_0$.Now, we go through the following hyperinductive procedure to produce hyperrational numbers $u_0, u_1, u_2, \ldots$ and $l_0, l_1, l_2, l_3, \ldots$.
(i) Set $u_0 = M$ and $l_0 = L$.
(ii) Suppose that we have already defined u_n and $l_n, n \in \mathbb{N}^{\#}$.
Consider the number $m_n = (u_n + l_n)/2$, i.e.,the average between u_n and l_n.
(1) If m_n is an upper bound for S, define $u_{n+1} = m_n$ and $l_{n+1} = l_n$.
(2) If m_n is not an upper bound for S, define $u_{n+1} = u_n$ and $l_{n+1} = m_n$.
Since $l_0 < M$, it is easy to prove by hyperinfinite induction that $(u_n)_{n \in \mathbb{N}^{\#}}$ is a non-increasing hypersequence, i.e.$u_{n+1} \leq u_n$ and $(l_n)_{n \in \mathbb{N}^{\#}}$ is a non-decreasing hypersequence, i.e. $l_{n+1} \geq l_n$.
Remark 10.5. Note that in the first case above, we have that

$$u_{n+1} - l_{n+1} = m_n - l_n = \frac{u_n + l_n}{2} - l_n = \frac{u_n - l_n}{2}. \quad (10.5)$$

In the second case, we also have that

$$u_{n+1} - l_{n+1} = u_n - m_n = u_n - \frac{u_n + l_n}{2} = \frac{u_n - l_n}{2}. \quad (10.6)$$

Now, this means that $u_1 - l_1 = \frac{1}{2}(M - L)$ and so $u_2 - l_2 = \frac{1}{2}(u_1 - l_1) = \frac{1}{2^2}(M - L)$, and in general by hyperinfinite induction one obtains

$$u_n - l_n = 2^{-n}(M - L). \quad (10.7)$$

Since $M > L$ so $M - L > 0$, and since $2^{-n} < n^{-1}$ we have for any $\varepsilon > 0, \varepsilon \approx 0$ that $2^{-n}(M - L) < \varepsilon$ for all sufficiently large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. Thus, $u_n - l_n < \varepsilon$ as well, and so

$$\#\text{-}\lim_{n \to \infty^{\#}}(u_n - l_n) = 0. \quad (10.8)$$

This defines two hypersequences of hyperrationals, and so we have hyperreal numbers

$l = (l_n)_{n \in \mathbb{N}^{\#}}$ and $u = (u_n)_{n \in \mathbb{N}^{\#}}$. It is easy to prove, by induction on $n \in \mathbb{N}^{\#}$ that:

(i) u_n is an upper bound for S for all $n \in \mathbb{N}^{\#}$ and

(ii) l_n is never an upper bound for S for any $n \in \mathbb{N}^{\#}$.

Thus u is an upper bound for S. To see that it is a least upper bound, notice that the #-limit of $(u_n - l_n)_{n \in \mathbb{N}^{\#}}$ is 0, and so $l = u$. Now suppose $b < u = l$ is a smaller upper bound

for S. Since $(l_n)_{n \in \mathbb{N}^{\#}}$ is monotonic increasing it is easy to see that $b < l_n$ for some $n \in \mathbb{N}^{\#}$.

But l_n is not an upper bound for S and so neither is b. Hence u is a least upper bound for S.

11.Basic analisys on external non-Archimedean field $\mathbb{R}^{\#}_{\mathbf{c}}$.

11.1.The #-limit of a function $f : \mathbb{R}^{\#}_{c} \to \mathbb{R}^{\#}_{c}$

Definition 11.1.The (ε, δ) definition of the #-limit of a function $f : D \to \mathbb{R}^{\#}_{c}$ is as follows: Let f be a $\mathbb{R}^{\#}_{c}$-valued function defined on a subset $D \subset \mathbb{R}^{\#}_{c}$ of the Cauchy hyperreal numbers. Let c be a limit point of D and let L be a hyperreal number. We say that

$$\#\text{-}\lim_{x \to_{\#} c} f(x) = L \quad (11.1)$$

if for every $\varepsilon \approx 0, \varepsilon > 0$ there exists a $\delta \approx 0, \delta > 0$ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$, symbolically:

$$\lim_{x \to_{\#} c} f(x) = L \Leftrightarrow (\forall \varepsilon(\varepsilon \approx 0 \wedge \varepsilon > 0) \exists \delta(\delta \approx 0 \wedge \delta > 0) \forall x \in D, 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon. \quad (11.2)$$

Definition 11.2.The function $f : \mathbb{R}^{\#}_{c} \to \mathbb{R}^{\#}_{c}$ is #-continuous (or micro continuous) at some

point c of its domain if the #-limit of $f(x)$, as x #-approaches c through the domain of f, exists and is equal to $f(c)$:

$$\#\text{-}\lim_{x \to_{\#} c} f(x) = f(c). \quad (11.3)$$

Theorem 11.1.If $\#\text{-}\lim_{x \to_{\#} x_0} f(x)$ exists; then it is unique that is; if $\#\text{-}\lim_{x \to_{\#} x_0} f(x) = L_1$ and $\#\text{-}\lim_{x \to_{\#} x_0} f(x) = L_2$, then $L_1 = L_2$.

Theorem 11.2. If $\#\text{-}\lim_{x \to_{\#} x_0} f_1(x) = L_1$ and $\#\text{-}\lim_{x \to_{\#} x_0} f_2(x) = L_2$ then

$$
\begin{aligned}
\#\text{-}\lim_{x\to_\# x_0}[f_1(x) \pm f_2(x)] &= L_1 \pm L_2,\\
\#\text{-}\lim_{x\to_\# x_0}[f_1(x) \times f_2(x)] &= L_1 \times L_2,\\
\#\text{-}\lim_{x\to_\# x_0}\frac{f_1(x)}{f_2(x)} &= \frac{L_1}{L_2}, L_2 \neq 0.
\end{aligned} \tag{11.4}
$$

Definition 11.3.(a) We say that $f(x)$ #-approaches the left-hand #-limit L as x #-approaches x_0 from the left,and write $\#\text{-}\lim_{x\to x_0-} f(x) = L$, if $f(x)$ is defined on some #-open interval (a, x_0) and, for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(x) - L| < \varepsilon$ if $x_0 - \delta < x < x_0$.

(b) We say that $f(x)$ #-approaches the right-hand #-limit L as x #-approaches x_0 from the right, and write $\#\text{-}\lim_{x\to_\# x_0+} f(x) = L$, if $f(x)$ is defined on some open interval (x_0, b) and, for each $\varepsilon > 0$, there is a $\delta > 0, \delta \approx 0$ such that $|f(x) - L| < \varepsilon, \varepsilon > 0, \varepsilon \approx 0$ if $x_0 < x < x_0 + \delta$.

Left- and right-hand #-limits are also called one-sided #-limits. We will often simplify the notation by writing $\#\text{-}\lim_{x\to_\# x_0-} f(x) = f(x_0-)$ and $\#\text{-}\lim_{x\to_\# x_0+} f(x) = f(x_0+)$.

Theorem 11.3. A function f has a #-limit at x_0 if and only if it has left- and right-hand #-limits at x_0; and they are equal. More specifically; $\#\text{-}\lim_{x\to_\# x_0} f(x) = L$ if and only if $f(x_0+) = f(x_0-) = L$.

Definition 11.4. We say that $f(x)$ approaches the #-limit L as x approaches $\infty^\#$, and write $\#\text{-}\lim_{x\to_\# \infty^\#} f(x) = L$, if f is defined on an interval $(a, \infty^\#)$ and, for each $\varepsilon > 0, \varepsilon \approx 0$, there is a number β such that $|f(x) = L| < \varepsilon$ if $x > \beta$.

Definition 11.5. We say that $f(x)$ approaches $\infty^\#$ as x approaches x_0 from the left, and write

$$
\#\text{-}\lim_{x\to_\# x_0-} f(x) = \infty^\# \text{ or } f(x_0-) = \infty^\# \tag{11.5}
$$

if f is defined on an interval (a, x_0) and, for each hyperreal number M, there is a $\delta \approx 0, \delta > 0$ such that $f(x) > M$ if $x_0 - \delta < x < x_0$.

Similarly we define: $\#\text{-}\lim_{x\to x_0-} f(x) = -\infty^\#, \#\text{-}\lim_{x\to_\# x_0+} f(x) = -\infty^\#, \#\text{-}\lim_{x\to_\# x_0+} f(x) = \infty^\#$.

Example 11.1. (i) $\#\text{-}\lim_{x\to_\# x_0-} x^{-1} = -\infty^\#$, (ii) $\#\text{-}\lim_{x\to_\# x_0+} x^{-1} = +\infty^\#$, (iii) $\#\text{-}\lim_{x\to_\# -\infty^\#} x^2 = \#\text{-}\lim_{x\to_\# \infty^\#} x^2 = \infty^\#$.

Remark 11.1. Throughout this paper, $\#\text{-}\lim_{x\to_\# x_0} f(x)$ exists" will mean that $\#\text{-}\lim_{x\to_\# x_0} f(x) = L$, where L is finite or hyperfinite.

To leave open the possibility that $L = \pm\infty^\#$, we will say that $\#\text{-}\lim_{x\to x_0} f(x)$ exists in the extended hyperreals.

This convention also applies to one-sided limits and limits as x approaches $\pm\infty^\#$.

11.2.Monotonic Functions $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$.

Definition 11.6.A function $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$ is nondecreasing on an interval $I \subset \mathbb{R}_c^\#$ if

$$
f(x_1) \leq f(x_2) \tag{11.6}
$$

whenever x_1 and x_2 are in I and $x_1 < x_2$, or nonincreasing on I if

$$
f(x_1) \geq f(x_2) \tag{11.7}
$$

whenever x_1 and x_2 are in I and $x_1 < x_2$.
In either case, f is on I. If $\leq$ can be replaced by $<$ in (11.6), f is increasing on I. If $\geq$ can be replaced by $>$ in (11.7), f is decreasing on I. In either of these two cases, f is strictly monotonic on I.

Theorem 11.4. Suppose that $f(x)$ is monotonic on (a,b) and define $\alpha = \inf_{a<x<b} f(x)$ and $\beta = \sup_{<x<b} f(x)$. Suppose that $\exists\alpha$ and $\exists\beta$, then:
(a) If f is nondecreasing, then $f(a+) = \alpha$ and $f(b-) = \beta$.
(b) If f is nonincreasing; then $f(a+) = \beta$ and $f(b-) = \alpha$.
Here $a+= -\infty^{\#}$ if $a = -\infty^{\#}$ and $b+= \infty^{\#}$ if $b = \infty^{\#}$.
(c) If $a < x_0 < b$, then $f(x_0+)$ and $f(x_0-)$ exist and are finite or hyperfinite; moreover, $f(x_0+) \leq f(x_0) \leq f(x_0-)$ if f is nondecreasing, and $f(x_0+) \geq f(x_0) \geq f(x_0-)$ if f is nonincreasing:

Proof (a) We first show that $f(a+) = \alpha$. If $M > \alpha$, there is an x_0 in (a,b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty^{\#}$, then $f(a+) = -\infty^{\#}$. If $\alpha > -\infty^{\#}$, let $M = \alpha+\varepsilon$, where $\varepsilon \approx 0, \varepsilon > 0$.
Then $\alpha \leq f(x) < \alpha+\varepsilon$, so (i) $|f(x)-\alpha| < \varepsilon$ if $a < x < x_0$.
If $a = -\infty^{\#}$, this implies that $f(-\infty^{\#}) = \alpha$. If a $> -\infty^{\#}$, let $\delta = x_0 - a$. Then (i) is equivalent to $|f(x)-\alpha| <\varepsilon$ if $a < x < a+\delta$, which implies that $f(a+) = \alpha$.
We now show that $f(b+) = \beta$. If $M < \beta$, there is an x_0 in (a,b) such that $f(x_0) > M$. Since $f(x)$ is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty^{\#}$, then $f(b-) = \infty^{\#}$. If $\beta < \infty^{\#}$, let $M = \beta-\varepsilon$, where $\varepsilon \approx \varepsilon > 0$. Then $\beta-\varepsilon < f(x) \leq \beta$, so (ii) $|f(x)-\beta| < \varepsilon$ if $x_0 < x < b$.
If $b = \infty^{\#}$, this implies that $f(\infty^{\#}) = \beta$. If b $< \infty^{\#}$, let $\delta = b-x_0$. Then (ii) is equivalent to $f(x) <$ if $b-\delta < x < b$, which implies that $f(b-) = \beta$.
(b) The proof is similar to the proof of (a).
(c) Suppose that $f(x)$ is nondecreasing. Applying (a) to $f(x)$ on (a,x_0) and (x_0,b) separately shows that $f(x_0-) = \sup_{a<x<x_0} f(x)$ and $f(x_0+) = \inf_{x_0<x<b} f(x)$.
However, if $x_1 < x_0 < x_2$, then $f(x_1) \leq f(x_0) \leq f(x_2)$ and hence, $f(x_0-) \leq f(x_0) \leq f(x_0+)$.

11.3. #-Limits Inferior and Superior

Definition 11.7.We say that: (i) f is bounded on a set $S \subseteq \mathbb{R}_c^{\#}$ if there is a constant $M \in \mathbb{R}, M < \infty$ such that $f(x) \leq M$ for all $x \in S$, (ii) f is hyperbounded on a set $S \subseteq \mathbb{R}_c^{\#}$ if f is not bounded on a set S and there is a constant $M \in \mathbb{R}_c^{\#}/\mathbb{R}, M < \infty^{\#}$ such that $f(x) \leq M$ for all $x \in S$.

Definition 11.8. Suppose that f is bounded or hyperbounded on $[a,x_0)$, where x_0 may be finite or hyperfinite or $\infty^{\#}$. For $a \leq x < x_0$, define (i) $S_f(x;x_0) = \sup_{x\leq t<x_0} f(t)$ and (ii) $I_f(x;x_0) = \inf_{x\leq t<x_0} f(t)$.
Then the left #-limit superior of $f(x)$ at x_0 is defined to be

$$\overline{\lim}_{x\to_{\#} x_0-} f(x) = \lim_{x\to_{\#} x_0-} S_f(x;x_0) \tag{11.8}$$

and the left limit inferior of $f(x)$ at x_0 is defined to be

$$\underline{\lim}_{x\to_{\#} x_0-} f(x) = \lim_{x\to_{\#} x_0-} I_f(x;x_0). \tag{11.9}$$

If $x_0 = \infty^{\#}$, we define $x_0- = \infty^{\#}$.

Theorem 11.5. If $f(x)$ is bounded or hyperbounded on $[a,x_0)$, then $\beta = \overline{\lim}_{x\to_{\#} x_0-} f(x)$ exists and is the unique hyperreal number with the following properties:

(a) If $\varepsilon > 0, \varepsilon \approx 0$, there is an a_1 in $[a, x_0)$ such that
(i) $f(x) < \beta + \varepsilon$ if $a_1 \le x < x_0$
(b) If $\varepsilon > 0, \varepsilon \approx 0$ and a_1 is in $[a, x_0)$, then
$f(\bar{x}) > \beta - \varepsilon$ for some $\bar{x} \in [a, x_0)$.
Proof. Since $f(x)$ is bounded or hyperbounded on $[a, x_0), S_f(x; x_0)$ is nonincreasing and bounded or hyperbounded on $[a, x_0)$. By applying Theorem 11.4(b) to $S_f(x; x_0)$, we conclude that β exists finite or hyperfinite.
Therefore, if $\varepsilon > 0, \varepsilon \approx 0$, there is an $\bar{a}$ in $[a, x_0)$ such that
(ii) $\beta - \varepsilon/2 < S_f(x; x_0) < \beta + \varepsilon/2$ if $\bar{a} \le x < x_0$.
Since $S_f(x; x_0)$ is an upper bound of $\{f(t) | x \le t < x_0\}, f(x) < S_f(x; x_0)$. Therefore, the second inequality in (ii) implies the inequality (i) with $a_1 = \bar{a}$. This proves (a).
To prove (b),let a_1 be given and define $x_1 = \max\{a_1, \bar{a}\}$. Then the first inequality in (ii) implies that (iii) $S_f(x; x_0) > \beta - \varepsilon/2$. Since $S_f(x; x_0)$ is the supremum of $\{f(t) | x_1 \le t < x_0\}$, there is an $\bar{x}$ in $[x_1, x_0)$ such that
$f(\bar{x}) > S_f(x; x_0) - \varepsilon/2$.This and (iii) imply that $f(\bar{x}) > \beta - \varepsilon/2$. Since $\bar{x}$ is in $[a_1, x_0)$, this proves (b).
Now we show that there cannot be more than one hyperreal number with properties (a) and (b). Suppose that $\beta_1 < \beta_2$ and β_2 has property (b); thus, if $\varepsilon \approx 0, \varepsilon > 0$ and a_1 is in $[a, x_0)$ there is an $\bar{x}$ in $[a_1, x_0)$ such that $f(\bar{x}) > \beta_2 - \varepsilon$. Letting $\varepsilon = \beta_2 - \beta_1$, we see that there is an $\bar{x}$ in $[a_1, b)$ such that $f(\bar{x}) > \beta_2 - (\beta_2 - \beta_1) = \beta_1$so β_1 cannot have property (a).Therefore, there cannot be more than one hyperreal number that satisfies both (a) and (b).
Theorem 11.6. If $f(x)$ is bounded or hyperbounded on $[a, x_0)$, then $\alpha = \underline{\lim}_{x \to x_0-} f(x)$ exists and there is the unique hyperreal number with the following properties:
(a) If $\varepsilon \approx 0, \varepsilon > 0$ there is an a_1 in $[a, x_0)$ such that
$f(x) > \alpha - \varepsilon$ if $a_1 \le x < x_0$.
(b) If $\varepsilon \approx 0, \varepsilon > 0$ and a_1 is in $[a, x_0)$, then
$f(\bar{x}) < \alpha + \varepsilon$ for some $\bar{x} \in [a, x_0)$.
Theorem 11.7. If $f(x)$ is bounded or hyperbounded on $[a, x_0)$, then
(i) $\underline{\lim}_{x \to_{\#} x_0-} f(x) \le \overline{\lim}_{x \to_{\#} x_0-} f(x)$;
(ii) $\underline{\lim}_{x \to_{\#} x_0-} f(-x) = -\overline{\lim}_{x \to_{\#} x_0-} f(x)$;
(iii) $\overline{\lim}_{x \to_{\#} x_0-} f(-x) = -\underline{\lim}_{x \to_{\#} x_0-} f(x)$;
(iv) $\underline{\lim}_{x \to_{\#} x_0-} f(x) = \overline{\lim}_{x \to_{\#} x_0-} f(x)$ if and only if $\lim_{x \to_{\#} x_0-} f(x)$ exists, in which case
$\lim_{x \to_{\#} x_0-} f(x) = \underline{\lim}_{x \to_{\#} x_0-} f(x) = \overline{\lim}_{x \to_{\#} x_0-} f(x)$
Theorem 11.8.Suppose that $f(x)$ and $g(x)$ are bounded or hyperbounded on $[a, x_0)$.
Then: (i) $\overline{\lim}_{x \to_{\#} x_0-} (f+g)(x) \le \overline{\lim}_{x \to_{\#} x_0-} f(x) + \overline{\lim}_{x \to x_0-} g(x)$;
(ii) $\underline{\lim}_{x \to_{\#} x_0-} (f+g)(x) \ge \underline{\lim}_{x \to_{\#} x_0-} f(x) + \underline{\lim}_{x \to_{\#} x_0-} g(x)$.
Theorem 11.9.The $\alpha = \lim_{x \to x_0-} f(x)$ exists i.e.,α is finite or hyperfinite
if and only if for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$
such that $|f(x_1) - f(x_2)| < \varepsilon$ if $x_0 - \delta < x_1, x_2 < x_0$.
Theorem 11.10.(i) Suppose that $f(x)$ is bounded or hyperbounded on an interval $(x_0, b]$,
then $\underline{\lim}_{x \to_{\#} x_0+} f(x) = \overline{\lim}_{x \to_{\#} x_0+} f(x)$ if and only if $\lim_{x \to x_0+} f(x)$ exists, in which case
$\lim_{x \to_{\#} x_0+} f(x) = \underline{\lim}_{x \to_{\#} x_0+} f(x) = \overline{\lim}_{x \to_{\#} x_0+} f(x)$.
(ii) Suppose that $f(x)$ is bounded or hyperbounded on an open interval containing x_0,

then $\lim_{x\to_{\#} x_0} f(x)$ exists if and only if
$\overline{\lim}_{x\to_{\#} x_0-} f(x) = \overline{\lim}_{x\to_{\#} x_0+} f(x) = \underline{\lim}_{x\to_{\#} x_0-} f(x) = \underline{\lim}_{x\to_{\#} x_0+} f(x)$.

11.4.The #-continuity of a function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_{\mathbf{c}}^{\#}$.

Definition 11.9. (i) We say that a function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_{\mathbf{c}}^{\#}$. is #-continuous at x_0 if f is defined on an open interval (a,b) containing x_0 and $\lim_{x\to_{\#} x_0-} f(x_0) = x_0$.
(ii) We say that f is #-continuous from the left at x_0 if f is defined on an open interval (a,x_0) and $f(x_0-) = f(x_0)$.
(iii) We say that f is #-continuous from the right at x_0 if f is defined on an open interval (x_0,b) and $f(x_0+) = f(x_0)$.

Theorem 11.11. (i) A function f is #-continuous at x_0 if and only if f is defined on an open
interval (a,b) containing x_0 and for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \tag{11.10}$$

whenever $|x - x_0| < \delta$.
(ii) A function f is #-continuous from the right at x_0 if and only if f is defined on an interval $[x_0,b)$ and for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that (11.10) holds whenever $x_0 \leq x < x_0 + \delta$.
(iii) A function f is #-continuous from the left at x_0 if and only if f is defined on an interval $(a,x_0]$ and for each $\varepsilon \approx 0, \varepsilon > 0$ there is a $\delta \approx 0, \delta > 0$ such that (11.10) holds whenever $x_0 - \delta < x \leq x_0$.
Note that from Definition 11.9 and Theorem 11.8, f is #-continuous at x_0 if and only if $f(x_0+) = f(x_0-) = f(x_0)$ or, equivalently, if and only if it is #-continuous from the right and left at x_0.

Definition 11.10. A function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_{\mathbf{c}}^{\#}$ is #-continuous on an open interval (a,b) if it is
#-continuous at every point in (a,b). If, in addition,

$$f(b-) = f(b) \tag{11.11}$$

or

$$f(a+) = f(a) \tag{11.12}$$

then f is #-continuous on $(a,b]$ or $[a,b)$, respectively. If f is #-continuous on (a,b) and (11.11) and (11.12) both hold, then f is #-continuous on $[a,b]$. More generally, if S is a subset of $\mathbf{dom}(f)$ consisting of finitely or countably or hyper finitely or hyper infinitely many disjoint intervals, then f is #-continuous on S if f is #-continuous on every interval in S.

Definition 11.11. A function $f : \mathbb{R}_c^{\#} \to \mathbb{R}_{\mathbf{c}}^{\#}$ is piecewise #-continuous on $[a,b]$ if
(i) $f(x_0+)$ exists for all x_0 in $[a,b)$;
(ii) $f(x_0-)$ exists for all x_0 in $(a,b]$;
(iii) $f(x_0+) = f(x_0-) = f(x_0)$ for all but except finitely or hyper finitely many points x_0 in (a,b).
If (iii) fails to hold at some x_0 in (a,b), f has a jump #-discontinuity at x_0. Also, f has a jump #-discontinuity at a if $f(a+) \neq f(a)$ or at b if $f(b-) \neq f(b)$.

Theorem 11.12. If f and g are #-continuous on a set S, then so are $f \pm g$, and fg. In addition, f/g is #-continuous at each x_0 in S such that $g(x_0) \neq 0$.
By hyper infinite induction, it can be shown that if $\forall n \in \mathbb{N}^{\#}$ $f_n(x)$ are #-continuous on a

set $S,$ then so are $\sum_{i\leq n} f_n(x)$. Therefore, $\forall n,m\in\mathbb{N}^{\#}$ any rational function $r(x)=\sum_{i\leq n} a_i x^i/\sum_{i\leq m} b_i x^i, b_i\neq 0$ is #-continuous for all values of x except those for which its denominator vanishes.

11.5.Removable #-discontinuities.

Definition 11.12.Let $f(x)$ be defined on a deleted #-neighborhood of x_0 and #-discontinuous (perhaps even undefined) at x_0. Then we say that $f(x)$ has a removable #-discontinuity at x_0 if $\lim_{x\to x_0} f(x_0)$ exists. In this case, the function

$$g(x)=\begin{cases} f(x) & \text{if} \quad x\in\mathbf{dom}(f) \text{ and } x\neq x_0 \\ \lim_{x\to x_0} f(x_0) & \text{if} \quad x=x_0 \end{cases} \tag{11.13}$$

is #-continuous at x_0.

11.6.Composite Functions $f:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_{\mathbf{c}}$.

Definition 11.13. Suppose that $f:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_{\mathbf{c}}$ and $g:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_{\mathbf{c}}$ are functions with domains $\mathbf{dom}(f)$ and $\mathbf{dom}(g)$ correspondingly. If $\mathbf{dom}(g)$ has a nonempty subset T such that $g(x)\in\mathbf{dom}(g)$ whenever $x\in T,$ then the composite function $f\circ g:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_{\mathbf{c}}$ is defined

on T by $(f\circ g)(x)=f(g(x))$

Theorem 11.10. Suppose that g is #-continuous at $x_0, g(x_0)$ is an #-interior point of $\mathbf{dom}(f)$ and f is #-continuous at $g(x_0)$. Then $f\circ g$ is #-continuous at x_0.

Proof. Suppose that $\varepsilon\approx 0,\varepsilon>0$. Since $g(x_0)$ is an #-interior point of $\mathbf{dom}(f)$ and $f(x)$ is #-continuous at $g(x_0)$, there is a $\delta_1\approx 0,\delta_1>0$ such that $f(t)$ is defined and
(i) $|f(t)-f(gx_0)|<\varepsilon$ if $|t-g(x_0)|<\delta_1$.
Since $g(x)$ is #-continuous at x_0, there is a $\delta\approx 0,\delta>0$ such that $g(x)$ is defined and
(ii) $|g(x)-g(x_0)|<\delta_1$ if $|x-x_0|<\delta$.
Now (i) and (ii) imply that $|f(g(x))-f(g(x_0))|<\delta$ if $|x-x_0|<\varepsilon$. Therefore, $f\circ g$ is #-continuous at x_0.

11.7.Bounded and Hyperbounded Functions $f:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_c$.

Definition 11.14. (i) A function $f:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_c$ is bounded below on a set $S\subset\mathbb{R}^{\#}_c$ if there is a finite or hyperfinite hyperreal number $m\in\mathbb{R}^{\#}_{c.\mathbf{fin}}$ such that $f(x)\geq m$ for all $x\in S$. If in this case the set $V=\{f(x)|x\in S\}$ has infimum $\alpha,$ we write $\alpha=\inf_{x\in S} f(x)$.
If there is a point $x_1\in S$ such that $f(x_1)=\alpha_{\text{,}}$, we say that α is the minimum of $f(x)$ on $S,$ and write $\alpha=\min_{x\in S} f(x)$
(ii) A function $f:\mathbb{R}^{\#}_c\to\mathbb{R}^{\#}_c$ is bounded above on $S\subset\mathbb{R}^{\#}_c$ if there is a finite or hyperfinite hyperreal number $M\in\mathbb{R}^{\#}_{c.\mathbf{fin}}$ such that $f(x)\leq M$ for all $x\in S$. If in this case, V has a supremum β, we write $\beta=\sup_{x\in S} f(x)$. If there is a point $x_1\in S$ such that $f(x_2)=\beta_{\text{,}}$, we say that β is the maximum of $f(x)$ on $S,$ and write $\alpha=\max_{x\in S} f(x)$.
(iii) If f is bounded above and below on a set S, we say that f is bounded on S.

Theorem 11.11. If f is #-continuous on a finite or hyperfinite #-closed interval $[a,b]$, then f is bounded on $[a,b]$.

Proof. Suppose that $t \in [a,b]$. Since f is #-continuous at t, there is an open interval I_t containing t such that

$$|f(x) - f(t)| < 1 if x \in I_t \cap [a,b] \qquad (11.14)$$

To see this, set $\varepsilon = 1$ in (11.10), Theorem 11.11. The collection $H = \{I_t | a \leq t \leq b\}$ is an open covering of $[a,b]$. Since $[a,b]$ is #-compact, the generalized Heine–Borel theorem implies that there are hyper finitely many points $t_1, t_2, \ldots, t_n, n \in \mathbb{N}^{\#}$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$. According to (11.14) with $t = t_i$, $|f(x) - f(t_i)| < 1$ if $x \in I_{t_i} \cap [a,b]$. Therefore,

$$f(x) = |(f(x) - f(t_i)) + f(t_i)| \leq |f(x) - f(t_i)| + |f(t_i)| \leq 1 + |f(t_i)| \qquad (11.15)$$

if $x \in I_{t_i} \cap [a,b]$. Let $M = 1 + \max_{1 \leq i \leq n} |f(t_i)|$. Since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$, (11.15) implies that $|f(x)| \leq M$ if $x \in [a,b]$.

Theorem 11.12. Suppose that f is #-continuous on a finite or hyperfinite closed interval $[a,b]$. Let $V_{a,b} = \{f(x) | x \in [a,b]\}$ and let

$$\alpha = \inf V_{a,b} = \inf_{a \leq x \leq b} f(x) \text{ and } \beta = \sup V_{a,b} = \sup_{a \leq x \leq b} f(x). \qquad (11.16)$$

Then α and β are respectively the minimum and maximum of f on $[a,b]$; that is there are points x_1 and x_2 in $[a,b]$ such that $\alpha = f(x_1)$ and $\beta = f(x_2)$.

Proof. We show that x_1 exists. Note that the set $V_{a,b}$ is admissible below (above), since f is #-continuous on $[a,b]$. Suppose that there is no x_1 in $[a,b]$ such that $f(x_1) = \alpha_{\text{c}}$. Then $f(x) > \alpha_{\text{c}}$ for all $x \in [a,b]$. We will show that this leads to a contradiction. Suppose that $t \in [a,b]$. Then $f(t) > \alpha$, so $f(t) > [f(t) + \alpha]/2 > \alpha$. Since f is #-continuous at t, there is an open interval I_t about t such that

$$f(x) > \frac{f(t) + \alpha}{2} \qquad (11.17)$$

if $x \in I_t \cap [a,b]$. The collection $H = \{I_t | a \leq t \leq b\}$ is an open covering of $[a,b]$. Since $[a,b]$ is #-compact, the generalized Heine–Borel theorem implies that there are hyper finitely many points $t_1, t_2, \ldots, t_n$ such that the intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b]$. Define $\alpha_1 = \min_{1 \leq i \leq n} [f(t_i) + \alpha]/2$. Then, since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$, (11.17) implies that $f(t) > \alpha_1, a \leq t \leq b$. But $\alpha_1 > \alpha$, so this contradicts the definition of α. Therefore, $f(x_1) = \alpha_{\text{c}}$ for some $x_1 \in [a,b]$.

11.8. Generalized Intermediate Value Theorem.

The next theorem shows that if f is continuous on a finite closed interval $[a,b]$, then f assumes every value between $f(a)$ and $f(b)$ as x varies from a to b.

Theorem 11.13.(**Generalized Intermediate Value Theorem**) Suppose that f is #-continuous on $[a,b]$, $f(a) \neq f(b)$ and $f(a) < \mu < f(b)$ Then $f(c) = \mu$ for some $c \in (a,b)$.

Proof. Suppose that $f(a) < \mu < f(b)$. The set $S = \{x | (a \leq x \leq b) \wedge (f(x) \leq \mu)\}$ is bounded and nonempty. Note that the set S is admissible above, since f is #-continuous on $[a,b]$ and therefore $\sup S$ exists. Let $c = \sup S$. We will show that $f(c) = \mu$. If $f(c) > \mu$,

then $c > a$ and, since f is #-continuous at c, there is an $\varepsilon > 0, \varepsilon \approx 0$ such that

$f(x) > \mu$ if $c - \varepsilon < x \leq c.$ Therefore, c is an upper bound for $S,$ which contradicts the definition of c as the supremum of $S.$ If $f(c) < \mu,$ then $c < b$ and there is an $\varepsilon > 0, \varepsilon \approx 0$ such that $f(x) < \mu$ for $c \leq x < c - \varepsilon,$ so c is not an upper bound for $S.$ This is also a contradiction. Therefore, $f(c) = \mu.$ The proof for the case where $f(b) < \mu < f(a)$ can be obtained by applying this result to $-f(x)$.
Lemma.11.1. If f is #-continuous at x_0 and $f(x_0) > \mu,$ then $f(x) > \mu$ for all x in some #-neighborhood of x_0.

11.9.Uniform #-Continuity.

Definition 11.15. A function f is uniformly #-continuous on a subset S of its domain if, for every $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(x) - f(x')| < \varepsilon$ whenever $|x - x'| < \delta$ and $x, x' \in S.$

We emphasize that in this definition δ depends only on and S and not on the particular choice of x and $x',$ provided that they are both in $S.$

Theorem 11.14. If f is #-continuous on a #-closed and bounded or hyperbounded interval $[a,b]$, then f is uniformly #-continuous on $[a,b]$.

Proof. Suppose that $\varepsilon > 0, \varepsilon \approx 0.$ Since f is #-continuous on $[a,b]$, for each $t \in [a,b]$ there is a positive number δ_t such that

$$|f(x) - f(t)| < \varepsilon/2 \qquad (11.18)$$

if $|x - t| < \delta_t$ and $x \in [a,b].$ If $I_t = (t - \delta_t, t + \delta_t)$, the collection $H = \{I_t | t \in [a,b]\}$ is an open covering of $[a,b].$ Since $[a,b]$ is #-compact, the generalized Heine–Borel theorem implies that there are hyper finitely many points $t_1, t_2, \ldots, t_n$ in $[a,b]$ such that $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b].$ Now define

$$\delta = \min\left\{\delta_{t_1}, \delta_{t_2}, \ldots, \delta_{t_n}\right\}. \qquad (11.19)$$

We will show that if

$$|x - x'| < \delta \text{ and } x, x' \in [a,b] \qquad (11.20)$$

then $|f(x) - f(x')| < \varepsilon.$ From the triangle inequality one obtains:

$$|f(x) - f(x')| = |(f(x) - f(t_r)) + (f(t_r) - f(x'))| \leq |f(x) - f(t_r)| + |f(t_r) - f(x')| \quad (11.21)$$

Since $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ cover $[a,b],$ x must be in one of these intervals. Suppose that $x \in I_{t_r}$ that is,

$$|x - t_r| \leq \delta_{t_r}. \qquad (11.22)$$

From (11.18) with $t = t_r,$

$$|f(x) - f(t_r)| \leq \frac{\varepsilon}{2}. \qquad (11.23)$$

From (11.20), (11.22), and the triangle inequality,

$$|x' - t_r| = |(x' - x) + (x - t_r)| \leq |x' - x| + |x - t_r| < \delta + \delta_{t_r} \leq 2\delta_{t_r}. \qquad (11.24)$$

Therefore, (11.18) with $t = t_r$ and x replaced by x' implies that

$$|f(x') - f(t_r)| \leq \frac{\varepsilon}{2}. \qquad (11.25)$$

Thus (11.25),(11.21) and (11.23) imply that $|f(x') - f(t_r)| \leq \varepsilon/2.$

11.10. Monotonic External Functions $f : \mathbb{R}^{\#}_{c} \to \mathbb{R}^{\#}_{c}$.

Theorem 11.15. If f is monotonic and nonconstant on $[a,b],$ then f is #-continuous on $[a,b]$ if and only if its range $\mathbf{range}\,(f) = \{f(x)|x \in [a,b]\}$ is the #-closed interval with endpoints $f(a)$ and $f(b)$.

Theorem 11.16. Suppose that f is increasing and #-continuous on $[a,b]$ and let $f(a) = c$ and $f(b) = d$. Then there is a unique function g defined on $[c,d]$ such that

$$g(f(x)) = x, a \leq x \leq b, \tag{11.26}$$

and

$$f(g(y)) = y, c \leq y \leq d. \tag{11.27}$$

Moreover, g is #-continuous and increasing on $[c,d]$:

The function g of Theorem 11.16 is the inverse of f, denoted by f^{-1}. Since (11.26) and (11.27) are symmetric in f and g, we can also regard f as the inverse of g, and denote it by g^{-1}.

11.11. The #-derivative of a $\mathbb{R}_c^\#$-valued function $f : D \to \mathbb{R}_c^\#$.

A function $f : D \to \mathbb{R}_c^\#, D \subset \mathbb{R}_c^\#$ is differentiable at an interior point $x_0 \in D$ of its domain $D \subset \mathbb{R}_c^\#$ if the difference quotient

$$\frac{f(x) - f(x_0)}{x - x_0}, x \neq x_0 \tag{11.28}$$

approaches a #-limit as x approaches x_0, in which case the #-limit is called the #-derivative of f at x_0, and is denoted by $f^{\#'}(x_0)$ or by $f'^{\#}(x_0)$ or by $d^\# f(x_0)/d^\# x$ i.e.,

$$d^\# f(x_0)/d^\# x \triangleq f'^{\#}(x_0) = \#\text{-}\lim_{x \to_\# x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{11.29}$$

If f is defined on an open set $S \subset \mathbb{R}_c^\#$, we say that f is #-differentiable on S if f is #-differentiable at every point of S. If f is #-differentiable on S, then $f'^{\#}$ is a function on S.

We say that f is #-continuously #-differentiable on S if $f'^{\#}(x)$ is #-continuous on S. If f is #-differentiable on a #-neighborhood of x_0, it is reasonable to ask if $f'^{\#}(x)$ is #-differentiable at x_0. If so, we denote the #-derivative of $f'^{\#}$ at x_0 by $f'^{\#}(x_0)$. This is the second #-derivative of f at x_0, and it is also denoted by $f^{(2)\#}(x_0)$. Continuing inductively, if $f^{(n-1)\#}$ is defined on a #-neighborhood of x_0, then the n-th #-derivative of f at x_0, denoted by $f^{(n)\#}(x_0)$, where $n \in \mathbb{N}^\#$ or by $d^{n\#} f(x_0)/d^\# x^n$ is the #-derivative of $f^{(n-1)\#}(x)$ at x_0. For convenience we define the zeroth #-derivative of f to be f itself; thus $f^{(0)\#} = f$.

Example11.1 If $n \in \mathbb{N}^\# \backslash \mathbb{N}$ is a positive hyperinteger and $f(x), = x^n$ then

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^n - x_0^n}{x - x_0} = \frac{x - x_0}{x - x_0}\left(Ext\text{-}\sum_{k=0}^{n-1} x^{n-k-1}\right). \tag{11.30}$$

Thus $f'^{\#}(x_0) = \#\text{-}\lim_{x \to_\# x_0} Ext\text{-}\sum_{k=0}^{n-1} x^{n-k-1} = nx^{n-1}$.

Lemma 11.2. If f is #-differentiable at x_0; then

$$f(x) = f(x_0) + \left[f'^{\#}(x_0) + E(x)\right](x - x_0), \tag{11.31}$$

where $E(x)$ is defined on a #-neighborhood of x_0 and $\#\text{-}\lim_{x \to_\# x_0} E(x) = E(x_0) = 0$.

Proof. Define

$$E(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} - f'^{\#}(x_0) & x \in \mathbf{Dom}(f) \text{ and } x \neq x_0 \\ 0 & x = x_0 \end{cases} \quad (11.32)$$

Solving (11.32) for $f(x)$ yields (11.31) if $x \neq x_0$, and (11.31) is obvious if $x = x_0$. Definition 11.29 implies that $\#\text{-}\lim_{x \to x_0} E(x) = 0$. We defined $E(x_0) = 0$ to make $E(x)$ #-continuous at x_0. Since the right side of (11.32) is #-continuous at x_0, so is the left. This yields the following theorem.

Theorem 11.17. If f is #-differentiable at x_0; then f is #-continuous at x_0.

Theorem 11.18. If f and g are #-differentiable at x_0, then so are $f \pm g$ and fg with

(a) $(f+g)'^{\#}(x_0) = f'^{\#}(x_0) + g'^{\#}(x_0)$;

(b) $(f-g)'^{\#}(x_0) = f'^{\#}(x_0) - g'^{\#}(x_0)$;

(c) $(fg)'^{\#}(x_0) = f'^{\#}(x_0)g(x_0) + f(x_0)g'^{\#}(x_0)$;

(d)The quotient f/g is #-differentiable at x_0 if $g(x_0) \neq 0$ with

$$\left(\frac{f}{g}\right)'^{\#}(x_0) = \frac{f'^{\#}(x_0)g(x_0) - g'^{\#}(x_0)f(x_0)}{[g(x_0)]^2}.$$

(e) If $n \in \mathbb{N}^{\#}$ and $f_i, 1 \leq i \leq n$ are #-differentiable at x_0, then so are $Ext\text{-}\sum_{i=1}^{n} f_i$ and

$$\left(Ext\text{-}\sum_{i=1}^{n} f_i(x_0)\right)'^{\#} = Ext\text{-}\sum_{i=1}^{n} f_i'^{\#}(x_0).$$

(f) If $n \in \mathbb{N}^{\#}$ and $f^{(n)\#}(x_0), g^{(n)\#}(x_0)$ exist, then so does $(f \times g)^{(n)\#}(x_0)$ and

$$(fg)^{(n)\#}(x_0) = Ext\text{-}\sum_{i=0}^{n} \binom{n}{i} f^{(i)\#}(x_0) g^{(n-i)\#}(x_0).$$

Proof. For the statements (a)-(d) the proof is straightforward. For the statements (e) and (f) immediately by hyper infinite induction.

Theorem 11.19. (The Chain Rule) Suppose that g is #-differentiable at x_0 and f is #-differentiable at $g(x_0)$. Then the composite function $h = f \circ g$ defined by $h(x) = f(g(x))$ is #-differentiable at x_0 with $h'^{\#}(x) = f'^{\#}(g(x_0))g'^{\#}(x_0)$.

Definition 11.16.If $f(x)$ is defined on $[x_0, b)$, the right-hand derivative of $f(x)$ at x_0 is defined to be

$$f_+'^{\#}(x_0) = \#\text{-}\lim_{x \to_\# x_0+} \frac{f(x)-f(x_0)}{x-x_0}, \quad (11.33)$$

if the #-limit exists, while if f is defined on $(a, x_0]$, the left-hand derivative of $f(x)$ at x_0 is defined to be

$$f_-'^{\#}(x_0) = \#\text{-}\lim_{x \to_\# x_0-} \frac{f(x)-f(x_0)}{x-x_0}, \quad (11.34)$$

if the #-limit exists.

Remark 11.2. Note that $f(x)$ is #-differentiable at x_0 if and only if $f_+'^{\#}(x_0)$ and $f_-'^{\#}(x_0)$ exist and are equal, in which case $f'^{\#}(x_0) = f_-'^{\#}(x_0) = f_+'^{\#}(x_0)$.

Definition 11.17.We say that $f(x_0)$ is a local extreme value of $f(x)$ if there is a $\delta > 0$, $\delta \approx 0$ such that $f(x) - f(x_0)$ does not change sign on

$$(x_0 - \delta x_0 + \delta) \cap \mathbf{dom}(f). \quad (11.35)$$

More specifically, $f(x_0)$ is a local maximum value of $f(x)$ if

$$f(x) \leq f(x_0) \quad (11.36)$$

or a local minimum value of $f(x)$ if

$$f(x) \geq f(x_0) \tag{11.37}$$

for all $x \in (x_0 - \delta x_0 + \delta) \cap \mathbf{dom}(f)$. The point x_0 is called a local extreme point of $f(x)$, or, more specifically, a local maximum or local minimum point of $f(x)$.

Theorem 11.20. If $f(x)$ is #-differentiable at a local extreme point $x_0 \in \mathbf{dom}(f)$ then $f'^{\#}(x_0) = 0$.

Theorem 11.21. (Generalized Rolle's Theorem) Suppose that f is #-continuous on the #-closed interval $[a,b]$ and #-differentiable on the #-open interval (a,b) and $f(a) = f(b)$. Then $f'^{\#}(c) = 0$ for some $c \in (a,b)$.

Theorem 11.22. (Intermediate Value Theorem for #-Derivatives) Suppose that $f(x)$ is #-differentiable on $[a,b]$, $f'^{\#}(a) \neq f'^{\#}(b)$ and $f'^{\#}(a) < \mu < f'^{\#}(b)$. Then $f'^{\#}(c) = \mu$ for some $c \in (a,b)$.

Theorem 11.23. (Generalized Mean Value Theorem) If f and g are #-continuous on the #-closed interval $[a,b]$ and #-differentiable on the open interval (a,b), then

$$[g(b) - g(a)]f'^{\#}(c) = [f(b) - f(a)]g'^{\#}(c) \tag{11.38}$$

for some $c \in (a,b)$.

Theorem 11.24.(Mean Value Theorem) If f is #-continuous on the #-closed interval $[a,b]$ and #-differentiable on the #-open interval (a,b), then

$$f'^{\#}(c) = \frac{f(b) - f(a)}{b - a} \tag{11.39}$$

for some $c \in (a,b)$.

Theorem 11.25. If $f'^{\#}(x)$ for all $x \in (a,b)$, then f is constant on (a,b).

Theorem 11.26. If $f'^{\#}(x)$ exists for all $x \in (a,b)$ and does not change sign on (a,b), then $f(x)$ is monotonic on (a,b) increasing, nondecreasing, decreasing, or nonincreasing as: (i) $f'^{\#}(x) > 0$, (ii) $f'^{\#}(x) \geq 0$, (iii) $f'^{\#}(x) < 0$, (iv) $f'^{\#}(x) \leq 0$, respectively, for all $x \in (a,b)$.

Theorem 11.27. If $\left|f'^{\#}(x)\right| < M, a < x < b$ then

$$|f(x) - f(x')| \leq M|x - x'|, \tag{11.40}$$

where $x, x' \in (a,b)$.

Definition 11.18.A function that satisfies an inequality like (11.40) for all x and x' in an interval is said to satisfy a Lipschitz condition on the interval.

Theorem 11.28. (Generalized L'Hospital's Rule) Suppose that f and g are #-differentiable and $g'^{\#}$ has no zeros on (a,b). Let #-$\lim_{x\to_{\#} b-} f(x) = $ #-$\lim_{x\to_{\#} b-} g(x)$ or #-$\lim_{x\to_{\#} b-} f(x) = \pm\infty^{\#}$and #-$\lim_{x\to_{\#} b-} g(x) = \pm\infty^{\#}$and suppose that

$$\# - \lim_{x\to_{\#} b-} \frac{f'^{\#}(x)}{g'^{\#}(x)} = L, \tag{11.41}$$

where $L \in \mathbb{R}^{\#}_{c}$ or $L = \pm\infty^{\#}$. Then

$$\# - \lim_{x\to_{\#} b-} \frac{f(x)}{g(x)} = L, \tag{11.42}$$

As we saw above in Lemma 11.2 if f is #-differentiable at x_0; then

$$f(x) = f(x_0) + f'^{\#}(x_0)(x - x_0) + E(x)(x - x_0), \quad (11.43)$$

where $\#\text{-}\lim_{x \to_{\#} x_0} E(x) = 0.$ To generalize this result, we first restate it: the polynomial $P_1(x) = f(x_0) + f'^{\#}(x_0)(x - x_0)$ which is of degree ≤ 1 and satisfies $P_1(x_0) = f(x_0)$, $P_1'^{\#}(x) = f'^{\#}(x_0)$, approximates $f(x)$ so well near x_0 such that

$$\#\text{-}\lim_{x \to_{\#} x_0} \frac{f(x) - P_1(x)}{x - x_0} = 0. \quad (11.44)$$

Now suppose that f has n #-derivatives at x_0 and $P_n(x)$ is the polynomial of degree $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ such that

$$P_n^{(r)\#}(x_0) = f^{(r)\#}(x_0), 0 \leq r \leq n. \quad (11.45)$$

Since $P_n(x)$ is a polynomial of hyperfinite degree n, it can be written as

$$P_n(x) = Ext\text{-}\sum_{i=0}^{n} a_i (x - x_0)^i \quad (11.46)$$

where $a_0, \ldots, a_n \in \mathbb{R}_c^{\#}$ are constants. Differentiating (11.46) gives $P_n^{(r)\#}(x_0) = r! a_r$, $0 \leq r \leq n$, so (11.45) determines a_r uniquely as $a_r = f^{(r)\#}(x_0)/r!, 0 \leq r \leq n$. Therefore,

$$P_n(x) = Ext\text{-}\sum_{r=0}^{n} \frac{f^{(r)\#}(x_0)}{r!}. \quad (11.47)$$

We call $P_n(x)$ the n-th Taylor hyper polynomial of $f(x)$ about x_0

Theorem 11.29. If $f^{(n)\#}(x_0)$ exists for some hyper integer $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ and $P_n(x)$ is the n-th Taylor hyper polynomial of f about x_0, then

$$\#\text{-}\lim_{x \to_{\#} x_0} \frac{f(x) - P_n(x)}{(x - x_0)^n} = 0. \quad (11.48)$$

Theorem 11.30. (Generalized Taylor's Theorem) Suppose that $f^{(n+1)\#}(x)$ exists on an #-open interval I about x_0, and let $x \in I$. Then the remainder $R_n(x) = f(x) - P_n(x)$ can be written as

$$R_n(x) = \frac{f^{(n+1)\#}(c)}{(n+1)!}(x - x_0)^n, \quad (11.49)$$

where c depends upon x and is between x and x_0.

11.12. The Riemann integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

The Riemann integral is defined as #-limit of Riemann hyperfinite sums of functions with respect to tagged partitions of an interval $[a,b] \subset \mathbb{R}_c^{\#}$ A tagged hyperfinite partition P of a closed interval $[a,b]$ on the real line is a hyperfinite sequence

$$a = x_0 \leq t_1 \leq x_1 \leq t_2 \leq x_2 \leq \ldots \leq x_{n-1} \leq t_n \leq x_n = b, \quad (11.50)$$

where $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. This partitions the interval $[a,b]$ into n sub-intervals $[x_{i-1}, x_i]$ indexed by $i \in \mathbb{N}^{\#}$, each of which is "tagged" with a distinguished point $t_i \in [x_{i-1}, x_i]$. Thus, any set of $n + 1 \in \mathbb{N}^{\#} \backslash \mathbb{N}$ points satisfying (11.50) defines a partition P of $[a,b]$, which we denote by $P = \{x_0, x_1, \ldots, x_n\}$. A Riemann hyperfinite sum of a function f with respect to such a tagged hyperfinite partition is defined as

$$I_n = \sum_{i=1}^{n} f(t_i)\Delta_i, \quad (11.51)$$

where $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. thus each term of the sum (11.51) is the area of a rectangle with height equal to the function value at the distinguished point of the given sub-interval, and width the same as the width of sub-interval, $\Delta_i = x_i - x_{i-1}$. The mesh(P) of such a tagged partition is the width of the largest sub-interval formed by the partition, $\max_{i=1...n} \Delta_i$.

Definition 11.19. The Riemann integral of a function f over the interval $[a,b]$ is equal to I if for every $\varepsilon > 0, \varepsilon \approx 0$ there exists $\delta > 0, \delta \approx 0$ such that for any partition with distinguished points on $[a,b]$ whose mesh is less than δ.

Upper and Lower Integrals.

Definition 11.20. f is bounded (hyperbounded) on $[a,b]$ and $P = \{x_0, x_1, \ldots, x_n\}$ is a hyperfinite partition of $[a,b]$, let

$$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x) \qquad (11.52)$$

and

$$m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x) \qquad (11.53)$$

The upper external hyperfinite sum of f over P is

$$S(P) = Ext\text{-}\sum_{j=1}^{n} M_j(x_j - x_{j-1}) \qquad (11.54)$$

and the upper external integral of f over $[a,b]$, denoted by

$$Ext\text{-}\overline{\int_a^b} f(x) d^{\#}x \qquad (11.55)$$

is the infimum of all hyperfinite upper sums.

The lower external hyperfinite sum of f over P is

$$s(P) = Ext\text{-}\sum_{j=1}^{n} m_j(x_j - x_{j-1}) \qquad (11.56)$$

and the lower external integral of f over $[a,b]$, denoted by

$$Ext\text{-}\underline{\int_a^b} f(x) d^{\#}x. \qquad (11.57)$$

is the supremum of all lower hyperfinite sums. If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

$$m(b-a) \leq s(P) \leq S(P) \leq M(b-a) \qquad (11.58)$$

for every hyperfinite partition P; thus, the set of upper hyperfinite sums of f over all partitions P of $[a,b]$ is bounded, as is the set of lower hyperfinite sums. Therefore, Theorems 1.1.3 and 1.1.8 imply that: if the quantity (11.55) and (11.57) exist then both are unique, and satisfy the inequalities

$$m(b-a) \leq Ext\text{-}\overline{\int_a^b} f(x) d^{\#}x \leq M(b-a) \qquad (11.59)$$

and

$$m(b-a) \leq Ext\text{-}\underline{\int_a^b} f(x) d^{\#}x \leq M(b-a). \qquad (11.60)$$

Theorem 11.31. Let f be bounded on $[a,b]$, and let P be a hyperfinite partition of $[a,b]$. Then (i) The upper hyperfinite sum $S(P)$ of f over P is the supremum of the set of all hyperfinite Riemann sums of f over P.

(ii) The lower hyperfinite sum $s(P)$ of f over P is the infimum of the set of all hyperfinite Riemann sums of f over P.

Proof (a) If $P = \{x_0, x_1, \ldots, x_n\}$, then $S(P) = Ext\text{-}\sum_{j=1}^{n} M_j(x_j - x_{j-1})$ where $M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x)$.

An arbitrary hyperfinite Riemann sum of f over P is of the following form

$$\sigma = Ext\text{-}\sum_{j=1}^{n} f(c_j)(x_j - x_{j-1}), \tag{11.61}$$

where $x_{j-1} \leq c_j \leq x_j$. Since $f(c_j) \leq M_j$, it follows that $\sigma \leq S(P)$. Now let $\varepsilon > 0, \varepsilon \approx 0$ and choose $\bar{c}_j \in [x_{j-1}, x_j]$ so that

$$f(\bar{c}_j) > M_j - \frac{\varepsilon}{n(x_j - x_{j-1})}, \tag{11.62}$$

where $1 \leq j \leq n \in \mathbb{N}^{\#} \backslash \mathbb{N}$. The hyperfinite Riemann sum $\bar{\sigma}$ produced in this way is

$$\bar{\sigma} = Ext\text{-}\sum_{j=1}^{n} f(\bar{c}_j)(x_j - x_{j-1}) > Ext\text{-}\sum_{j=1}^{n} \left[M_j - \frac{\varepsilon}{n(x_j - x_{j-1})} \right](x_j - x_{j-1}) = S(P) - \varepsilon. \tag{11.63}$$

Now Theorem 1.1.3 implies that $S(P)$ is the supremum of the set of hyperfinite Riemann sums of f over P.

The Riemann–Stieltjes Integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

Definition 11.21. Let f and g be defined on $[a,b]$. We say that f is Riemann–Stieltjes integrable with respect to g on $[a,b]$, if there is a number $L \in \mathbb{R}_c^{\#}$ with the following property: For every $\varepsilon > 0, \varepsilon \approx 0$, there is a $\delta > 0, \delta \approx 0$ such that

$$\left| Ext\text{-}\sum_{j=1}^{n} f(c_j)[g(x_j) - g(x_{j-1})] - L \right| < \varepsilon \tag{11.64}$$

provided only that $P = \{x_0, x_1, \ldots, x_n\}, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ is a hyperfinite partition of $[a,b]$ such that $\|P\| < \delta$ and $x_{j-1} \leq c_j \leq x_j, j \in n$. In this case, we say that L is the external Riemann–Stieltjes integral of f with respect to g over $[a,b]$, and write

$$Ext\text{-}\int_a^b f(x) d^{\#} g(x) = L. \tag{11.65}$$

11.13 Existence of the integral of a $\mathbb{R}_c^{\#}$-valued external function $f(x)$.

Lemma 11.3 Suppose that

$$|f(x)| \leq M, a \leq x \leq b \tag{11.66}$$

and let P' be a hyperfinite partition of $[a,b]$ obtained by adding $r \in \mathbb{N}^{\#} \backslash \mathbb{N}$ points to a partition $P = \{x_0, x_1, \ldots, x_n\}, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ of $[a,b]$. Then

$$S(P) \geq S(P') \geq S(P) - 2Mr\|P\| \tag{11.67}$$

and

$$s(P) \leq s(P') \leq s(P) + 2Mr\|P\|. \tag{11.68}$$

Theorem 11.32. If $f(x)$ is bounded on $[a,b]$, then

$$Ext\text{-}\underline{\int}_a^b f(x)d^{\#}x \leq Ext\text{-}\overline{\int}_a^b f(x)d^{\#}x. \quad (11.69)$$

Theorem 11.33. If f is integrable on $[a,b]$, then

$$Ext\text{-}\underline{\int}_a^b f(x)d^{\#}x = Ext\text{-}\overline{\int}_a^b f(x)d^{\#}x = Ext\text{-}\int_a^b f(x)d^{\#}x. \quad (11.70)$$

Theorem 11.34.If f is bounded (or hyperbounded) on $[a,b]$ and

$$Ext\text{-}\underline{\int}_a^b f(x)d^{\#}x = Ext\text{-}\overline{\int}_a^b f(x)d^{\#}x = L, \quad (11.71)$$

then $f(x)$ is integrable on $[a,b]$ and

$$Ext\text{-}\int_a^b f(x)d^{\#}x = L. \quad (11.72)$$

Theorem 11.35.A bounded (hyperbounded) function f is integrable on $[a,b]$ if and only if

$$Ext\text{-}\underline{\int}_a^b f(x)d^{\#}x = Ext\text{-}\overline{\int}_a^b f(x)d^{\#}x. \quad (11.73)$$

Theorem 11.36.If f is bounded (hyperbounded) on $[a,b]$, then f is integrable on $[a,b]$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a partition P of $[a,b]$ for which

$$S(P) - s(P) < \varepsilon. \quad (11.74)$$

Theorem 11.37.If f is #-continuous on $[a,b]$, then f is integrable on $[a,b]$.
Theorem 11.38. If f is monotonic on $[a,b]$, then f is integrable on $[a,b]$.
Theorem 11.39.(a) If f and g are integrable on $[a,b]$, then so is $f+g$, and

$$Ext\text{-}\int_a^b [f(x)+g(x)]d^{\#}x = Ext\text{-}\int_a^b f(x)d^{\#}x + Ext\text{-}\int_a^b g(x)d^{\#}x \quad (11.75)$$

(b) If $f_i, 1 \leq i \leq n \in \mathbb{N}^{\#}$ are integrable on $[a,b]$, then so is $Ext\text{-}\sum_{j=1}^{n} f_i(x)$, and

$$Ext\text{-}\int_a^b \left[Ext\text{-}\sum_{j=1}^{n} f_i(x) \right] d^{\#}x. = Ext\text{-}\sum_{j=1}^{n} \left(Ext\text{-}\int_a^b f_i(x)d^{\#}x \right) \quad (11.76)$$

Theorem 11.40.(a) If f is integrable on $[a,b]$ and $c \in \mathbb{R}_c^{\#}$ is a constant, then cf is integrable on $[a,b]$ and

$$Ext\text{-}\int_a^b c\, f(x)d^{\#}x = c\left(Ext\text{-}\int_a^b f(x)d^{\#}x \right). \quad (11.77)$$

(b) If $f_i, 1 \leq i \leq n \in \mathbb{N}^{\#}$ are integrable on $[a,b]$ and $c_i \in \mathbb{R}_c^{\#}$ are constants, then

$$Ext\text{-}\int_a^b \left[Ext\text{-}\sum_{j=1}^{n} c_i\, f_i(x) \right] d^{\#}x. = Ext\text{-}\sum_{j=1}^{n} c_i \left(Ext\text{-}\int_a^b f_i(x)d^{\#}x \right). \quad (11.78)$$

Theorem 11.41. If f is and g integrable on $[a,b]$ and $f(x) \leq g(x)$ for $x \in [a,b]$,then

$$Ext\text{-}\int_a^b f(x)d^{\#}x \leq Ext\text{-}\int_a^b g(x)d^{\#}x. \qquad (11.79)$$

Theorem 11.42. If $f(x)$ is integrable on $[a,b]$, then so is $|f(x)|$, and

$$\left|Ext\text{-}\int_a^b f(x)d^{\#}x\right| \leq Ext\text{-}\int_a^b |f(x)|d^{\#}x. \qquad (11.80)$$

Theorem 11.43. If $f(x)$ and $g(x)$ are integrable on $[a,b]$, then so is the product $f(x)g(x)$.

Theorem 11.44.(First Mean Value Theorem for Integrals) Suppose that $u(x)$ is #-continuous and $v(x)$ is integrable and nonnegative on $[a,b]$. Then

$$Ext\text{-}\int_a^b f(x)v(x)d^{\#}x = v(c)\left(Ext\text{-}\int_a^b f(x)d^{\#}x\right) \qquad (11.81)$$

for some $c \in [a,b]$.

Theorem 11.45.If $f(x)$ is integrable on $[a,b]$ and $a \leq a_1 < b_1 \leq b,$ then $f(x)$ is integrable on $[a_1,b_1]$.

Theorem 11.46.If $f(x)$ is integrable on $[a,b]$ and $[b,c]$ then $f(x)$ is integrable on $[a,b]$ and

$$Ext\text{-}\int_a^c f(x)d^{\#}x = Ext\text{-}\int_a^b f(x)d^{\#}x + Ext\text{-}\int_b^c f(x)d^{\#}x. \qquad (11.82)$$

Theorem 11.47.If $f(x)$ is integrable on $[a,b]$ and $a \leq c \leq b,$ then the function $F(x)$ defined by

$$F(x) = Ext\text{-}\int_c^x f(t)d^{\#}t \qquad (11.83)$$

satisfies a Lipschitz condition on $[a,b]$, and is therefore #-continuous on $[a,b]$.

Theorem 11.48.If $f(x)$ is integrable on $[a,b]$ and $a \leq c \leq b,$ then $F(x) = Ext\text{-}\int_c^x f(t)d^{\#}t$ is #-differentiable at any point $x_0 \in [a,b]$, where $f(x)$ is #-continuous, with

$$F'^{\#}(x_0) = f(x_0). \qquad (11.84)$$

If $f(x)$ is #-continuous from the right at $a,$ then $F'^{\#}_{+}(a) = f(a)$. If $f(x)$ is #-continuous from the left at $b,$ then $F'^{\#}_{+}(b) = f(b)$.

Theorem 11.49.Suppose that $F(x)$ is #-continuous on the #-closed interval $[a,b]$ and #-differentiable on the #-open interval $(a,b),$ and $f(x)$ is integrable on $[a,b]$. Suppose also that $F'^{\#}(x) = f(x), a < x < b.$ Then

$$Ext\text{-}\int_a^b f(x)d^{\#}x = F(b) - F(a). \qquad (11.85)$$

If $f'^{\#}(x)$ is integrable on $[a,b]$, then

$$Ext\text{-}\int_a^b f'^{\#}(x)d^{\#}x = f(b) - f(a). \qquad (11.86)$$

Definition 11.22. A function $F(x)$ is an #-antiderivative of $f(x)$ on $[a,b]$ if $F(x)$ is #-continuous on $[a,b]$ and #-differentiable on $[a,b]$, with $F'^{\#}(x) = f(x), a < x < b$.
Theorem 11.50.If $F(x)$ is an #-antiderivative of $f(x)$ on $[a,b]$, then so is $F(x) + c$ for any constant c. Conversely,if $F_1(x)$ and $F_2(x)$ are #-antiderivatives of f on $[a,b]$, then $F_1(x) - F_2(x)$ is constant on $[a,b]$.
Theorem 11.51.(Fundamental Theorem of Calculus) If $f(x)$ is #-continuous on $[a,b]$, then $f(x)$ has an #-antiderivative on $[a,b]$. Moreover, if $F(x)$ is any #-antiderivative of f on $[a,b]$, then

$$Ext\text{-}\int_a^b f(x)d^{\#}x = F(b) - F(a). \qquad (11.87)$$

Theorem 11.52. (Integration by Parts) If $u'^{\#}(x)$ and $v'^{\#}(x)$ are integrable on $[a,b]$, then

$$Ext\text{-}\int_a^b u(x)v'^{\#}(x)d^{\#}x = u(x)v(x)|_a^b - Ext\text{-}\int_a^b u(x)'^{\#}v(x)d^{\#}x. \qquad (11.88)$$

Theorem 11.53.Suppose that the transformation $x = \varphi(t)$ maps the interval $c \leq t \leq d$ into the interval $a \leq x \leq b$, with $\varphi(c) = \alpha$ and $\varphi(c) = \beta$, and let $f(x)$ be #-continuous on $[a,b]$. Let $\varphi'^{\#}(t)$ be integrable on $[c,d]$. Then

$$Ext\text{-}\int_\alpha^\beta f(x)d^{\#}x = Ext\text{-}\int_c^d f(\varphi(t))\varphi'^{\#}(t)d^{\#}t. \qquad (11.89)$$

Theorem 11.54. Suppose that $\varphi'^{\#}(t)$ is integrable and $\varphi(t)$ is monotonic on $[c,d]$, and the transformation $x = \varphi(t)$ maps $[c,d]$ onto $[a,b]$. Let $f(x)$ be bounded (hyperbounded) on $[a,b]$. Then $g(t) = f(\varphi(t))\varphi'^{\#}(t)$ is integrable on $[c,d]$ if and only if $f(x)$ is integrable over $[a,b]$,and in this case

$$Ext\text{-}\int_a^b f(x)d^{\#}x = Ext\text{-}\int_c^d f(\varphi(t))\varphi'^{\#}(t)d^{\#}t. \qquad (11.90)$$

11.14.Improper integrals.

Definition 11.22. We say $f(x)$ is locally integrable on an interval I if $f(x)$ is integrable on every finite or hyperfinite #-closed subinterval of I.
Definition 11.23.If f is locally integrable on $[a,b]$, we define

$$Ext\text{-}\int_a^b f(x)d^{\#}x = \#\text{-}\lim_{c \to_{\#} b-}\left(Ext\text{-}\int_a^c f(x)d^{\#}x \right). \qquad (11.91)$$

Remark 11.3.The #-limit in (91.91) always exists if $[a,b]$ is finite or hyperfinite and f is locally integrable and bounded (hyperbounded) on $[a,b]$. In this case, Definitions 11.70 and 11.91 assign the same value to $Ext\text{-}\int_a^b f(x)d^{\#}x$ no matter how f

is defined. However, the #-limit may also exist in cases where $b = \infty^{\#}$ or $b < \infty^{\#}$ and f is hyper unbounded as x approaches b from the left.

Definition 11.24.In the cases mentioned above, Definition 11.91 assigns a value to an integral that does not exist in the sense of Definition 11.70, and $Ext\text{-}\int_a^b f(x)d^{\#}x$ is said to be an improper integral that #-converges to the #-limit in (11.91). We also say in this case that f is integrable on $[a,b]$ and that $Ext\text{-}\int_a^b f(x)d^{\#}x$ exists.

If the #-limit in (11.91) does not exist (finite or hyperfinite), we say that the improper integral $Ext\text{-}\int_a^b f(x)d^{\#}x$ #-diverges, and f is nonintegrable on $[a,b)$. In particular, if $\#\text{-}\lim_{c\to_{\#}b-}\left(Ext\text{-}\int_a^c f(x)d^{\#}x\right) = \pm\infty^{\#}$ we say that #-diverges to $\infty^{\#}$, and we write

$$Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#} \tag{11.92}$$

or

$$Ext\text{-}\int_a^b f(x)d^{\#}x = -\infty^{\#}, \tag{11.93}$$

whichever the case may be.Similar comments apply to the next two definitions.

Definition 11.25.If $f(x)$ is locally integrable on $(a,b]$, we define

$$Ext\text{-}\int_a^b f(x)d^{\#}x = \#\text{-}\lim_{c\to_{\#}a+}\left(Ext\text{-}\int_c^b f(x)d^{\#}x\right). \tag{11.94}$$

provided that the #-limit exists (finite or hyperfinite). To include the case where $a = -\infty^{\#}$, we adopt the convention that $-\infty^{\#} += -\infty^{\#}$.

Definition 11.26.If $f(x)$ is locally integrable on (a,b), we define

$$Ext\text{-}\int_a^b f(x)d^{\#}x = Ext\text{-}\int_a^{\alpha} f(x)d^{\#}x + Ext\text{-}\int_{\alpha}^b f(x)d^{\#}x, \tag{11.95}$$

where $a < \alpha < b$, provided that both improper integrals on the right exist i.e.,finite or hyperfinite.

Remark 11.4.Note that the existence and value of $Ext\text{-}\int_a^b f(x)d^{\#}x$ according to Definition 11.26 do not depend on the particular choice of $\alpha \in (a,b)$.

Remark 11.5.When we wish to distinguish between improper integrals and integrals in the sense of Definition 11.70, we will call the latter proper integrals.

Theorem 11.55.Suppose that $f_1,f_2,\ldots,f_n$ are locally integrable on $[a,b)$ and that $Ext\text{-}\int_a^b f_1(x)d^{\#}x,\ldots,Ext\text{-}\int_a^b f_n(x)d^{\#}x$ #-converge. Let $c_1,c_2,\ldots,c_n$ be constants.Then $Ext\text{-}\int_a^b\left(Ext\text{-}\sum_{i=1}^n c_if_i(x)\right)d^{\#}x$ #-converges and

$$Ext\text{-}\int_a^b\left(Ext\text{-}\sum_{i=1}^n c_if_i(x)\right)d^{\#}x = Ext\text{-}\sum_{i=1}^n c_i\left(Ext\text{-}\int_a^b f_i(x)d^{\#}x\right). \tag{11.96}$$

11.15.Improper integrals of nonnegative functions $f : D \to \mathbb{R}_c^{\#}$. Absolute Integrability.

Theorem 11.56. If $f(x)$ is nonnegative and locally integrable on $[a,b)$, then

$Ext\text{-}\int_a^b f(x)d^{\#}x$ converges if the function

$$F(x) = Ext\text{-}\int_a^x f(x)d^{\#}x \quad (11.97)$$

is bounded (hyperbounded) on $[a,b)$, and $Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#}$ if it is not.

Theorem 11.57.(Comparison Test) If f and g are locally integrable on $[a,b)$ and $0 \leq f(x) \leq g(x), 0 \leq x < b,$ then (a) $Ext\text{-}\int_a^b f(x)d^{\#}x < \infty^{\#}$ if $Ext\text{-}\int_a^b g(x)d^{\#}x < \infty^{\#}$ and (b) $Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#}$ if $Ext\text{-}\int_a^b g(x)d^{\#}x = \infty^{\#}$.

Theorem 11.58. Suppose that f and g are locally integrable on $[a,b),\ g(x) > 0$ and $f(x) \geq 0$ on some subinterval $[a_1,b) \subset [a,b)$, and

$$\#\text{-}\lim_{c \to_{\#} b-} \frac{f(x)}{g(x)} = M. \quad (11.98)$$

(a) If $0 < M < \infty^{\#},$ then $Ext\text{-}\int_a^b f(x)d^{\#}x$ and $Ext\text{-}\int_a^b g(x)d^{\#}x$ converge or diverge together.

(b) If $M = \infty^{\#}$ and $Ext\text{-}\int_a^b g(x)d^{\#}x = \infty^{\#}$, then $Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#}$.

(c) If $M = 0$ and $Ext\text{-}\int_a^b g(x)d^{\#}x < \infty^{\#}$, then $Ext\text{-}\int_a^b f(x)d^{\#}x < \infty^{\#}$.

Definition 11.27. We say that f is absolutely integrable on $[a,b)$ if f is locally inte grable on $[a,b)$ and $Ext\text{-}\int_a^b |f(x)|d^{\#}x < \infty^{\#}$. In this case we also say that $Ext\text{-}\int_a^b f(x)d^{\#}x$ #-converges absolutely or is absolutely #-convergent.

Theorem 11.59.If f is locally integrable on $[a,b)$ and $Ext\text{-}\int_a^b |f(x)|d^{\#}x < \infty^{\#},$ then $Ext\text{-}\int_a^b f(x)d^{\#}x$ #-converges: that is, an absolutely #-convergent integral is #-convergent.

Theorem 11.60. (Dirichlet's Test) Suppose that f is #-continuous and its #-antiderivative $F(x) = Ext\text{-}\int_a^x f(x)d^{\#}x$ is bounded (hyperbounded) on $[a,b)$.
Let $g'^{\#}$ be absolutely integrable on $[a,b)$, and suppose that

$$\#\text{-}\lim_{c \to_{\#} b-} g(x) = 0. \quad (11.99)$$

Then $Ext\text{-}\int_a^x f(x)g(x)d^{\#}x$ #-converges.

Theorem 11.61. Suppose that $u(x)$ is #-continuous on $[a,b)$ and $Ext\text{-}\int_a^x u(x)d^{\#}x$ #-diverges. Let $v(x)$ be positive and #-differentiable on $[a,b),$ and suppose that #-$\lim_{c \to_{\#} b-} v(x) = \infty^{\#}$ and $v'^{\#}/v^2$ is absolutely integrable on $[a,b)$. Then $Ext\text{-}\int_a^x u(x)v(x)d^{\#}x$ #-diverges.

Theorem 11.62.Suppose that $g(x)$ is monotonic on $[a,b)$ and $Ext\text{-}\int_a^b f(x)d^{\#}x = \infty^{\#}$.
Let $f(x)$ be locally integrable on $[a,b)$ and

$$Ext\text{-}\int_{x_j}^{x_{j+1}} |f(x)|d^{\#}x \geq \rho, j \geq 0 \quad (11.100)$$

for some positive ρ where $\{x_j\}_{j \in \mathbb{N}^{\#}}$ is an increasing hyper infinite sequence of points in $[a,b)$ such that #-$\lim_{j \to_{\#} \infty^{\#}} x_j = b$ and $x_{j+1}xj \leq M, j \geq 0$, for some $M.$ Then

$$Ext\text{-}\int_a^b |f(x)g(x)|d^{\#}x = \infty^{\#}. \quad (11.101)$$

11.16. Change of Variable in an Improper Integral

Theorem 11.63. Suppose that $\varphi(t)$ is monotonic and $\varphi'^{\#}(t)$ is locally integrable on either of the half-open intervals $I = [c,d)$ or $(c,d]$, and let $x = \varphi(t)$ map I onto either of the half-open intervals $J = [a,b)$ or $J = (a,b]$. Let f be locally integrable on J. Then the improper integrals

$$Ext\text{-}\int_a^b f(x)d^{\#}x \text{ and } Ext\text{-}\int_a^b f(\varphi)|\varphi'^{\#}(t)|d^{\#}t \qquad (11.102)$$

#-diverge or #-converge together, in the latter case to the same value. The same conclusion holds if $\varphi(t)$ and $\varphi'^{\#}(t)$ have the stated properties only on the #-open interval (a,b), the transformation $x = \varphi(t)$ maps (c,d) onto (a,b), and f is locally integrable on (a,b).

11.17. Generalized integrability criterion due to Lebesgue.

The main result of this section is an integrability criterion due to Lebesgue that does not require computation, but has to do with how badly #-discontinuous a function may be and still be integrable.

Definition 11.28. If $f(x)$ is bounded (hyperbounded) on $[a,b]$, the oscillation $W_f[a,b]$ of $f(x)$ on $[a,b]$ is defined by

$$W_f[a,b] = \sup_{a\leq x,x'\leq b} |f(x) - f(x')| \qquad (11.103)$$

which can also be written as

$$W_f[a,b] = \sup_{a\leq x\leq b} f(x) - \inf_{a\leq x\leq b} f(x). \qquad (11.104)$$

Definition 11.29. If $a < x < b$, the oscillation $w_f(x)$ of $f(x)$ at x is defined by

$$w_f(x) = \#\text{-}\lim_{h\to_{\#} 0+} W_f(x-h,x+h) \qquad (11.105)$$

The corresponding definitions for $x = a$ and $x = b$ are

$$w_f(a) = \#\text{-}\lim_{h\to_{\#} 0+} W_f(a,a+h) \text{ and } w_f(b) = \#\text{-}\lim_{h\to_{\#} 0+} W_f(b-h,b). \qquad (11.106)$$

Note that for a fixed $x \in (a,b)$, $W_f(x-h,x+h)$ is a nonnegative and nondecreasing function of h for $0 < h < \min\{x-a,b-x\}$, therefore, $w_f(x)$ exists and is nonnegative.

Theorem 11.64. Let f be defined on $[a,b]$. Then f is #-continuous at $x_0 \in [a,b]$ if and only if $w_f(x) = 0$; #-continuity at a or b means #-continuity from the right or left, respectively.

Definition 11.30. A subset S of the $\mathbb{R}_c^{\#}$ is of Lebesgue measure zero if for every $\varepsilon > 0, \varepsilon \approx 0$, there is a hyperfinite or hyper infinite sequence of open intervals $I_1, I_2, \ldots$ such that

$$S \subset \bigcup_j I_j \qquad (11.107)$$

and

$$Ext\text{-}\sum_{j=1}^{n} L(I_j) < \varepsilon, n \geq 1. \qquad (11.108)$$

Note that any subset of a set of Lebesgue measure zero is also of Lebesgue measure zero.

Example 1. Any hyperfinite set $S = \{x_i\}_{i\in\mathbf{n}}, \mathbf{n} \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is of Lebesgue measure zero, since we can choose #-open intervals $I_1, I_2, \ldots, I_{\mathbf{n}}$ such that $x_j \in I_j$ and $L(I_j) < \varepsilon/\mathbf{n}$, $1 \leq j \leq \mathbf{n}$.

Definition 11.31.An infinite set $S \subset \mathbb{R}_c^\#$ is hyper denumerable if its members can be listed in a hyper infinite sequence (that is, in a one-to-one correspondence with the positive hyper integers); thus, $S = \{x_i\}_{i \in \mathbb{N}^\#}$. An infinite set that does not have this property is hyper non hyper denumerable.

Example 2. Any denumerable set $S = \{x_i\}_{i \in \mathbb{N}^\#}$is of Lebesgue measure zero, since if $\varepsilon > 0, \varepsilon \approx 0$, it is possible to choose open intervals $I_1, I_2, \ldots,$ so that $x_j \in I_j$ and $L(I_j) < 2^{-j}\varepsilon, j \geq 1$. Then (11.108) holds since $Ext\text{-}\sum_{j=1}^{\mathbf{n}} 2^{-j} = 1 - 2^{-\mathbf{n}} < 1$.

Theorem 11.64. If $w_f(x) < \varepsilon, \varepsilon \approx 0$, for $a \leq x \leq b$, then there is a $\delta > 0, \delta \approx 0$ such that $W_f[a,b] < \varepsilon$, provided that $a_1, b_1 \subset [a,b]$ and $b_1 - a_1 < \delta$.

Theorem 11.65.Let f be bounded (hyperbounded) on $[a,b]$ and define $E_\rho = \{x \in [a,b] | w_f(x) > \rho\}$. Then E_ρ is #-closed; and f is integrable on $[a,b]$ if and only if for every pair of positive numbers ρ and δ, E_ρ can be covered by hyper finitely many open intervals $I_1, I_2, \ldots, I_{\mathbf{p}}, \mathbf{p} \in \mathbb{N}^\# \backslash \mathbb{N}$ such that

$$Ext\text{-}\sum_{j=1}^{\mathbf{p}} L(I_j) < \delta. \tag{11.109}$$

Theorem 11.66. A bounded (hyperbounded) function f is integrable on a finite or hyperfinite interval $[a,b]$ if and only if the set S of #-discontinuities of f in $[a,b]$ is of Lebesgue measure zero.

12.Hyper infinite external sequences and series

12.1.Hyper infinite external sequences

An hyper infinite sequence (or hypersequence) of $\mathbb{R}_c^\#$-real numbers is a $\mathbb{R}_c^\#$-valued function defined on a set of hyperintegers $\{n | n \in \mathbb{N}^\# \wedge n \geq k \in \mathbb{N}\}$. We call the values of the function the terms of the hypersequence. We denote a hypersequence by listing its terms in order; thus, $\{s_n\}_k^{\infty^\#} = \{s_k, s_{k+1}, \ldots\}$. We often write $\{s_n\}_{n \in \mathbb{N}^\#}$or simple $\{s_n\}$ for a shot.

Definition 12.1. A hyper infinite sequence $\{s_n\}_k^{\infty^\#}$ converges to a limit $s \in \mathbb{R}_c^\#$ if for every $\varepsilon \approx 0, \varepsilon > 0$ there is an hyperinteger $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that

$$|s_n - s| < \varepsilon \text{ if } n \geq N \tag{12.1}$$

In this case we say that $\{s_n\}$ is #-convergent and write

$$\#\text{- } \lim_{n \to_\# \infty^\#} s_n = s. \tag{12.2}$$

A hyper infinite sequence that does not #-converge diverges, or is #-divergent.

Theorem 12.1. The #-limit of a #-convergent hypersequence is unique:

Proof. Suppose that $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = s_1$ and $\#\text{-}\lim_{n \to_\# \infty^\#} s_n = s_2$. We must show that $s = s'$. Let $\varepsilon \approx 0, \varepsilon > 0$. From Definition 10.1, there are hyperintegers N_1 and N_2 such that $|s_n - s_1| < \varepsilon$ if $n \geq N_1$, and $|s_n - s_2| < \varepsilon$ if $n \geq N_2$. These inequalities both hold if $n \geq N = \max(N_1, N_2)$, which implies that: $|s_1 - s_2| < 2\varepsilon$. Since this inequality holds for every $\varepsilon \approx 0, \varepsilon > 0$ and $|s_1 - s_2|$ is independent of ε, we conclude that $|s_1 - s_2| = 0$; that is, $s_1 = s_2$.

Definition 12.2.A hypersequence $\{s_n\}$ is bounded above if there is a hyperreal number
$b \in \mathbb{R}_c^\#$ such that $s_n \le b$ for all $n \in \mathbb{N}^\#$; bounded below if there is a real number $a \in \mathbb{R}_c^\#$ such that $s_n \ge a$ for all $n \in \mathbb{N}^\#$;or bounded if there is a real number $r \in \mathbb{R}_c^\#$ such that $|s_n| \le r$ for all $n \in \mathbb{N}^\#$.

Theorem 12.2. Any #-convergent hypersequence $\{s_n\}$ is bounded or hyperbounded.

Proof. By taking $\varepsilon = 1$ in Eq.(12.1), we see that if $\#\text{-}\lim_{n\to_\# \infty^\#} s_n = s$, then there is an hyperinteger $N \in \mathbb{N}^\#\backslash\mathbb{N}$ such that $\left| s_n - s \right| < 1$ if $n \ge N$. Therefore,
$s_n = |(s_n - s) + s| \le |s_n - s| + |s| < 1 + |s|$ if $n \ge N$; and
$s_n \le \max\{(\max_{1\le i\le N-1}\{|s_0|, |s_1|, \ldots, |s_{N-1}|\}), 1 + |s|\}$ for all $n \in \mathbb{N}^\#$, so $\{s_n\}$ is bounded.

Definition 12.3.(Sequences Diverging to $\pm\infty^\#$). We say that

$$\#\text{-}\lim_{n\to_\# \infty^\#} s_n = +\infty^\#$$

if for any hyperreal number $a, s_n > a$ for any $n \ge N \in \mathbb{N}^\#\backslash\mathbb{N}$. Similarly,

$$\#\text{-}\lim_{n\to_\# \infty^\#} s_n = -\infty^\#$$

if for any hyperreal number $a, s_n < a$ for any $n \ge N \in \mathbb{N}^\#\backslash\mathbb{N}$. However, we do not regard $\{s_n\}$ as #-convergent unless $\#\text{-}\lim_{n\to_\# \infty^\#} s_n$
is finite or hyperfinite, as required by Definition 12.1. To emphasize this distinction, we say that $\{s_n\}$ diverges to $\infty^\#(-\infty^\#)$ if $\#\text{-}\lim_{n\to_\# \infty^\#} s_n = \infty^\#(-\infty^\#)$.

Theorem 12.3. Assume that a nonempty set $S \subset \mathbb{R}_c^\#$ of real $\mathbb{R}_c^\#$-numbers has a supremum $\sup(S)$, then $\sup S$ is the unique hyperreal number $\beta \in \mathbb{R}_c^\#$ such that
(a) $x \le \beta$ for all $x \in S$
(b) if $\varepsilon > 0, \varepsilon \approx 0$ (no matter how infinite small) there is an $x_0 \in S$ such that $x_0 > \beta - \varepsilon$.

Proof. We first show that $\beta = \sup S$ has properties (a) and (b). Since β is an upper bound of S, it must satisfy (a). Since any hyperreal number α less than β can be written as $\alpha = \beta - \varepsilon$ with $\varepsilon = \beta - \alpha > 0$, (b) is just another way of saying that no number less than β is an upper bound of S. Hence, $\beta = \sup S$ satisfies (a) and (b). Now we show that there cannot be more than one hyperreal number with properties (a) and (b).
Suppose that $\beta_1 < \beta_2$ and β_2 has property (b); thus, if $\varepsilon > 0$, there is an $x_0 \in S$ such that $x_0 > \beta_2 - \varepsilon$. Then, by taking $\varepsilon = \beta_2 - \beta_1$, we see that there is an $x_0 \in S$ such that $x_0 > \beta_2 - (\beta_2 - \beta_1) = \beta_1$, so β_1 cannot have property (a). Therefore, there cannot be more than one hyperreal number that satisfies both (a) and (b).

Definition 12.4. A hypersequence $\{s_n\}_{n\in\mathbb{N}^\#}$ is nondecreasing if $s_n \ge s_{n-1}$ for all $n \in \mathbb{N}^\#$, or nonincreasing if $s_n \le s_{n-1}$ for all $n \in \mathbb{N}^\#$. A monotonic hyper infinite sequence is a hyper infinite sequence that is either nonincreasing or nondecreasing. If $s_n > s_{n-1}$ for all $n \in \mathbb{N}^\#$, then $\{s_n\}_{n\in\mathbb{N}^\#}$ is increasing, while if $s_n < s_{n-1}$ for all $n \in \mathbb{N}^\#$, $\{s_n\}_{n\in\mathbb{N}^\#}$ is decreasing.

Theorem 12.4.(a) If $\{s_n\}_{n\in\mathbb{N}^\#}$ is nondecreasing and there exists $\sup\{s_n | n \in \mathbb{N}^\#\}$ then

$$\#\text{-}\lim_{n\to_\# \infty^\#} s_n = \sup\{s_n | n \in \mathbb{N}^\#\}.$$

(b) If $\{s_n\}_{n\in\mathbb{N}^\#}$ is nonincreasing and there exists $\inf\{s_n | n \in \mathbb{N}^\#\}$ then

$$\#\text{-}\lim_{n\to_\# \infty^\#} s_n = \inf\{s_n | n \in \mathbb{N}^\#\}.$$

Proof. (a) Let $\beta = \sup\{s_n | n \in \mathbb{N}^\#\}$. . If $\beta < +\infty^\#$, Theorem 12.3 implies that if $\varepsilon > 0$ then $\beta - \varepsilon < s_N \le \beta$ for some hyperinteger $N \in \mathbb{N}^\#\backslash\mathbb{N}$. Since $s_N \le s_n \le \beta$ if $n \ge N$, it follows

that $\beta - \varepsilon < s_n \leq \beta$ if $n \geq N$. This implies that $|s_n - \beta| < \varepsilon$ if $n \geq N$, so $\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_n = \beta$, by definition of the #-limit. If $\beta = +\infty^{\#}$
and b is any hyperreal number, then $s_N > b$ for some hyperinteger N. Then $s_n > b$ for $n \geq N$, so $\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_n = +\infty^{\#}$.

Theorem 12.5.(Generalized Nested Intervals Theorem)
Let $\{I_n\}_{n \in \mathbb{N}^{\#}} = \{[a_n, b_n]\}_{n \in \mathbb{N}^{\#}}, [a_n, b_n] \subset \mathbb{R}_c^{\#}$ be a hyper infinite sequence of #-closed intervals satisfying each of the following conditions:
(i) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots$,
(ii) $b_n - a_n \to_{\#} 0$ as $n \to \infty^{\#}$.
Then $\bigcap_{n=1}^{\infty^{\#}} I_n$ consists of exactly one hyperreal number $x \in \mathbb{R}_c^{\#}$. Moreover both hyper infinite sequences $\{a_n\}$ and $\{b_n\}$ #-converge to x.

Proof. See proof to Theorem 8.11.

Theorem 12.6.(Generalized Bolzano-Weierstrass Theorem) Every bounded (hyperbounded) hyper infinite sequence $\{s_n\}_{n \in \mathbb{N}^{\#}}$ has a #-convergent sub hyper infinite sequence.

Proof.Let $\{s_n\}_{n \in \mathbb{N}^{\#}}$ be a bounded hyper infinite sequence. Then, there exists an interval $[a_1, b_1]$ such that: (i) $a_1, b_1 \in \mathbb{Q}^{\#}$ and (ii) $a_1 \leq s_n \leq b_1$ for all $n \in \mathbb{N}^{\#}$.
Either $\left[a_1, \frac{a_1+b_1}{2}\right]$ or $\left[\frac{a_1+b_1}{2}, b_1\right]$ contains hyperinfinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$.
That is, there exists hyper infinitely many $n \in \mathbb{N}^{\#}$ such that a_n is in $\left[a_1, \frac{a_1+b_1}{2}\right]$, or there exists hyper infinitely many $n \in \mathbb{N}^{\#}$ such that a_n is in $\left[\frac{a_1+b_1}{2}, b_1\right]$.
If $\left[a_1, \frac{a_1+b_1}{2}\right]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$, let $[a_2, b_2] = \left[a_1, \frac{a_1+b_1}{2}\right]$. Otherwise, let $[a_2, b_2] = \left[\frac{a_1+b_1}{2}, b_1\right]$.
Either $\left[a_2, \frac{a_2+b_2}{2}\right]$ or $\left[\frac{a_2+b_2}{2}, b_2\right]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$.
If $\left[a_2, \frac{a_2+b_2}{2}\right]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$, let $[a_3, b_3] = \left[a_2, \frac{a_2+b_2}{2}\right]$. Otherwise, let $[a_3, b_3] = \left[\frac{a_2+b_2}{2}, b_2\right]$.
By hyper infinite induction, we can continue this construction and obtain a hyper infinite sequence of intervals $\{[a_n, b_n]\}_{n \in \mathbb{N}^{\#}}$ such that:
(i) for each $n \in \mathbb{N}^{\#}$, interval $[a_n, b_n]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$,
(ii) for each $n \in \mathbb{N}^{\#}, [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ and
(iii) for each $n \in \mathbb{N}^{\#}, b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n)$.
The nested intervals theorem implies that the intersection $\bigcap_{n \in \mathbb{N}^{\#}} [a_n, b_n]$ of all of the intervals $[a_n, b_n]$ is a single point s. We will now construct a sub hyper infinite sequence of
$\{s_n\}_{n \in \mathbb{N}^{\#}}$ which will #-converge to s.
Since $[a_1, b_1]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_1 \in \mathbb{N}^{\#}$ such that s_{k_1} is in $[a_1, b_1]$. Since $[a_2, b_2]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_2 \in \mathbb{N}^{\#}, k_2 > k_1$ such that s_{k_2} is in $[a_2, b_2]$. Since $[a_3, b_3]$ contains hyper infinitely many terms of $\{s_n\}_{n \in \mathbb{N}^{\#}}$, there exists $k_3 \in \mathbb{N}^{\#}, k_3 > k_2$ such that s_{k_3} is in $[a_3, b_3]$. Continuing this process by hyper infinite induction, we obtain a hyper infinite sequencev$\{s_{k_n}\}_{n \in \mathbb{N}^{\#}}$ such that $s_{k_n} \in [a_n, b_n]$ for each $n \in \mathbb{N}^{\#}$. The hypersequence $\{s_{k_n}\}_{n \in \mathbb{N}^{\#}}$ is a sub hyper infinite sequence of $\{s_n\}_{n \in \mathbb{N}^{\#}}$ since $k_{n+1} > k_n$ for each $n \in \mathbb{N}^{\#}$. Since $\#\text{-}\lim_{n \to \infty^{\#}} a_n = s$ and $\#\text{-}\lim_{n \to \infty^{\#}} b_n = s$ and $a_n \leq s_n \leq b_n$ for each $n \in \mathbb{N}^{\#}$, the squeeze theorem implies that that $\#\text{-}\lim_{n \to \infty^{\#}} s_n = s$.

12.2.Hyper infinite external series of constant.

Definition 12.5. If $\{a_n\}_k^{\infty^\#}$ is an hyper infinite external sequence of Cauchy hyperreal numbers, the symbol

$$Ext\text{-}\sum_{n=k}^{\infty^\#} a_n \tag{12.3}$$

is an hyper infinite series, and a_n is the n-th term of the hyper infinite series.

We say that $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$ #-converges to the sum $A \in \mathbb{R}_c^\#$, and write

$$Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = A \tag{12.4}$$

if the hyper infinite sequence $\{A_n\}_k^{\infty^\#}$ defined by

$$A_n = Ext\text{-}\sum_{i=k}^{i=n} a_n \tag{12.5}$$

$n \in \mathbb{N}^\#$, #-converges to A. The hyperf inite sum A_n is the n-th partial sum of $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$

If $\{A_n\}_k^{\infty^\#}$ diverges, we say that $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$ diverges; in particular, if $\mathbf{lim}_{n \to_\# \infty^\#} A_n = \infty^\#$ or $-\infty^\#$, we say that $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$ #-diverges to $\infty^\#$ or $-\infty^\#$, and write

$$Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = \infty^\# \text{ or } Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = -\infty^\#. \tag{12.6}$$

A divergent hyperinfinite series that does not diverge to $\pm\infty^\#$ is said to oscillate, or be oscillatory.

Example 12.1 Consider the hyper infinite series

$$Ext\text{-}\sum_{n=0}^{\infty^\#} r^n, -1 < r < 1. \tag{12.7}$$

Here $a_n = r^n, n \geq 0, n \in \mathbb{N}^\#$ and

$$A_n = 1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1 - r} \tag{12.8}$$

which #-converges to $1 = 1/(1-r)$ as $n \to \infty^\#$; thus, we write

$$Ext\text{-}\sum_{n=0}^{\infty^\#} r^n = 1/(1-r), -1 < r < 1.$$

An hyperinfinite series can be viewed as a generalization of a gyperfinite sum $A_N = Ext\text{-}\sum_{n=k}^{N} a_n$ Therefore, #-$\mathbf{lim}_{N \to \infty^\#} A_N = A$.

Theorem 12.7. The sum of a #-convergent hyper infinite series is unique:

Theorem 12.8. Let $\sum_{n=k}^{\infty^{\#}} a_n = A$ and $\sum_{n=k}^{\infty^{\#}} b_n = B$ where A and B are finite or hyperfinite.

Then

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} (a_n \pm b_n) = A \pm B \tag{12.9}$$

and

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} (c \times a_n) = c \times A \tag{12.10}$$

if $c \in \mathbb{R}_c^{\#}$ is a constant.

Theorem 12.9. (Cauchy's #-convergence criterion for hyper infinite series) A hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ #-converges if and only if for every $\varepsilon > 0, \varepsilon \approx 0$ there is an gyperinteger $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$ such that

$$\left| Ext\text{-}\sum_{n}^{m} a_n \right| < \varepsilon \tag{12.11}$$

if $m \geq n \geq N$.

Corollary 12.1. If $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ #-converges; then #-$\mathbf{lim}_{N \to \infty^{\#}} a_n = 0$.

Corollary 12.2. If $Ext\text{-}\sum_{n}^{\infty^{\#}} a_n$ #-converges; then for each $\varepsilon > 0, \varepsilon \approx 0$ there is an gyperinteger $K \in \mathbb{N}^{\#} \backslash \mathbb{N}$ such that $\left| Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n \right| < \varepsilon$ if $k \geq K$, that is

$$\#\text{-}\lim_{k \to \infty^{\#}} \left(Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n \right) = 0. \tag{12.12}$$

12.3.Hyper Infinite Series of Nonnegative Terms.

The theory of series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ with terms that are nonnegative for sufficiently large $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ is simpler than the general theory, since such a series either #-converges to a finite or hyperfinite #-limit or diverges to $\infty^{\#}$, as the next theorem shows.

Theorem 12.10.If $a_n \geq 0$ for $n \geq k$, then $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ #-converges if its partial sums are bounded or hyper bounded, or #-diverges to $\infty^{\#}$ if they are not. These are the only possibilities and, in either case, $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \{A_n | n \geq k\}$, where $A_n = Ext\text{-}\sum_{i=k}^{n} a_i$.

Theorem 12.11.(The Comparison Test) Suppose that

$$0 \leq a_n \leq b_n, n \geq k. \tag{12.13}$$

Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n < \infty^{\#}$. (b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n = \infty^{\#}$.

Theorem 12.12.(The Integral Test) Let

$$c_n = f(n), n \geq k, \tag{12.14}$$

where f is positive; nonincreasing; and locally #-integrable on $[k, \infty^{\#})$. Then

$$Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#} \tag{12.15}$$

if and only if

$$Ext\text{-}\int_{k}^{\infty^{\#}} f(x)d^{\#}x < \infty^{\#}. \tag{12.16}$$

Example 12.2. The integral test implies that the hyper infinnite series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} n^{-p}$ converge if $p > 1$ and diverge if $0 < p \leq 1,$ because the same is true of the integral $Ext\text{-}\int_{a}^{\infty^{\#}} x^{-p}d^{\#}x, a > 1.$

The next theorem is often applicable where the integral test is not.

Theorem 12.13.Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k.$ Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n < \infty^{\#}$ and $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_n}{b_n} < \infty^{\#}.$

(b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n = \infty^{\#}$ and $\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_n}{b_n} > 0.$

Corollary 12.3. Suppose that $a_n \geq 0$ and $b_n > 0$ for $n \geq k$, and $\#\text{-}\lim_{n\to\infty^{\#}} \frac{a_n}{b_n} = L.$ where $0 < L < \infty^{\#}$.Then $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n$ #-converge or #-diverge together.

Theorem 12.14.Suppose that $a_n > 0, b_n > 0,$ and

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}. \tag{12.17}$$

Then (a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n < \infty^{\#}$.(b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $Ext\text{-}\sum_{n=k}^{\infty^{\#}} b_n = \infty^{\#}$.

Theorem 12.15.(The Ratio Test) Suppose that $a_n > 0$ for $n \geq k.$ Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}$ if $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} < 1.$(b) $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = \infty^{\#}$ if $\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} > 1.$If

$$\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} \leq 1 \leq \overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} \tag{12.18}$$

then the test is inconclusive; that is, $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n$ may #-converge or #-diverge.

Proof.(a) If $\overline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} < 1,$ there is a number r such that $0 < r < 1$ and $\frac{a_{n+1}}{a_n} < r$ for $n \in \mathbb{N}^{\#}$ sufficiently large. This can be rewritten as $\frac{a_{n+1}}{a_n} < \frac{r^{n+1}}{r^n}$

Since $Ext\text{-}\sum_{n=k}^{\infty^{\#}} r^n < \infty^{\#}$Theorem 12.14 (a) with $b_n = r^n$ implies that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n < \infty^{\#}.$

(b) If $\underline{\#\text{-}\lim}_{n\to\infty^{\#}} \frac{a_{n+1}}{a_n} > 1,$there is a number r such that $r > 1$ and $\frac{a_{n+1}}{a_n} > r$ for $n \in \mathbb{N}^{\#}$ sufficiently large. This can be rewritten as $\frac{a_{n+1}}{a_n} > \frac{r^{n+1}}{r^n}$.Since

$Ext\text{-}\sum_{n=k}^{\infty^\#} r^n = \infty^\#$Theorem 12.14 (b) with $b_n = r^n$ implies that $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = \infty^\#$.

To see that no conclusion can be drawn if (12.18) holds,consider hyper infinite series

$$Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = Ext\text{-}\sum_{n=k}^{\infty^\#} n^{-p}. \tag{12.19}$$

This series #-converges if $p > 1$ or #-diverges if $p \leq 1$, however,

$$\underline{\#\text{-}\lim}_{n\to\infty^\#} \frac{a_{n+1}}{a_n} = \overline{\#\text{-}\lim}_{n\to\infty^\#} \frac{a_{n+1}}{a_n} = 1. \tag{12.20}$$

Corollary 12.4.Suppose that $a_n > 0$ for $n \geq k$ and $\#\text{-}\lim_{n\to\infty^\#} \frac{a_{n+1}}{a_n} = L$. Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n < \infty^\#$ if $L < 1$. (b) $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = \infty^\#$ if $L > 1$.

The test is inconclusive if $L = 1$.

Theorem 12.16.(Generalized Raabe's Test) Suppose that $a_n > 0$ for large $n \in \mathbb{N}^\# \backslash \mathbb{N}$.

Let $M = \overline{\#\text{-}\lim}_{n\to\infty^\#}\left(\frac{a_{n+1}}{a_n} - 1\right)$ and $m = \underline{\#\text{-}\lim}_{n\to\infty^\#}\left(\frac{a_{n+1}}{a_n} - 1\right)$. Then

(a) $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n < \infty^\#$ if $M < -1$. (b) $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = \infty^\#$ if $m > -1$.

The test is inconclusive if $m \leq -1 \leq M$.

Theorem 12.17.(Generalized Cauchy's Root Test)bSuppose that $a_n \geq 0$ for $n \geq k \in \mathbb{N}^\# \backslash \mathbb{N}$, then

(a) $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n < \infty^\#$ if $\overline{\#\text{-}\lim}_{n\to\infty^\#} \sqrt[n]{a_n} < 1$. (b) $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n = \infty^\#$ if $\overline{\#\text{-}\lim}_{n\to\infty^\#} \sqrt[n]{a_n} > 1$.

The test is inconclusive if $\overline{\#\text{-}\lim}_{n\to\infty^\#} \sqrt[n]{a_n} = 1$.

12.4.Absolute and Conditional #-Convergence.

Definition 12.6.A series $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$ #-converges absolutely, or is absolutely #-convergent if $Ext\text{-}\sum_{n=k}^{\infty^\#} |a_n| < \infty^\#$.

Theorem 12.18. If $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$ #-converges absolutely; then $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n$ #-converges.

Theorem 12.19. (Dirichlet's Test for Hyper Infinite Series) The hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$ is #-converges if the following conditions are satisfied

(i) $\#\text{-}\lim_{n\to\infty^\#} a_n = 0$,

(ii)

$$Ext\text{-}\sum_{n=k}^{\infty^\#} |a_{n+1} - a_n| < \infty^\# \tag{12.21}$$

and

(iii) for all $n \geq k$

$$Ext\text{-}\sum_{i=k}^{n} b_n \leq M \tag{12.22}$$

for some constant M.

Proof. Let $B_n, n \geq k$ be the partial sum

$$B_n = Ext\text{-}\sum_{i=k}^{n} b_n \tag{12.23}$$

Let us consider the partial sums $S_n, n \geq k$ of $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$, where

$$S_n = Ext\text{-}\sum_{i=k}^{n} a_n b_n \tag{12.24}$$

By substituting $b_k = B_k$ and $b_n = B_n - B_{n-1}, n \geq k+1$, into (12.24), we obtain

$$S_n = a_k b_k + Ext\text{-}\sum_{i=k+1}^{n} a_i (B_i - B_{i-1}), \tag{12.25}$$

which we rewrite as

$$S_n = a_n B_n + Ext\text{-}\sum_{i=k}^{n-1} (a_i - a_{i+1}) B_i. \tag{12.26}$$

Now (12.26) can be viewed as

$$S_n = T_{n-1} + a_n B_n, \tag{12.27}$$

where $T_{n-1} = Ext\text{-}\sum_{i=k}^{n-1} (a_i - a_{i+1}) B_i$; that is, $\{T_n\}$ is the hyper infinite sequence of partial sums of the hyper infinite series

$$Ext\text{-}\sum_{i=k}^{\infty^\#} (a_i - a_{i+1}) B_i. \tag{12.28}$$

Since $|(a_i - a_{i+1}) B_i| \leq M |a_i - a_{i+1}|$ from (12.22), the comparison test and (12.21) imply that the series (12.28) #-converges absolutely. Theorem 12.18 now implies that $\{T_n\}_{n \in \mathbb{N}^\#}$ #-converges. Let $T = \#\text{-}\lim_{n \to \infty^\#} T_n$. Since B_n is bounded (hyperbounded) and $\#\text{-}\lim_{n \to \infty^\#} a_n = 0$, we infer from (12.27) that

$$\#\text{-}\lim_{n \to \infty^\#} S_n = \#\text{-}\lim_{n \to \infty^\#} T_{n-1} + \#\text{-}\lim_{n \to \infty^\#} a_n B_n = T. \tag{12.29}$$

Therefore, $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$ is #-converges.

Corollary 12.4.(Abel’s Test for Hyper Infinite Series) The series $Ext\text{-}\sum_{n=k}^{\infty^\#} a_n b_n$ #-converges if $a_{n+1} \leq a_n$ for $n \geq k$, $\#\text{-}\lim_{n \to \infty^\#} a_n = 0$ and $Ext\text{-}\sum_{i=k}^{n} b_n \leq M$, for some constant M.

Corollary 12.5.(Alternating Hyper Infinite Series Test) The series $Ext\text{-}\sum_{n=0}^{\infty^{\#}}(-1)^n a_n$

#-converges if $0 \leq a_{n+1} \leq a_n$ and #-$\lim_{n\to\infty^{\#}} a_n = 0$.

Proof. Let $b_n = (-1)^n$, then $\{|B_n|\}_{n\in\mathbb{N}^{\#}}$ is a hyper infinite sequence of zeros and ones and therefore bounded. The conclusion now follows from Abel's test.

12.5.Grouping Terms in a Hyper Infinite Series.

The terms of a hyper finite sum can be grouped arbitrarily by it hyper finite (but not by countable set of it finite subsets) subsets by inserting corresponding parentheses, see Appendix C. According to the next theorem, the same is true of an hyper infinite series that #-converges or #-diverges to $\pm\infty^{\#}$.

Theorem 12.20. Suppose that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} a_n = A$, where $-\infty^{\#} \leq A \leq \infty^{\#}$. Let $\{n_j\}_{n\in\mathbb{N}^{\#}}$ be an increasing hyper infite sequence of integers, with $n_1 \geq k$. Define

$$\begin{aligned} b_1 &= Ext\text{-}\sum_{n=k}^{n_1} a_n, \\ b_2 &= Ext\text{-}\sum_{n=n_1+1}^{n_2} a_n, \\ &\cdots\cdots\cdots\cdots \\ b_r &= Ext\text{-}\sum_{n=n_{r-1}+1}^{n_r} a_n \end{aligned} \tag{12.30}$$

Then

$$Ext\text{-}\sum_{j=1}^{\infty^{\#}} b_{n_j} = A. \tag{12.31}$$

12.6.Rearrangement of hyper infite series.

A hyperfinite sum is not changed by rearranging its terms ,see Appendix C. According to the next theorem, we see that every rearrangement of an absolutely #-convergent hyper infite series has the same sum, but that conditionally #-convergent series fail, spectacularly, to have this property.

Theorem 12.21. If $Ext\text{-}\sum_{n=1}^{\infty^{\#}} b_n$ is a rearrangement of an absolutely #-convergent series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ then $Ext\text{-}\sum_{n=1}^{\infty^{\#}} b_n$ also #-converges absolutely, and to the same sum.

Theorem 12.22. If $\{a_{n_i}\}_{i\in\mathbb{N}^{\#}}$ and $\{a_{m_j}\}_{j\in\mathbb{N}^{\#}}$ are respectively the subsequences of all positive and negative terms in a conditionally #-convergent series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$

then

$$Ext\text{-}\sum_{i=1}^{\infty^{\#}} a_{n_i} = \infty^{\#} \text{ and } Ext\text{-}\sum_{j=1}^{\infty^{\#}} a_{m_j} = -\infty^{\#}. \tag{12.32}$$

Theorem 12.23.Suppose that $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ is conditionally #-convergent and μ and ν are arbitrarily given in the extended hyperreals; with $\mu \leq \nu$. Then the terms of $Ext\text{-}\sum_{n=1}^{\infty^{\#}} a_n$ can be rearranged to form a series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} b_n$ with partial sums $B_n = Ext\text{-}\sum_{i=1}^{n} b_i$ such that

$$\overline{\lim_{n \to_{\#} \infty^{\#}}} B_n = \nu \text{ and } \underline{\lim_{n \to_{\#} \infty^{\#}}} B_n = \mu. \tag{12.33}$$

12.7.Multiplication of hyper infite Series.

Given two hyper infite series $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n$ we can arrange all possible products $a_i b_j,\ i,j \geq 0$ in a two-dimensional array:

$$\begin{array}{ccccc} a_0b_0 & a_0b_1 & a_0b_2 & a_0b_3 & \cdots \\ a_1b_0 & a_1b_1 & a_1b_2 & a_1b_3 & \cdots \\ a_2b_0 & a_2b_1 & a_2b_2 & a_2b_3 & \cdots \\ a_3b_0 & a_3b_1 & a_3b_2 & a_3b_3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \end{array} \tag{12.34}$$

where the subscript on a is constant in each row and the subscript on b is constant in each column. Any sensible definition of the product

$$\left(Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n\right)\left(Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n\right) \tag{12.35}$$

clearly must involve every product in this array exactly once; thus, we might define the product of the two series to be the series $Ext\text{-}\sum_{n=0}^{\infty^{\#}} p_n$, where $\{p_n\}_{i\in\mathbb{N}^{\#}}$ is a hyper infite sequence obtained by ordering the products in (12.34) according to some method that chooses every product exactly once.

Theorem 12.24.Let $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n = A$ and $Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n = B,$where A and B are finite or hyperfinite, and at least one term of each series is nonzero. Then $Ext\text{-}\sum_{n=0}^{\infty^{\#}} p_n = A \times B$ for every hyper infinite sequence $\{p_n\}_{i\in\mathbb{N}^{\#}}$ obtained by ordering the products in

(12.34) if and only if $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n$ #-converge absolutely:

Moreover, in this case, $Ext\text{-}\sum_{n=0}^{\infty^{\#}} p_n$ #-converges absolutely.

Definition 12.7.The Cauchy product of $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n$ is $Ext\text{-}\sum_{n=0}^{\infty^{\#}} c_n$, where

$$c_n = Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j}. \quad (12.36)$$

Thus, c_n is the external sum of all products $a_l b_k$, where $i \geq 0, j \geq 0$, and $i + j = n$; thus,

$$c_n = Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j} = Ext\text{-}\sum_{j=0}^{n} b_j a_{n-j}. \quad (12.37)$$

Theorem 12.25. If $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n$ and $Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n$ #-converge absolutely to sums A and B, then the Cauchy product $Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j}$ #-converges absolutely to AB.

Theorem 12.26. Let $f(\alpha) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{\alpha^n}{n!}$ and $f(\beta) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{\beta^n}{n!}$, then

$$f(\alpha)f(\beta) = f(\alpha + \beta). \quad (12.38)$$

Proof. From Eq.(12.37) we obtain

$$c_n = Ext\text{-}\sum_{n=0}^{m} \frac{\alpha^{n-m}\beta^m}{(n-m)!m!} = \frac{1}{n!}\left(Ext\text{-}\sum_{n=0}^{m} \binom{n}{m} \alpha^{n-m}\beta^m\right) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{(\alpha+\beta)^n}{n!} \quad (12.39)$$

Thus

$$f(\alpha)f(\beta) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{(\alpha+\beta)^n}{n!} = f(\alpha + \beta). \quad (12.40)$$

12.8.Double Hyper Infinite Sequences.

Definition 12.8. A double hyper infinite sequence of hyperreal numbers $\mathbb{R}_c^{\#}$ (complex numbers $\mathbb{C}_c^{\#} = \mathbb{R}_c^{\#} + i\mathbb{R}_c^{\#}$) is a $\mathbb{R}_c^{\#}$-valued ($\mathbb{C}_c^{\#}$-valued) function $s : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{R}_c^{\#}$ or $s : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{C}_c^{\#}$. We shall use the notation $\{s_{n,m}\}_{n,m \in \mathbb{N}^{\#}}$ or simply s_{nm}.

Definition 12.9. We say that a double hyper infinite sequence $s_{n,m}$ #-converges to $a \in \mathbb{C}_c^{\#}$ and we write $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m} = a$, if the following condition is satisfied: for every $\varepsilon > 0, \varepsilon \approx 0$, there exists $N \in \mathbb{N}^{\#}$ such that $|s_{n,m} - a| < \varepsilon$ if $n, m \geq N$.

Theorem 12.27. (Uniqueness of Double #-Limits). A double hyper infinite $\mathbb{C}_c^{\#}$-valued sequence has at most one #-limit.

Definition 12.10. A double hyper infinite sequence $s_{n,m}$ is called bounded (hyper bounded) if there exists finite (hyperfinite) number $M \in \mathbb{R}_c^{\#}$, $M > 0$ such that $|s_{n,m}| \leq M, \forall n, m \in \mathbb{N}^{\#}$.

Theorem12.28. A #-convergent double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence is bounded or hyper bounded.

Definition 12.11.A double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence $s_{n,m}$ is called a Cauchy sequence if and only if for every $\varepsilon > 0, \varepsilon \approx 0,$ there exists a hypernatural number $N \in \mathbb{N}^{\#} \backslash \mathbb{N}$ such that $|s_{p,q} - s_{n,m}| < \varepsilon,\ \forall p(p \geq n \geq N)$ and $\forall q(q \geq m \geq N)$.

Theorem 12.28.(Cauchy Convergence Criterion for Double hyper infinite Sequences). A double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence $s_{n,m}, n, m \in \mathbb{N}^{\#}$ #-converges if and only if it is a Cauchy sequence.

Definition 12.12.Let $s_{n,m}$ be a double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence.

(i) If $s_{n,m} \leq s_{j,k},\ \forall n \forall j \forall m \forall k(n \leq j \wedge m \leq k),\ n,m,j,k \in \mathbb{N}^{\#},$ we say the sequence $s_{n,m}$ is increasing.

(ii) $s_{n,m} \geq s_{j,k},\ \forall n \forall j \forall m \forall k(n \leq j \wedge m \leq k),\ n,m,j,k \in \mathbb{N}^{\#},$ we say the sequence $s_{n,m}$ is decreasing.

(ii) If $s_{n,m}$ is either increasing or decreasing, then we say it is monotone.

Definition 12.13.For a double sequence $s_{n,m}$, the #-limits

$\#\text{-}\lim_{n \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{m \to_{\#} \infty^{\#}} s_{n,m} \right)$

and $\#\text{-}\lim_{m \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m} \right)$ are called repeated #-limits.

Theorem 12.29.Let $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m} = a.$ Then $\#\text{-}\lim_{m \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m} \right) = a$ if and only if $\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m}$ exists for each $m \in \mathbb{N}^{\#}$.

Theorem 12.30. Let $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m} = a.$ Then the repeated #-limits $\#\text{-}\lim_{n \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{m \to_{\#} \infty^{\#}} s_{n,m} \right)$ and $\#\text{-}\lim_{m \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m} \right)$ exist and both are equal to a if and only if (i) $\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m}$ exists for each $m \in \mathbb{N}^{\#}$, and (ii) $\#\text{-}\lim_{m \to_{\#} \infty^{\#}} s_{n,m}$ exists for each $n \in \mathbb{N}^{\#}$.

Theorem 12.31.If $s_{n,m}$ is a double sequence such that the repeated #-limit $\#\text{-}\lim_{m \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m} \right) = a$ and the #-limit $\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m}$ exists uniformly in $m \in \mathbb{N}^{\#}$, then the double #-limit $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m} = a$.

Theorem 12.32.(Monotone Convergence Theorem). A monotone double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence is #-convergent if and only if it is bounded (hyper bounded). Further: (i) If $s_{n,m}$ is increasing and bounded (hyper bounded) above, then

$\#\text{-}\lim_{m \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m} \right) = \#\text{-}\lim_{n \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{m \to_{\#} \infty^{\#}} s_{n,m} \right) = \#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m}.$

(ii) If $s_{n,m}$ is decreasing and bounded (hyper bounded) below, then

$\#\text{-}\lim_{m \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{n \to_{\#} \infty^{\#}} s_{n,m} \right) = \#\text{-}\lim_{n \to_{\#} \infty^{\#}} \left(\#\text{-}\lim_{m \to_{\#} \infty^{\#}} s_{n,m} \right) = \#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m}.$

Theorem12.33.(The Sandwich Theorem). Suppose that $x_{n,m}, s_{n,m},$ and $y_{n,m}$ are double $\mathbb{R}_c^{\#}$-valued hyper infinite sequences such that $x_{n,m} \leq s_{n,m} \leq y_{n,m}, \forall n, m \in \mathbb{N}^{\#}$, and $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} x_{n,m} = \#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} y_{n,m}.$

Then $s_{n,m}$ is #-convergent and $\#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} x_{n,m} = \#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} y_{n,m} = \#\text{-}\lim_{n,m \to_{\#} \infty^{\#}} s_{n,m}.$

Definition 12.14. Let $s_{n,m}$ be a double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence and let $(k_1, r_1) < (k_2, r_2) < \ldots < (k_n, r_n) < \ldots$ be a strictly increasing sequences of pairs of hypernatural numbers. Then the sequence s_{k_n, r_m} is called a subsequence of $s_{n,m}$.

Theorem12.34.If a double $\mathbb{C}_c^{\#}$-valued hyper infinite sequence $s_{n,m}$ #-converges to number $a \in \mathbb{C}_c^{\#}$, then any hyper infinite subsequence of $s_{n,m}$ also #-converges to a.

Theorem12.35.If the repeated #-limits of a double sequence $s_{n,m}$ exist and satisfy $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{n,m}\right) = \#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{n,m}\right) = a,$ then the

repeted #-limits for any subsequence s_{p_n,q_m} exist and satisfy
$\#\text{-}\lim_{m\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{n\to_{\#}\infty^{\#}} s_{p_n,q_m}\right) = \#\text{-}\lim_{n\to_{\#}\infty^{\#}}\left(\#\text{-}\lim_{m\to_{\#}\infty^{\#}} s_{p_n,q_m}\right) = a.$

Theorem12.36. Every double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence has a monotone hyper infinite subsequence.

Theorem12.37.(Bolzano-Weierstrass Theorem). A bounded (hyper bounded) double $\mathbb{R}_c^{\#}$-valued hyper infinite sequence sequence has a #-convergent monotone subsequence.

12.9.External Double Hyper Infinite Series.

Definition 12.15. Let $z : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{C}_c^{\#}$ be external hyper infinite double sequence of complex numbers $\mathbb{C}_c^{\#}$ and let $s_{n,m}$ be the double hyper infinite sequence defined by the equation

$$s_{n,m} = Ext\text{-}\sum_{i=1}^{n}\left(Ext\text{-}\sum_{j=1}^{m} z_{i,j}\right). \tag{12.41}$$

The pair (z,s) is called a double hyper infinite series and is denoted by the symbol

$$Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m} \tag{12.42}$$

or, more briefly by $Ext\text{-}\sum_{n,m=1}^{\infty^{\#}} z_{n,m}$. Each number $z_{n,m}$ is called a term of the double series and each $s_{n,m}$ is called a partial sum.

Definition 12.16.We say that the double series $Ext\text{-}\sum_{n,m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent to the sum s if $\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} s_{n,m} = s.$ If no such #-limit exists, we say that the double series $Ext\text{-}\sum_{n,m=1}^{\infty^{\#}} z_{n,m}$ is #-divergent.

Definition 12.17. The hyper infinite series

$$Ext\text{-}\sum_{n=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{m=1}^{\infty^{\#}} z_{n,m}\right) \tag{12.43}$$

and

$$Ext\text{-}\sum_{m=1}^{\infty^{\#}}\left(Ext\text{-}\sum_{n=1}^{\infty^{\#}} z_{n,m}\right) \tag{12.44}$$

are called repeated hyper infinite series.

Theorem 12.38.If the double hyper infinite series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent, then

$$\#\text{-}\lim_{n,m\to_{\#}\infty^{\#}} z_{n,m} = 0. \tag{12.45}$$

Theorem 12.39.(Cauchy #-Convergence Criterion for Double hyper infinite Series.)

A double hyper infinite series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ #-converges if and only if its sequence of partial sums $s_{n,m}$ is Cauchy.

Theorem12.40.If the double series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ #-converges to s_1 and $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} u_{n,m}$ #-converges to s_2, then: (i) $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m} + Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} u_{n,m} = s_1 + s_2$.

(ii) $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} c \times z_{n,m} = c \times \left(Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m} \right)$.

Theorem12.41.Suppose that the double series $Ext\text{-}\sum_{n=1,m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent, with sum s. Then the repeated series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} \left(Ext\text{-}\sum_{m=1}^{\infty^{\#}} z_{n,m} \right)$ and $Ext\text{-}\sum_{m=1}^{\infty^{\#}} \left(Ext\text{-}\sum_{n=1}^{\infty^{\#}} z_{n,m} \right)$ are both #-convergent with sum s if and only if for every $m \in \mathbb{N}^{\#}$, the series $Ext\text{-}\sum_{n=1}^{\infty^{\#}} z_{n,m}$ is #-convergent, and for every $n \in \mathbb{N}^{\#}$, the series $Ext\text{-}\sum_{m=1}^{\infty^{\#}} z_{n,m}$ is #-convergent.

12.10.Interchanging the order of summation of hyper infinite sum.

Theorem 12..Assum that

$$Ext\text{-}\sum_{i=1}^{\infty^{\#}} \left(Ext\text{-}\sum_{k=1}^{\infty^{\#}} |a_{jk}| \right) < \infty^{\#}. \qquad (12.46)$$

Then

$$Ext\text{-}\sum_{i=1}^{\infty^{\#}} \left(Ext\text{-}\sum_{k=1}^{\infty^{\#}} |a_{jk}| \right) = Ext\text{-}\sum_{k=1}^{\infty^{\#}} \left(Ext\text{-}\sum_{j=1}^{\infty^{\#}} |a_{jk}| \right) \qquad (12.47)$$

13.Hyper infinite sequences and series of $\mathbb{R}_c^{\#}$-valued functions.

13.1.Uniform #-Convergence

If $f_k, f_{k+1}, \ldots, f_n, \ldots, n \in \mathbb{N}^{\#}$ are $\mathbb{R}_c^{\#}$-valued functions defined on a subset $D \subset \mathbb{R}_c^{\#}$ of the hyperreals, we say that $\{f_n\}_{n \in \mathbb{N}^{\#}}$ is an hyper infinite sequence of functions on D. If the sequence of values $\{f_n(x)\}_{n \in \mathbb{N}^{\#}}$ #-converges for each x in some subset S of D, then $\{f_n\}_{n \in \mathbb{N}^{\#}}$ defines a #-limit function on S. The formal definition is as follows.

Definition 13.1. Suppose that $\{f_n\}_{n \in \mathbb{N}^{\#}}$ is a hyper infinite sequence of functions on $D \subset \mathbb{R}_c^{\#}$ and the hyper infinite sequence of values $\{f_n(x)\}_{n \in \mathbb{N}^{\#}}$ #-converges for each x in some subset S of D. Then we say that $\{f_n\}_{n \in \mathbb{N}^{\#}}$ #-converges pointwise on S to the

#-limit function f, defined by

$$f(x) = \#\text{-}\lim_{n\to\infty^\#} f_n(x), x \in S. \quad (13.1)$$

Definition 13.2.Let f be a function defined on $S \subset \mathbb{R}_c^\#$ and there exist $\sup_{x\in S}|f(x)|$, then we set

$$\|f\|_S = \sup_{x\in S}|f(x)|. \quad (13.2)$$

Lemma 13.1. If g and h are defined on S, then $\|g+h\|_S \le \|g\|_S + \|h\|_S$ and $\|g \times h\|_S \le \|g\|_S \times \|h\|_S$. Moroever if either g or h is bounded on S, then $\|g-h\|_S \ge \|g\|_S - \|h\|_S$.

Definition 13.2. A hyper infinite sequence $\{f_n\}_{n\in\mathbb{N}^\#}$ of functions defined on a set S #-converges uniformly to the #-limit function f on S if $\#\text{-}\lim_{n\to\infty^\#}\|f_n - f\|_S = 0$.
Thus, f_n #-converges uniformly to f on S if for each $\varepsilon > 0, \varepsilon \approx 0$, there is an integer $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that

$$\|f_n - f\| < \varepsilon \text{ if } n \ge N. \quad (13.3)$$

Theorem 13.1. Let $f_n, n \in \mathbb{N}^\#$ be hyper infinite sequence defined on S. Then
(a) f_n #-converges pointwise to f on S if and only if there is, for each $\varepsilon > 0, \varepsilon \approx 0$, and $x \in S$, an integer $N \in \mathbb{N}^\# \backslash \mathbb{N}$ which may depend on x as well as ε such that $|f_n(x) - f(x)| < \varepsilon$ if $n \ge N$;
(b) f_n #-converges uniformly to f on S if and only if there is for each $\varepsilon > 0, \varepsilon \approx 0$, an integer $N \in \mathbb{N}^\# \backslash \mathbb{N}$ which depends only on and not on any particular x in S such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in S$ if $n \ge N$.

Theorem 13.2. If f_n #-converges uniformly to f on S, then f_n #-converges pointwise to f on S. The converse is false; that is pointwise #-convergence does not imply uniform #-convergence.

Theorem 13.3. (Cauchy's Uniform #-Convergence Criterion) A sequence of functions f_n #-converges uniformly on a set S if and only if for each $\varepsilon > 0, \varepsilon \approx 0$, there is an integer $N \in \mathbb{N}^\# \backslash \mathbb{N}$ such that

$$\|f_n - f_m\|_S < \text{ if } n, m \ge N. \quad (13.4)$$

Theorem 13.4. If f_n #-converges uniformly to f on S and each f_n is #-continuous at a point $x_0 \in S$; then so is f. Similar statements hold for #-continuity from the right and left.

Theorem 13.5. Suppose that f_n #-converges uniformly to f on $S = [a,b]$. Assume that f and all f_n are #-integrable on $[a,b]$. Then

$$Ext\text{-}\int_a^b f(x)d^\#x = \#\text{-}\lim_{n\to\infty^\#}\left(Ext\text{-}\int_a^b f_n(x)d^\#x\right). \quad (13.5)$$

Proof. Since

$$\left|Ext\text{-}\int_a^b f(x)d^\#x - Ext\text{-}\int_a^b f_n(x)d^\#x\right| \le Ext\text{-}\int_a^b |f(x) - f_n(x)|d^\#x \le (b-a)\|f - f_n\|_S \quad (13.6)$$

and $\#\text{-}\lim_{n\to\infty^\#}\|f - f_n\|_S = 0$, the Eq.(13.5) follows.

Theorem 13.6. Suppose that $f_n(x)$ #-converges pointwise to f and each $f_n(x)$ is #-integrable on $[a,b]$.Then
(a) If the #-convergence is uniform, then $f(x)$ is #-integrable on $[a,b]$ and (13.5) holds.
(b) If the sequence $\|f_n\|_{[a,b]}$ is bounded and $f(x)$ is #-integrable on $[a,b]$, then (13.5) holds.

Theorem 13.7. Suppose that $f_n'^{\#}(x)$ is #-continuous on $[a,b]$ for all $n \in \mathbb{N}^{\#}$ and $\left\{f_n'^{\#}\right\}_{n\in\mathbb{N}^{\#}}$ #-converges uniformly on $[a,b]$ Suppose also that $\left\{f_n(x_0)\right\}_{n\in\mathbb{N}^{\#}}$ #-converges for some $x_0 \in [a,b]$. Then $\left\{f_n(x)\right\}_{n\in\mathbb{N}^{\#}}$ #-converges uniformly on $[a,b]$ to a #-differentiable #-limit function $f(x)$ and

$$f'^{\#}(x) = \#\text{-}\lim_{n\to\infty^{\#}} f_n'^{\#}(x), x \in (a,b), \tag{13.7}$$

while

$$f_+'^{\#}(a) = \#\text{-}\lim_{n\to\infty^{\#}} f_n'^{\#}(a+), f_-'^{\#}(b) = \#\text{-}\lim_{n\to\infty^{\#}} f_n'^{\#}(b-). \tag{13.8}$$

13.2.Hyper Infinite Series of Functions.

Definition 13.3. If $\{f_j(x)\}_{j=k}^{\infty^{\#}}$ is a hyper infinite sequence of $\mathbb{R}_c^{\#}$-valued functions defined on a set $D \subset \mathbb{R}_c^{\#}$ of hyperreals, then

$$Ext\text{-}\sum_{j=k}^{\infty^{\#}} f_j(x) \tag{13.9}$$

is an hyper infinite series of functions on D. The partial sums of , $Ext\text{-}\sum_{j=k}^{\infty^{\#}} f_j(x)$ are defined by

$$F_n(x) = Ext\text{-}\sum_{j=k}^{n} f_j(x), n \in \mathbb{N}^{\#}. \tag{13.10}$$

If $F_n(x)$ #-converges pointwise to a function F on a subset $S \subset D$, we say that $Ext\text{-}\sum_{j=k}^{n} f_j(x)$ #-converges pointwise to the sum $F(x)$ on S, and write

$$F(x) = Ext\text{-}\sum_{j=k}^{\infty^{\#}} f_j(x). \tag{13.11}$$

If $F_n(x)$ #-converges uniformly to $F(x)$ on S, we say that $Ext\text{-}\sum_{j=k}^{n} f_j(x)$ #-converges uniformly to $F(x)$ on S.

Example 13.1. The functions $f_j(x) = x^j, j \in \mathbb{N}^{\#}$ define the hyper infinite series $Ext\text{-}\sum_{j=0}^{\infty^{\#}} x^j$ on $D = (-\infty^{\#}, \infty^{\#})$. The n-th partial sum of the series is $F_n(x) = Ext\text{-}\sum_{j=0}^{n} x^j$, or, in closed form,

$$F_n(x) = \begin{cases} \dfrac{1-x^n}{1-x} & x \neq 1 \\ n+1 & x = 1 \end{cases} \tag{13.12}$$

Therefore $\{F_n\}$ #-converges pointwise toif $|x| < 1$ and #-diverges if $|x| > 1$, hence, we get $F(x) = Ext\text{-}\sum_{j=0}^{\infty^{\#}} x^j = (1-x)^{-1}, -1 < x < 1.$ Since the difference $F(x) - F_n(x) =$ $= \dfrac{x^n}{1-x}$ can be made arbitrarily infinite large by taking x infinite close to 1, $\|F - F_n\|_{(-1,1)} = \infty^{\#}$ so the #-convergence is not uniform on $(-1,1)$. Neither is it uniform on any interval $(-1,r]$ with $1 < r < 1,$ since $\|F - F_n\|_{[-r,r]} = r^n/(1-r)$ and #-$\lim_{n\to\infty^{\#}} r^n = \infty^{\#}.$ Put another way, the series #-converges uniformly on #-closed subsets of $(-1,1)$.

Theorem 13.8.(Cauchy's Uniform #-Convergence Criterion) A hyper infinite series $Ext\text{-}\sum_{i=0}^{\infty^{\#}} f_i(x)$ #-converges uniformly on a set $S \subset \mathbb{R}_c^{\#}$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is an hyperinteger $N \in \mathbb{N}^{\#}$ such that

$$\left\| Ext\text{-}\sum_{n}^{m} f_i(x) \right\|_S < \varepsilon \qquad (13.13)$$

if $m \geq n \geq N$.

Corollary 13.1. If $Ext\text{-}\sum_{i=0}^{\infty^{\#}} f_i(x)$ #-converges uniformly on $S,$ then #-$\lim_{n\to\infty^{\#}} \|f_n\|_S = 0.$

Theorem 13.9.(Weierstrass's Test) The hyper infinite series $Ext\text{-}\sum_{i=0}^{\infty^{\#}} f_i(x)$ #-converges uniformly on S if

$$\|f_n\|_S \leq M_n, n \geq k, \qquad (13.14)$$

where $Ext\text{-}\sum_{n=k}^{\infty^{\#}} M_n < \infty^{\#}.$

Theorem 13.10.(Dirichlet's Test for Uniform #-Convergence) The hyper infinite vseries $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x)g_n(x)$ #-converges uniformly on S if f_n #-converges uniformly to zero on $S,$ $Ext\text{-}\sum_{n=k}^{\infty^{\#}} (f_{n+1}(x) - f_n(x))$ #-converges absolutely uniformly on $S,$ and

$$\left\| Ext\text{-}\sum_{i=k}^{n} g_i(x) \right\|_S \leq M, \qquad (13.14)$$

where $n \geq k,$ for some constant $M.$

Corollary 13.2.The hyper infinite series $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x)g_n(x)$ #-converges uniformly on S if $f_{n+1}(x) \leq f_n(x), x \in S, n \geq k, \{f_n\}$ #-converges uniformly to zero on $S,$ and

$$\left\| Ext\text{-}\sum_{i=k}^{n} g_i(x) \right\|_S \leq M, \qquad (13.15)$$

for some constant $M.$

13.3.#-Continuity, #-Differentiability, and Integrability of hyper infinite Series.

Theorem 13.11.If $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x)$ #-converges uniformly to $F(x)$ on S and each f_n is

#-continuous at a point x_0 in S, then so is $F(x)$. Similar statements hold for #-continuity from the right and left.

Theorem 13.12.Suppose that $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x)$ #-converges uniformly to $F(x)$ on $S = [a,b]$ Assume that $F(x)$ and $f_n(x), n \geq k,$ are integrable on $[a,b]$. Then

$$Ext\text{-}\int_a^b F(x)d^{\#}x = Ext\text{-}\sum_{n=k}^{\infty^{\#}} \left(Ext\text{-}\int_a^b f_n(x)d^{\#}x \right). \quad (13.16)$$

Theorem 13.13.Suppose that f_n is #-continuously #-differentiable on $[a,b]$ for each $n \geq k,$ $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x_0)$ #-converges for some $x_0 \in [a,b]$ and $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(x)$ #-converges uniformly on $[a,b]$. $Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n(x)$ #-converges uniformly on $[a,b]$ to a #-differentiable function $F(x)$, and $F'^{\#}(x) = Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(x), a < x < b,$while $F'^{\#}(a+) = Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(a+)$ and $F'^{\#}(b-) = Ext\text{-}\sum_{n=k}^{\infty^{\#}} f_n'^{\#}(b-)$.

14.Hyper Infinite Power Series.

14.1.The convergence properties of hyper infinite power series.

Definition 14.1. A hyper infinite series of the form

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x - x_0)^n \quad (14.1)$$

where $x_0 \in \mathbb{R}_c^{\#}$ and $a_n \in \mathbb{R}_c^{\#}, n \in \mathbb{N}^{\#}$ is called a hyper infinite power series in $(x - x_0)$. The following theorem summarizes the #-convergence properties of hyper infinite power series.

Theorem 14.1.In connection with the hyper infinite power series (14.1) define R in the extended hyperreals by

$$\frac{1}{R} = \overline{\#\text{-}\lim_{n\to\infty^{\#}}} \sqrt[n]{|a_n|} \quad (14.2)$$

In particular, $R = 0$ if $\overline{\#\text{-}\lim_{n\to\infty^{\#}}} \sqrt[n]{|a_n|} = \infty^{\#}$, and $R = \infty^{\#}$ if $\overline{\#\text{-}\lim_{n\to\infty^{\#}}} \sqrt[n]{|a_n|} = 0$.Then the hyper infinite power series #-converges:

(a) only for $x = x_0$ if $R = 0$

(b) for all $x \in \mathbb{R}_c^{\#}$ if $R = \infty^{\#}$, and absolutely uniformly in every bounded set;

(c) for $x \in (x_0 - R, x_0 + R)$ if $0 < R < 1$, and absolutely uniformly in every closed subset of this interval.

The series #-diverges if $|x - x_0| > R$. No general statement can be made concerning #-convergence at the endpoints $x = x_0 + R$ and $x = x_0 + R$: the series may #-converge absolutely or conditionally at both; #-converge conditionally at one and #-diverge at the other; or #-diverge at both.

Theorem 14.2. The radius of #-convergence of $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x - x_0)^n$ is given by

$$\frac{1}{R} = \#\text{-}\lim_{n\to\infty^\#}\left|\frac{a_{n+1}}{a_n}\right| \tag{14.3}$$

if the #-limit exists in the extended hyperreals.

Example 14.1. For the hyper infinite power series

$$Ext\text{-}\sum_{n=0}^{\infty^\#}\frac{x^n}{n!} \tag{14.4}$$

one obtains that

$$\#\text{-}\lim_{n\to\infty^\#}\left|\frac{a_{n+1}}{a_n}\right| = \#\text{-}\lim_{n\to\infty^\#}\frac{n!}{(n+1)!} = \#\text{-}\lim_{n\to\infty^\#}\frac{1}{n+1} = 0. \tag{14.4'}$$

Therefore, $R = \infty^\#$; that is, the series #-converges for all $x \in \mathbb{R}_c^\#$, and absolutely uniformly
in every bounded set.

Theorem 14.3. A hyper infinite power series

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^\#}a_n(x-x_0)^n \tag{14.5}$$

with positive radius of #-convergence R is #-continuous and #-differentiable in its interval of #-convergence; and its #-derivative can be obtained by #-differentiating term by term; that is;

$$f'^\#(x) = Ext\text{-}\sum_{n=0}^{\infty^\#}na_n(x-x_0)^{n-1} \tag{14.6}$$

which can also be written as

$$f'^\#(x) = Ext\text{-}\sum_{n=0}^{\infty^\#}(n+1)a_{n+1}(x-x_0)^n \tag{14.7}$$

This hyper infinite series also has radius of #-convergence R.

Theorem 14.4. A hyper infinite power series

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^\#}a_n(x-x_0)^n \tag{14.8}$$

with positive radius of #-convergence R has #-derivatives of all orders in its interval of #-convergence, which can be obtained by repeated term by term #-differentiation thus,

$$\begin{aligned} f^{(n)\#}(x) &= Ext\text{-}\sum_{n=k}^{\infty^\#}n(n-1)\cdots(n-k+1)a_n(x-x_0)^n = \\ &= Ext\text{-}\sum_{n=k}^{\infty^\#}\left[\left(Ext\text{-}\prod_{j=n-k+1}^{n}j\right)a_n(x-x_0)^n\right]. \end{aligned} \tag{14.9}$$

The radius of #-convergence of each of these hyper infinite series is R.

Corollary 14.1. (Uniqueness of hyper infinite Power Series) If

$$Ext\text{-}\sum_{n=0}^{\infty^\#}a_n(x-x_0)^n = Ext\text{-}\sum_{n=0}^{\infty^\#}b_n(x-x_0)^n \tag{14.10}$$

for all x in some interval $(x_0 - r, x_0 + r)$ then

$$a_n = b_n, n \geq 0. \tag{14.11}$$

Corollary 14.2. If

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x-x_0)^n, |x-x_0| < R \qquad (14.12)$$

then

$$a_n = \frac{f^{(n)\#}(x)}{n!}. \qquad (14.13)$$

Theorem 14.5. If x_1 and x_2 are in the interval of #-convergence of

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x-x_0)^n \qquad (14.14)$$

Then

$$Ext\text{-}\int_{x_1}^{x_2} f(x)d^{\#}x = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{a_n}{n+1}\left[(x_2-x_0)^{n+1} - (x_1-x_0)^{n+1}\right] \qquad (14.15)$$

that is, a hyper infinite power series may be integrated term by term between any two points in its interval of #-convergence.

Theorem 14.6.Suppose that $f(x)$ is hyper infinitely #-differentiable on an interval I and

$$\#\text{-}\lim_{n\to\infty^{\#}} \frac{r^n}{n!}\left\|f^{(n)\#}(x)\right\|_I = 0. \qquad (14.16)$$

Then, if $x_0 \in I^0$, the hyper infinite Taylor series

$$Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{f^{(n)\#}(x)}{n!}(x-x_0)^n \qquad (14.17)$$

#-converges uniformly to $f(x)$ on $I_r = I \cap [x_0 - r, x_0 + r]$.

Theorem 14.7.If

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n(x-x_0)^n, |x-x_0| < R_1 \qquad (14.18)$$

and

$$g(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} b_n(x-x_0)^n, |x-x_0| < R_2 \qquad (14.19)$$

and α and β are constants, then

$$\alpha f(x) + \beta g(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (\alpha a_n + \beta b_n)(x-x_0)^n, |x-x_0| < R, \qquad (14.20)$$

where $R \geq \min\{R_1, R_2\}$.

Theorem 14.8.If $f(x)$ and $g(x)$ are given by Eq.(14.19) and Eq.(14.20) correspondingly, then

$$f(x)g(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} c_n(x-x_0)^n, |x-x_0| < R, \qquad (14.21)$$

where

$$c_n = Ext\text{-}\sum_{j=0}^{n} a_j b_{n-j} = \sum_{j=0}^{n} a_{n-j} b_j, \qquad (14.22)$$

$n \in \mathbb{N}^{\#}$ and $R \geq \min\{R_1, R_2\}$.

Theorem 14.9.(Generalized Abel’s Theorem) Let $f(x)$ be defined by a hyper infinite power series

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n (x - x_0)^n, |x - x_0| < R \tag{14.23}$$

with finite or hyperfinite radius of #-convergence $R \in \mathbb{R}_c^{\#}$.

(a) If $Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n R^n$ #-converges, then

$$\#\text{-} \lim_{x \to_{\#} (x_0 + R)-} f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n R^n. \tag{14.24}$$

(b) If $Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n a_n R^n$ #-converges, then

$$\#\text{-} \lim_{x \to_{\#} (x_0 - R)+} f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n a_n R^n. \tag{14.25}$$

14.2.The $\mathbb{R}_c^{\#}$-valued #-exponential $Ext\text{-}\exp(x)$

We define the #-exponential $Ext\text{-}\exp(x)$ function as the solution of the differential equation

$$f'^{\#}(x) = f(x), f(0) = 1. \tag{14.26}$$

We solve it by setting

$$f(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n x^n, f'^{\#}(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} n a_n x^n. \tag{14.27}$$

Therefore

$$Ext\text{-}\exp(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{x^n}{n!} \tag{14.28}$$

From Eq.(12.40) and Eq.(14.28) we get

$$(Ext\text{-}\exp(x))(Ext\text{-}\exp(y)) = Ext\text{-}\exp(x+y), \tag{14.29}$$

for any $x, y \in \mathbb{R}_c^{\#}$. We often denote #-exponential $Ext\text{-}\exp(x)$ by $Ext\text{-}e^x$

$$Ext\text{-}e^x . \tag{14.30}$$

14.3.The $\mathbb{R}_c^{\#}$-valued Trigonometric Functions $Ext\text{-}\sin(x)$ and $Ext\text{-}\cos(x)$.

We define the $\mathbb{R}_c^{\#}$-valued Trigonometric Functions $Ext\text{-}\sin(x)$ and $Ext\text{-}\cos(x)$ by

$$Ext\text{-}\sin(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{14.31}$$

and

$$Ext\text{-}\cos(x) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{14.32}$$

It can be shown that the series (14.30) and (14.31) #-converge for all $x \in \mathbb{R}_c^{\#}$ and #-differentiating yields

$$[Ext\text{-}\sin(x)]'^{\#} = Ext\text{-}\sum_{n=0}^{\infty^{\#}}(-1)^n \frac{x^{2n}}{(2n)!} = Ext\text{-}\cos(x) \quad (14.33)$$

and

$$[Ext\text{-}\cos(x)]'^{\#} = Ext\text{-}\sum_{n=1}^{\infty^{\#}}(-1)^n \frac{x^{2n-1}}{(2n-1)!} = -Ext\text{-}\sum_{n=0}^{\infty^{\#}}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = \\ = -[Ext\text{-}\sin(x)]. \quad (14.34)$$

14.4. $\mathbb{R}_c^{\#}$-valued functions of several variables.

In this subsection we study $\mathbb{R}_c^{\#}$-valued functions defined on subsets of the n-dimensional external linear space $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$ which consists of all external and internal hyperfinite (or finite) sequences (called a vector) $\mathbf{X} = \{x_i\}_{i=1}^{i=n} = \{x_i\}_{i\in n}$ of hyperreal numbers, called the coordinates or components of vector $\mathbf{X}$.

Definition 14.2. The vector sum of $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{Y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{X} + \mathbf{Y} = \{x_i + y_i\}_{i=1}^{i=n}. \quad (14.35)$$

If $a \in \mathbb{R}_c^{\#}$ is a hyperreal number, the scalar multiple of $\mathbf{X}$ by a is

$$a \cdot \mathbf{X} = \{ax_i\}_{i=1}^{i=n}. \quad (14.36)$$

Theorem 14.10.If $\mathbf{X}, \mathbf{Y},$ and $\mathbf{Z}$ are in $\mathbb{R}_c^{\#n}$ and $a, b \in \mathbb{R}_c^{\#}$ are hyperreal numbers, then

(i) $\mathbf{X} + \mathbf{Y} = \mathbf{Y} + \mathbf{X}$ - vector addition is commutative

(ii) $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$ - vector addition is associative

(iii) There is a unique vector $\mathbf{0}$, called the zero vector, such that $\mathbf{X} + 0 = \mathbf{X}$ for all $\mathbf{X} \in \mathbb{R}_c^{\#n}$

(iv) For each $\mathbf{X} \in \mathbb{R}_c^{\#n}$ there is a unique vector $-\mathbf{X}$ such that $\mathbf{X} + (-\mathbf{X}) = \mathbf{0}$

(v) $a \cdot (b \cdot \mathbf{X}) = (ab) \cdot \mathbf{X}$

(vi) $(a + b) \cdot \mathbf{X} = a \cdot \mathbf{X} + b \cdot \mathbf{X}$

(vii) $a \cdot (\mathbf{X} + \mathbf{Y}) = a \cdot \mathbf{X} + a \cdot \mathbf{Y}$

(viii) $1 \cdot \mathbf{X} = \mathbf{X}$.

Clearly, $\mathbf{0} = \{0\}_{i=1}^{i=n}$ and, if $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$, then $-\mathbf{X} = \{-x_i\}_{i=1}^{i=n}$.

We write $\mathbf{X} + (-\mathbf{Y})$ as $\mathbf{X} - \mathbf{Y}$. The point $\mathbf{0}$ is called the origin.

Definition 14.3. The length of the vector $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ is

$$\|\mathbf{X}\| = \left(Ext\text{-}\textstyle\sum_{i=1}^{n} x_i^2\right)^{1/2}. \quad (14.37)$$

The distance between points $\mathbf{X}$ and $\mathbf{Y}$ is $\|\mathbf{X} - \mathbf{Y}\|$; in particular, $\|\mathbf{X}\|$ is the distance between $\mathbf{X}$ and the origin. If $\|\mathbf{X}\| = 1$, then $\mathbf{X}$ is a unit vector.

Definition 14.4.The inner product $\mathbf{X} \cdot \mathbf{Y}$ of $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ and $\mathbf{Y} = \{y_i\}_{i=1}^{i=n}$ is

$$\mathbf{X} \cdot \mathbf{Y} = Ext\text{-}\textstyle\sum_{i=1}^{n} x_i y_i. \quad (14.38)$$

Theorem 14.11. (Schwarz's Inequality) If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}_c^{\#n}$ then

$$\|\mathbf{X} \cdot \mathbf{Y}\| \leq \|\mathbf{X}\| \|\mathbf{Y}\|, \quad (14.39)$$

with equality if and only if one of the vectors is a scalar multiple of the other:

Theorem 14.12. (Triangle Inequality) If $\mathbf{X}, \mathbf{Y} \in \mathbb{R}_c^{\#n}$ then

$$\|\mathbf{X} + \mathbf{Y}\| \leq \|\mathbf{X}\| + \|\mathbf{Y}\|, \quad (14.40)$$

with equality if and only if one of the vectors is a nonnegative multiple of the other.
Corollary 14.3. If $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathbb{R}_c^{\#n}$, then

$$\|\mathbf{X}-\mathbf{Z}\|\leq\|\mathbf{X}-\mathbf{Y}\|+\|\mathbf{Y}-\mathbf{Z}\|. \quad (14.41)$$

Corollary 14.4. If $\mathbf{X},\mathbf{Y}\in\mathbb{R}_c^{\#n}$, then

$$\|\mathbf{X}-\mathbf{Y}\|\geq|\|\mathbf{X}\|-\|\mathbf{Y}\||. \quad (14.42)$$

Theorem 14.13.If $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathbb{R}_c^{\#n}$ and $a\in\mathbb{R}_c^{\#}$ is a scalar, then
(i) $\|a\mathbf{X}\|=|a|\|\mathbf{X}\|$
(ii) $\|\mathbf{X}\|\geq 0,$ with equality if and only if $\mathbf{X}=\mathbf{0}$
(iii) $\|\mathbf{X}-\mathbf{Y}\|\geq 0,$ with equality if and only if $\mathbf{X}=\mathbf{Y}$
(iv) $\mathbf{X}\cdot\mathbf{Y}=\mathbf{Y}\cdot\mathbf{X}$
(v) $\mathbf{X}\cdot(Y+Z)=\mathbf{X}\cdot\mathbf{Y}+\mathbf{X}\cdot\mathbf{Z}$
(vi) $(c\mathbf{X})\cdot\mathbf{Y}=\mathbf{X}\cdot(c\mathbf{Y})=c(\mathbf{X}\cdot\mathbf{Y})$
Definition 14.5.Non-Archimedian metric space (X,d) is a set X together with a $\mathbb{R}_c^{\#}$-valued function $d:X\times X\to\mathbb{R}_c^{\#}$ (called a metric or non-Archimedian distance function) which assigns a hyperreal number $d(x,y)$ to every pair x,y belongs X satisfying the properties:
1.$d(x,y)\geq 0$ and $d(x,y)=0$ iff $x=y$,
2.$d(x,y)=d(y,x)$,
3.$d(x,y)+d(y,z)\geq d(x,z)$.
Remark 14.1. Note that external linear space $\mathbb{R}_c^{\#n}$ endroved with distance function $d(\mathbf{X},\mathbf{Y})=\|\mathbf{X}-\mathbf{Y}\|$ satisfying the properties 1-3 mentioned above in Definition 14.5.

14.5.Line Segments in $\mathbb{R}_c^{\#n},n\in\mathbb{N}^{\#}$.

Definition 14.6. Suppose that $\mathbf{X}_0,\mathbf{U}\in\mathbb{R}_c^{\#n}$ and $\mathbf{U}\neq\mathbf{0}$. Then the line through $\mathbf{X}_0$ in the direction of $\mathbf{U}$ is the set of all points in $\mathbb{R}_c^{\#n}$of the form

$$\mathbf{X}(\mathbf{X}_0,\mathbf{U})=\mathbf{X}_0+t\mathbf{U},-\infty^{\#}<t<\infty^{\#}. \quad (14.43)$$

A set of points of the form

$$\mathbf{X}=\mathbf{X}_0+t\mathbf{U},t_1\leq t\leq t_2 \quad (14.44)$$

is called a line segment. In particular, the line segment from $\mathbf{X}_0$ to $\mathbf{X}_1$ is the set of points of the form

$$\mathbf{X}=\mathbf{X}_0+t(\mathbf{X}_1-\mathbf{X}_0)=t\mathbf{X}_1+(1-t)\mathbf{X}_0,0\leq t\leq 1. \quad (14.45)$$

Definition 14.7. A hyper infinite sequence of points $\mathbf{X}_n,n\in\mathbb{N}^{\#}$ in $\mathbb{R}_c^{\#n}$ #-converges to the #-limit $\overline{\mathbf{X}}$ if

$$\#\text{-}\lim_{n\to_{\#}\infty^{\#}}\|\mathbf{X}_n-\overline{\mathbf{X}}\|=0. \quad (14.46)$$

In this case we write $\#\text{-}\lim_{n\to_{\#}\infty^{\#}}\mathbf{X}_n=\overline{\mathbf{X}}$.
Theorem 14.14. Let $\overline{\mathbf{X}}=\{x_i\}_{i=1}^{i=n}$ and $\mathbf{X}_m=\{x_{i_m}\}_{i=1}^{i=n},m\geq 1.$Then $\#\text{-}\lim_{n\to_{\#}\infty^{\#}}\mathbf{X}_m=\overline{\mathbf{X}}$ if and only if $\#\text{-}\lim_{m\to_{\#}\infty^{\#}}x_{i_m}=\bar{x}_i,1\leq i\leq n;$that is a hyper infinite sequence $\{\mathbf{X}_m\}$ of points in $\mathbb{R}_c^{\#n}$ #-converges to a #-limit $\overline{\mathbf{X}}$ if and only if the hyper infinite sequences of components of $\{\mathbf{X}_m\}$ #-converge to the respective components of $\overline{\mathbf{X}}$.
Theorem 14.15.(Cauchy's #-Convergence Criterion) A hyper infinite sequence $\{\mathbf{X}_m\}$ in $\mathbb{R}_c^{\#n}$ #-converges if and only if for each $\varepsilon>0,\varepsilon\approx 0,$ there is an hyperinteger $N\in\mathbb{N}^{\#}\backslash\mathbb{N}$ such that

$$\|\mathbf{X}_n - \mathbf{X}_m\| < \varepsilon \tag{14.47}$$

if $n, m \geq N$.

Definition 14.8.If A is a subset of a metric space $\mathbb{R}_c^{\#n}$ then x is a #-limit point of A if it is the #-limit of an eventually non-constant hyper infinite sequence $\{a_i\}_{i\in\mathbb{N}^\#}$ of points of A.

Definition 14.9.A subset A is said to be a #-closed subset of $\mathbb{R}_c^{\#n}$ if it contains all its #-limit points.

Example 14.1.(i) $\mathbb{R}_c^\#$ with the canonical metric $d(x,y) = |x-y|$, since in $\mathbb{R}_c^\#$ every hyperreal number is a #-limit point of the hyper infinite sequence $\{q_i\}_{i\in\mathbb{N}^\#}$ of hyperrationals $q_i \in \mathbb{Q}^\#, i \in \mathbb{N}^\#$.

(ii) The empty set is #-closed.

(iii) Any finite set is #-closed.

(iv) Any hyperfinite set is #-closed.

(v) The closed interval $[a,b]$, where $a,b \in \mathbb{R}_c^\#$, is #-closed subset of $\mathbb{R}_c^\#$ with its canonical metric.

(vi) Let Δ be a set $\Delta = \{\varepsilon \| \varepsilon| \approx 0\}$. A set Δ is #-closed subset of $\mathbb{R}_c^\#$, since in Δ every hyperreal number $\delta \in \Delta$ is a #-limit point of the hyper infinite sequence $\{q_i\}_{i\in\mathbb{N}^\#}$ of hyperrationals $q_i \in \Delta \cap \mathbb{Q}^\#, i \in \mathbb{N}^\#$.

Definition 14.10.An #-neighbourhood of a point p in a metric space (X,d) is the set $N_\varepsilon(p) = \{x \in X | d(x,p) < \varepsilon, \varepsilon \approx 0\}$

Definition 14.11.A subset A of a metric space (X,d) is called #-open in X if every point of A has an #-neighbourhood which lies completely in A.

Example 14.2. (i) Any open interval (a,b) is an #-open set in $\mathbb{R}_c^\#$ with its canonical metric $d(x,y) = |x-y|$.

(ii) A set $\Delta = \{\varepsilon \| \varepsilon| \approx 0\}$ is #-open subset of $\mathbb{R}_c^\#$, since every point of Δ obviously has a #-neighbourhood which lies completely in Δ.

Remark 14.2.Note that a set $\Delta = \{\varepsilon \| \varepsilon| \approx 0\}$ are #-open and #-closed simultaneously.

Definition 14.12.A subset A of a non-Archimedian metric space X is admissible if A is exactly #-closed or exactly #-open but not #-open and #-closed simultaneously.

Theorem 14.16.(i) The union (of an arbitrary number) of #-open admissible sets is #-open.(ii) The intersection of finitely or hyper finitely many #-open admissible sets is #-open.

Proof. (i) Let $x \in \cup A_i = A$. Then $x \in A_i$ for some i. Since this is #-open, x has an #-neighbourhood lying completely inside A_i and this is also inside A.

(ii) It is enough to show this for just two #-open sets A and B. So suppose $x \in A \cap B$. Then $x \in A$ and so has an #-neighbourhood $N_{\varepsilon_1}(p), \varepsilon_1 \approx 0$ lying in A. Similarly x has an #-neighbourhood $N_{\varepsilon_2}(p), \varepsilon_2 \approx 0$ lying in B. So if $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ the #-neighbourhood $N_\varepsilon(p)$ lies in both A and B and hence in $A \cap B$. By hyper infinite induction statement (ii) holds in general.

Theorem 14.17. Any admissible subset A of a metric space X is #-closed if and only if its complement $X \backslash A$ is admissible and is #-open subset of a metric space X.

Proof. 1.Suppose A is admissible and A is #-closed. We need to show that $X \backslash A$ is #-open.

So suppose that x belongs $X \backslash A$. Then some #-neighbourhood of x does not meet A (otherwise x would be a #-limit point of A and hence in A). Thus this #-neighbourhood

of x lies completely in $X\backslash A$ which is what we needed to prove.
2.Conversely, suppose that $X\backslash A$ is #-open. We need to show that A contains all its #-limit points. So suppose x is a #-limit point of A and that $x \notin A$. Then $x \in X\backslash A$ and hence
has an #-neighbourhood subset $X\backslash A$. But this is an #-neighbourhood that does not meet A
and we have a contradiction.

Definition 14.13.If $\mathbf{S}$ is a nonempty subset of $\mathbb{R}_c^{\#n}$, then

$$d(\mathbf{S}) = \sup\{\|\mathbf{X}-\mathbf{Y}\| | \mathbf{X},\mathbf{Y} \in \mathbf{S}\} \tag{14.48}$$

is the diameter of $\mathbf{S}$. If $d(S) < \infty^{\#}, \mathbf{S}$ is bounded or hyperbounded.If $d(S) = \infty^{\#}, \mathbf{S}$ is hyperunbounded.

Theorem 14.18. (Principle of Nested Sets) If $\mathbf{S}_1, \mathbf{S}_2, \ldots,$ are #-closed nonempty subsets of $\mathbb{R}_c^{\#n}$ such that

$$\forall r(r \in \mathbb{N}^{\#})[\mathbf{S}_{r+1} \subset \mathbf{S}_r] \tag{14.49}$$

and

$$\#\text{-}\lim_{r \to_{\#} \infty^{\#}} d(\mathbf{S}_r) = 0, \tag{14.50}$$

then the intersection

$$\Lambda = \bigcap_{r=1}^{\infty^{\#}} \mathbf{S}_r \tag{14.51}$$

contains exactly one point:

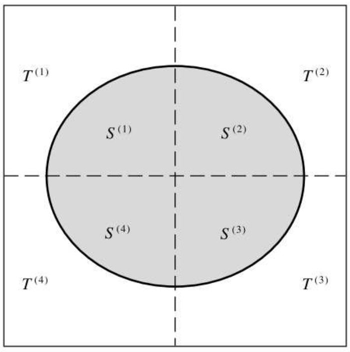
Proof.Let $\{\mathbf{X}_r\}$ be a hyper infinite sequence such that $\mathbf{X}_r \in \mathbf{S}_r, r \geq 1$. Because of (14.49), $\mathbf{X}_r \in \mathbf{S}_k$ if $r \geq k$, so $\|\mathbf{X}_r - \mathbf{X}_s\| < d(\mathbf{S}_k)$ if $r,s \geq k$.
From (14.50) and Theorem 14.15., $\mathbf{X}_r$ #-converges to a #-limit $\overline{\mathbf{X}}$. Since $\overline{\mathbf{X}}$ is a #-limit point of every $\mathbf{S}_k$ and every $\mathbf{S}_k$ is #-closed, $\overline{\mathbf{X}}$ is in every $\mathbf{S}_k$. Therefore, $\overline{\mathbf{X}} \in \Lambda$, so $\Lambda \neq \varnothing$.
Moreover, $\overline{\mathbf{X}}$ is the only point in Λ, since if $\mathbf{Y} \in \Lambda$, then $\|\overline{\mathbf{X}} - \mathbf{Y}\| < d(\mathbf{S}_k), k \geq 1,$and (14.50) implies that $\mathbf{Y} = \overline{\mathbf{X}}$.

Definition 14.14. If $\mathbf{S}$ is a nonempty admissible subset of $\mathbb{R}_c^{\#n}$ we say that $\mathbf{S}$ is a #-compact set in $\mathbb{R}_c^{\#n}$ if is a #-closed and bounded or hyperbounded set.

Definition 14.15. Collection $\mathbf{H}$ of admissible #-open sets is an #-open covering of a set S if $S \subset \cup\{H | H \in \mathbf{H}\}$.

Theorem 14.19.(Heine-Borel Theorem) If H is an #-open covering of a #-compact subset S, then S can be covered by hyper finitely many sets from H.

Proof.The proof is by contradiction. We first consider the case where $n = 2$.Suppose that there is a #-open covering $\mathbf{H}$ for $\mathbf{S}$ from which it is impossible to select a hyperfinite
subcovering. Since $\mathbf{S}$ is bounded or hyperbounded, $\mathbf{S}$ is contained in a #-closed square $T = \{\{x,y\} | a_1 \leq x \leq a_1 + L, a_2 \leq x a_2 + L\}$ with sides of length L (Pic. 14.5.1).



Pic.14.5.1.

Bisecting the sides of T as shown by the dashed lines in Figure 14.5.1 leads to four #-closed squares, $T^{(1)}, T^{(2)}, T^{(3)},$, and $T^{(4)}$, with sides of length $L/2$. Let $S^{(i)} = \mathbf{S} \cap T, 1 \leq i \leq 4.$ Each $S^{(i)}$, being the intersection of admissible #-closed sets,is #-closed, and $S = \cup_{i=1}^{4} S^{(i)}$. Moreover, $\mathbf{H}$ covers each $S^{(i)}$, but at least one $S^{(i)}$cannot be covered by any finite or hyperfinite subcollection of $\mathbf{H}$, since if all the $S^{(i)}$could be, then so could S. Let S_1 be a set with this property, chosen from $S^{(1)}, S^{(2)}, S^{(3)},$ and $S^{(4)}$.
We are now back to the situation we started from: a #-compact set S_1 covered by $\mathbf{H}$, but not by any hyperfinite subcollection of $\mathbf{H}$. However, S_1 is contained in a square T_1 with sides of length $L/2$ instead of L. Bisecting the sides of T_1 and repeating the argument, we obtain a subset S_2 of S_1 that has the same properties as S, except that it is contained in a square with sides of length $L/4$. Continuing in this way produces a hyper infinite sequence of nonempty #-closed sets $S_0 = S, S_1, S_2, \ldots,$ such that $S_k \supset S_{k+1}$ and $d(S_k) \leq L/2^{k-1/2}, k \geq 0$. From Theorem 14.18, there is a point $\overline{\mathbf{X}}$ in $\cap_{k=1}^{\infty^{\#}} S_k$. Since $\overline{\mathbf{X}} \in S$, there is an open set H in $\mathbf{H}$ that contains $\overline{\mathbf{X}}$, and this H must also contain some #-neighborhood of $\overline{\mathbf{X}}$. Since every $\mathbf{X}$ in S_k satisfies the inequality $\|\mathbf{X} - \overline{\mathbf{X}}\| < 2^{-k+1/2}L,$ it follows that $S_k \subset H$ for $k \in \mathbb{N}^{\#}/\mathbb{N}$ sufficiently large. This contradicts our assumption on $\mathbf{H}$, which led us to believe that no S_k could be covered by a hyperfinite
number of sets from $\mathbf{H}$. Consequently, this assumption must be false: $\mathbf{H}$ must have a finite or hyperfinite subcollection that covers S. This completes the proof for $n = 2$.
The idea of the proof is the same for $n > 2$. The counterpart of the square T is the hypercube with sides of length L :

$$T = \{(x_1, x_2, \ldots, x_n) | a_i \leq x_i \leq a_i + L, 1 \leq i \leq n\}.$$

Halving the intervals of variation of the n coordinates $x_1, x_2, \ldots, x_n$ divides T into 2^n closed hypercubes with sides of length $L/2$:

$$T^{(i)} = \{(x_1, x_2, \ldots, x_n) | b_i \leq x_i \leq b_i + L/2, 1 \leq i \leq n\},$$

where $b_i = a_i$ or $b_i = a_i + L/2$. If no hyperfinite subcollection of $\mathbf{H}$ covers S, then at least one of these smaller hypercubes must contain a subset of S that is not covered by any hyperfinite subcollection of S. Now the proof proceeds as for $n = 2$.

14.6.#-Neighborhoods and #-open sets in $\mathbb{R}_c^{\#n}, n \in \mathbb{N}^{\#}$. Connected Sets and Regions in $\mathbb{R}_c^{\#n}$.

Definition 14.16. Assum that A is admissible subset of $\mathbb{R}_c^{\#n}$.
(i) The #-interior #-$int(A)$ of a set A is the largest open subset A,

(ii) The #-closure $\#\text{-}cl(A)$ of a set A is the smallest #-closed set containing A.

Theorem 14.20. 1. $\#\text{-}cl(\varnothing) = \varnothing$

2. $A \subset \#\text{-}cl(A)$ for any set A.

3. $\#\text{-}cl(A \cup B) = \#\text{-}cl(A) \cup \#\text{-}cl(B)$ for any sets A and B.

4. $cl(\#\text{-}cl(A)) = \#\text{-}cl(A)$ for any set A.

Proof.1. and 2. follow from the definition.

To prove 3 note that $\#\text{-}cl(A) \cup \#\text{-}cl(B)$ is a #-closed set which contains $A \cup B$ and so $\#\text{-}cl(A) \subset \#\text{-}cl(A \cup B)$. Similarly, $\#\text{-}cl(B) \subset \#\text{-}cl(A \cup B)$ and so $\#\text{-}cl(A) \cup \#\text{-}cl(B) \subset \#\text{-}cl(A \cup B)$ and the result follows.

To prove 4 we have $\#\text{-}cl(A) \subset \#\text{-}cl(\#\text{-}cl(A))$ from 2. Also $\#\text{-}cl(A)$ is a #-closed set which contains $\#\text{-}cl(A)$ and hence it contains $\#\text{-}cl(\#\text{-}cl(A))$.

Example 14.3.For $\mathbb{R}_c^{\#}$ with its usual topology induced by its canonical metric $d(x,y) = |x-y|$, #-cl$((a,b)) = [a,b]$ and $\#\text{-}int([a,b]) = (a,b)$.

Definition 14.17. If $\varepsilon > 0, \varepsilon \approx 0$, the ε-neighborhood of a point $\mathbf{X}_0$ in $\mathbb{R}_c^{\#n}$ is the set

$$N_\varepsilon(\mathbf{X}_0) = \{\mathbf{X} | \|\mathbf{X}-\mathbf{X}_0\| < \varepsilon\}. \tag{14.52}$$

Definition 14.18.If $\mathbf{X}_0$ is a point in in $\mathbb{R}_c^{\#n}$ and $r > 0$, the sphere of radius r about $\mathbf{X}_0$ is the set $S_r(\mathbf{X}_0) = \{\mathbf{X} | \|\mathbf{X}-\mathbf{X}_0\| = r\}$

Definition 14.19.If $\mathbf{X}_0$ is a point in in $\mathbb{R}_c^{\#n}$ and $r > 0$, the #-open n-ball of radius r about $\mathbf{X}_0$ is the set $B_r(\mathbf{X}_0) = \{\mathbf{X} | \|\mathbf{X}-\mathbf{X}_0\| < r\}$. Thus, ε-neighborhoods are #-open n-balls.

If $\mathbf{X}_1$ is in $B_r(\mathbf{X}_0)$ and $\|\mathbf{X}-\mathbf{X}_1\| < \varepsilon = r - \|\mathbf{X}-\mathbf{X}_0\|$, then $\mathbf{X}$ is in $B_r(\mathbf{X}_0)$.

Thus, $B_r(\mathbf{X}_0)$ contains an ε-neighborhood of each of its points, and is therefore #-open.

The #-closure of $B_r(\mathbf{X}_0)$ is the #-closed n-ball of radius r about $\mathbf{X}_0$, defined by $\#\text{-}cl(B_r(\mathbf{X}_0)) = \{\mathbf{X} | \|\mathbf{X}-\mathbf{X}_0\| < r\}, r = \|\mathbf{X}_1 - \mathbf{X}_0\|$.

Proposition14.1. If $\mathbf{X}_1$ and $\mathbf{X}_2$ are in S$_r(X_0)$ for some $r > 0$, then so is every point on the line segment from $\mathbf{X}_1$ to $\mathbf{X}_2$.

Definition 14.20.A subset $S \subset \mathbb{R}_c^{\#n}$ is #-connected if it is impossible to represent S as the union of two disjoint nonempty sets such that neither contains a #-limit point of the other; that is, if S cannot be expressed as $S = A \cup B$, where

$$A \neq \varnothing, B \neq \varnothing, \#\text{-}cl(A) \cap B = \varnothing, \#\text{-}cl(B) \cap A = \varnothing. \tag{14.53}$$

If S can be expressed in this way, then S is #-disconnected.

Definition 14.21. A region S in $\mathbb{R}_c^{\#n}$ is the union of an #-open #-connected set with some, all, or none of its #-boundary; thus, $\#\text{-}int(S)$ is #-connected, and every point of S is a #-limit point of $\#\text{-}int(S)$.

14.7.The #-limits and #-continuity $\mathbb{R}_c^{\#}$-valued functions of $n \in \mathbb{N}^{\#}$ variables.

We denote the domain of a function *f* by $\mathbf{D}_f$ and the value of *f* at a point $\mathbf{X} = \{x_i\}_{i=1}^{i=n}$ by $f(\mathbf{X})$ or $f\left(\{x_i\}_{i=1}^{i=n}\right)$.

Definition 14.22. We say that $f(\mathbf{X})$ #-approaches the #-limit L as $\mathbf{X}$ #-approaches $\mathbf{X}_0$ and write

$$\#\text{-}\lim_{\mathbf{X}\to_\# \mathbf{X}_0} f(\mathbf{X}) = L \tag{14.54}$$

if $\mathbf{X}_0$ is a #-limit point of $\mathbf{D}_f$ and, for every $\varepsilon > 0, \varepsilon \approx 0$, there is a $\delta > 0, \delta \approx 0$, such that $|f(\mathbf{X}) - L < \varepsilon|$ for all $\mathbf{X} \in \mathbf{D}_f$ such that $0 < \|\mathbf{X}-\mathbf{X}_0\| < \delta$.

Theorem 14.21.If $\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0} f(\mathbf{X})$ exists, then it is unique.

Theorem 14.22. Suppose that f and g are defined on a set $D \subset \mathbb{R}_c^{\#n}$, $\mathbf{X}_0$ is a #-limit point

of D, and $\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0} f(\mathbf{X}) = L_1, \#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0} g(\mathbf{X}) = L_2$. Then

$$\begin{gathered}\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0}(f \pm g)(\mathbf{X}) = L_1 \pm L_2,\\ \#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0}(f \times g)(\mathbf{X}) = L_1 L_2,\\ \text{and}\\ \#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0}(f/g)(\mathbf{X}) = L_1 L_2,\end{gathered} \tag{14.55}$$

if $L_2 \neq 0$.

Definition 14.23. We say that $f(\mathbf{X})$ #-approaches $\infty^{\#}$ as $\mathbf{X}$ #-approaches $\mathbf{X}_0$ and write

$$\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0} f(\mathbf{X}) = \infty^{\#} \tag{14.56}$$

if $\mathbf{X}_0$ is a #-limit point of $\mathbf{D}_f$ and, for every hyperreal number M, there is a $\delta > 0$, $\delta \approx 0$, such that $f(\mathbf{X}) > M$ whenever $0 < \|\mathbf{X} - \mathbf{X}_0\| < \delta$ and $\mathbf{X} \in \mathbf{D}_f$. We say that

$$\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0} f(\mathbf{X}) = -\infty^{\#} \tag{14.57}$$

if $\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0}[-f(\mathbf{X})] = \infty^{\#}$.

Definition 14.24. If $\mathbf{D}_f$ is hyperunbounded, we say that

$$\#\text{-}\lim_{\|\mathbf{X}\|\to_{\#}\infty^{\#}} f(\mathbf{X}) = L, \tag{14.58}$$

where L finite or hyperfinite if for every $\delta > 0, \delta \approx 0$, there is a number $R \in \mathbb{R}_c^{\#}$ such that $|f(\mathbf{X}) - L < \varepsilon|$ whenever $\|\mathbf{X}\| > R$ and $\mathbf{X} \in \mathbf{D}_f$.

Definition 14.25. If $\mathbf{X}_0 \in \mathbf{D}_f$ and is a #-limit point of $\mathbf{D}_f$, then we say that f is #-continuous at $\mathbf{X}_0$ if

$$\#\text{-}\lim_{\mathbf{X}\to_{\#}\mathbf{X}_0} f(\mathbf{X}) = f(\mathbf{X}_0). \tag{14.59}$$

Theorem 14.23.Suppose that $\mathbf{X}_0 \in \mathbf{D}_f$ and is a #-limit point of $\mathbf{D}_f$. Then f is #-continuous at $\mathbf{X}_0$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(\mathbf{X}) - f(\mathbf{X}_0)| < \varepsilon$ whenever $\|\mathbf{X} - \mathbf{X}_0\| < \delta$ and $\mathbf{X} \in \mathbf{D}_f$.

Definition 14.26.We will say that f is #-continuous on S if f is #-continuous at every point of S.

Theorem 14.24. If f and g are #-continuous on a set $S \subset \mathbb{R}_c^{\#n}$, then so are $f \pm g$, and fg. Also, f/g is #-continuous at each $\mathbf{X}_0 \in S$ such that $g(\mathbf{X}_0) \neq 0$.

Definition 14.27. Suppose that $g_1, g_2, \ldots, g_n, n \in \mathbb{N}^{\#}$ are $\mathbb{R}_c^{\#}$-valued functions defined on a subset $T \subset \mathbb{R}_c^{\#n}$, and define the vector-valued function G on T by

$$G(\mathbf{U}) = (g_1(\mathbf{U}), g_2(\mathbf{U}), \ldots, g_n(\mathbf{U})), \mathbf{U} \in T. \tag{14.60}$$

Then $g_1, g_2, \ldots, g_n$ are the component functions of $G = (g_1, g_2, \ldots, g_n)$.

We say that

$$\#\text{-}\lim_{\mathbf{U}\to_{\#}\mathbf{U}_0} G(\mathbf{U}) = \mathbf{L} = (L_1, \ldots, L_n) \tag{14.61}$$

if $\#\text{-}\lim_{\mathbf{U}\to_{\#}\mathbf{U}_0} g_i(\mathbf{U}) = L_i, 1 \leq i \leq n$ and that G is #-continuous at $\mathbf{U}_0$ if $g_1, g_2, \ldots, g_n$ are each #-continuous at $\mathbf{U}_0$.

Theorem 14.25.For a vector-valued function $G, \#\text{-}\lim_{\mathbf{U}\to_{\#}\mathbf{U}_0} G(\mathbf{U}) = \mathbf{L}$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $\|G(\mathbf{U}) - \mathbf{L}\| < \varepsilon$ whenever $0 < \|\mathbf{U} - \mathbf{U}_0\| < \delta$ and $\mathbf{U} \in \mathbf{D}_G$.Similarly, G is #-continuous at $\mathbf{U}_0$ if and only if for each $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $\|G(\mathbf{U}) - G(\mathbf{U}_0)\| < \varepsilon$ whenever $\|\mathbf{U} - \mathbf{U}_0\| < \delta$ and $\mathbf{U} \in \mathbf{D}_G$.

Theorem 14.26. Let f be a $\mathbb{R}_c^{\#}$-valued function defined on a subset of $\mathbb{R}_c^{\#n}$, and let the vector-valued function $G = (g_1, g_2, \ldots, g_n)$ be defined on a domain $\mathbf{D}_G$ in $\mathbb{R}_c^{\#n}$. Let the set $T = \{\mathbf{U} | \mathbf{U} \in \mathbf{D}_G \text{ and } G(U) \in D_f\}$, be nonempty; and define the $\mathbb{R}_c^{\#}$-valued composite function $h = f \circ G$ on T by $h(\mathbf{U}) = f(G(\mathbf{U})), \mathbf{U} \in T.$ Now suppose that $\mathbf{U}_0 \in T$ and is a #-limit point of T, G is #-continuous at $\mathbf{U}_0$, and f is #-continuous at $\mathbf{X}_0 = G(\mathbf{U}_0)$. Then h is #-continuous at $\mathbf{U}_0$.

Theorem 14.27. If f is #-continuous on a #-compact set $S \subset \mathbb{R}_c^{\#n}$, then f is bounded or hyperbounded on S.

Theorem 14.28. Let f be #-continuous on a compact set $S \subset \mathbb{R}_c^{\#n}$ and $\alpha = \inf_{\mathbf{X} \in S} f(\mathbf{X})$, $\beta = \sup_{\mathbf{X} \in S} f(\mathbf{X})$. Then $f(\mathbf{X}_1) = \alpha$ and $f(\mathbf{X}_2) = \beta$ for some $\mathbf{X}_1$ and $\mathbf{X}_2$ in S.

Theorem 14.29. (Intermediate Value Theorem) Let f be #-continuous on a region $S \subset \mathbb{R}_c^{\#n}$. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are in S and $f(\mathbf{A}) < u < f(\mathbf{B})$. Then $f(\mathbf{C}) = u$ for some $\mathbf{C} \in S$.

Definition 14.28. f is uniformly #-continuous on a subset S of its domain in $\mathbb{R}_c^{\#n}$ if for every $\varepsilon > 0, \varepsilon \approx 0$ there is a $\delta > 0, \delta \approx 0$ such that $|f(X) - f(X_0)| < \varepsilon$ whenever $\|\mathbf{X} - \mathbf{X}_0\| < \delta$ and $\mathbf{X}, \mathbf{X}_0 \in S$.

Theorem 14.30. If f is #-continuous on a #-compact set $S \subset \mathbb{R}_c^{\#n}$ then f is uniformly #-continuous on S.

14.8. Partial #-Derivatives and the #-Differential

Definition 14.29. Let Φ be a unit vector and $\mathbf{X}$ a point in $\mathbb{R}_c^{\#\mathbf{n}}$. The directional #-derivative of $f(\mathbf{X})$ at $\mathbf{X}$ in the direction of Φ is defined by

$$\frac{\partial'^{\#} f(\mathbf{X})}{\partial'^{\#} \Phi} = \#\text{-}\lim_{t \to_{\#} 0} \frac{f(\mathbf{X} + t\Phi) - f(\mathbf{X})}{t} \tag{14.62}$$

if the #-limit exists. That is, $\partial'^{\#} f(\mathbf{X}) / \partial'^{\#} \Phi$ is the ordinary derivative of the function $H(t) = f(\mathbf{X} + t\Phi)$ at $t = 0$, if $H'^{\#}(t)$ exists. The directional #-derivatives that we are most interested in are those in the directions of the unit vectors $\mathbf{E}_i, 1 \leq i \leq n,$ where all components of $\mathbf{E}_i$ are zero except for the i-th, which is 1.

Definition 14.30. Since $\mathbf{X}$ and $\mathbf{X} + t\mathbf{E}_i$ differ only in the i-th coordinate, $\partial'^{\#} f(\mathbf{X}) / \partial'^{\#} \mathbf{E}_i$ is called the partial #-derivative of f with respect to x_i at $\mathbf{X}$. It is also denoted by $\partial'^{\#} f(\mathbf{X}) / \partial'^{\#} x_i$ or $f_{x_i}'^{\#}(\mathbf{X})$, thus,

$$\frac{\partial'^{\#} f(\mathbf{X})}{\partial'^{\#} x_i} = f_{x_i}'^{\#}(\mathbf{X}) = \#\text{-}\lim_{t \to_{\#} 0} \frac{f(\{x_i + t\}_{i \in n}) - f(\{x_i\}_{i \in n})}{t} \tag{14.63}$$

If $\mathbf{X} = (x, y)$, then we denote the partial #-derivatives accordingly; thus,

$$\frac{\partial'^{\#} f(x,y)}{\partial'^{\#} x} = f_x'^{\#}(x,y) = \#\text{-}\lim_{h \to_{\#} 0} \frac{f(x+h,y) - f(x,y)}{h} \tag{14.64}$$

and

$$\frac{\partial'^{\#} f(x,y)}{\partial'^{\#} y} = f_y'^{\#}(x,y) = \#\text{-}\lim_{h \to_{\#} 0} \frac{f(x,y+h) - f(x,y)}{h}. \tag{14.65}$$

Theorem 14.31. If $f_{x_i}'^{\#}(\mathbf{X})$ and $g_{x_i}'^{\#}(\mathbf{X})$ exist, then

$$\frac{\partial'^{\#}(f+g)(\mathbf{X})}{\partial'^{\#}x_i} = f'^{\#}_{x_i}(\mathbf{X}) + g'^{\#}_{x_i}(\mathbf{X}), \frac{\partial'^{\#}(f\times g)(\mathbf{X})}{\partial'^{\#}x_i} = f'^{\#}_{x_i}(\mathbf{X})g(\mathbf{X}) + g'^{\#}_{x_i}(\mathbf{X})f(\mathbf{X}), \quad (14.66)$$

and, if $g(\mathbf{X}) \neq 0$,

$$\frac{\partial'^{\#}(f/g)(\mathbf{X})}{\partial'^{\#}x_i} = \frac{g(\mathbf{X})f'^{\#}_{x_i}(\mathbf{X}) - f(\mathbf{X})g'^{\#}_{x_i}(\mathbf{X})}{[g(\mathbf{X})]^2}. \quad (14.67)$$

If $f'^{\#}_{x_i}(\mathbf{X})$ exists at every point of a set $D \subset \mathbb{R}_c^{\#\mathbf{n}}$, then it defines a function $f'^{\#}_{x_i}(\mathbf{X})$ on D. If this function has a partial #-derivative with respect to x_j on a subset of D, we denote the partial #-derivative by

$$\frac{\partial'^{\#}}{\partial'^{\#}x_j}\left(\frac{\partial'^{\#}f(\mathbf{X})}{\partial'^{\#}x_i}\right) = \frac{\partial^{2\prime\#}f(\mathbf{X})}{\partial'^{\#}x_j\partial'^{\#}x_i} = f'^{\#}_{x_ix_j}(\mathbf{X}). \quad (14.68)$$

The function obtained by differentiating $f(\mathbf{X})$ successively with respect to $x_{i_1}, x_{i_2}, \ldots, x_{i_r}$ is denoted by

$$\frac{\partial^{r\prime\#}f(\mathbf{X})}{\partial'^{\#}x_{i_r}\partial'^{\#}x_{i_{r-1}}\ldots\partial'^{\#}x_{i_1}} = f'^{\#}_{x_{i_1}\ldots x_{i_{r-1}}x_{i_r}}(\mathbf{X}) \quad (14.69)$$

it is an r th-order partial derivative of $f(\mathbf{X})$.

Theorem 14.32.Suppose that $f, f'^{\#}_x, f'^{\#}_y$, and $f'^{\#}_{xy}$ exist on a #-neighborhood Ω of (x_0, y_0), and $f'^{\#}_{xy}$ is #-continuous at (x_0, y_0). Then $f'^{\#}_{yx}$ (x_0, y_0) exists, and

$$f'^{\#}_{yx}(x_0, y_0) = f'^{\#}_{xy}(x_0, y_0). \quad (14.70)$$

Theorem 14.33.Suppose that f and all its partial #-derivatives of order r are #-continuous on an #-open subset S of $\mathbb{R}_c^{\#\mathbf{n}}$. Then

$$f'^{\#}_{x_{i_1},x_{i_2},\ldots,x_{i_r}}(\mathbf{X}) = f'^{\#}_{x_{j_1},x_{j_2},\ldots,x_{j_r}}(\mathbf{X}), \mathbf{X} \in \mathbf{S}, \quad (14.71)$$

if each of the variables $x_1, x_2, \ldots, x_n$ appears the same number of times in $\{x_{i_1}, x_{i_2}, \ldots, x_{i_r}\}$ and $\{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\}$. If this number is r_k, we denote the common value of the two sides of (14.71) by

$$\frac{\partial^{r\prime\#}f(\mathbf{X})}{\partial'^{\#}x_{i_r}\partial'^{\#}x_{i_{r-1}}\ldots\partial'^{\#}x_{i_1}}. \quad (14.72)$$

Definition 14.31. A function $f(\mathbf{X})$ is #-differentiable at $\mathbf{X}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$ if there are constants $m_1, m_2, \ldots, m_n$ such that

$$\#\text{-}\lim_{\|\mathbf{X}-\mathbf{X}_0\|\to_{\#} 0} \frac{f(\mathbf{X}) - f(\mathbf{X}_0) - \left(Ext\text{-}\sum_{i=1}^{n} m_i(x_i - x_{i0})\right)}{\|\mathbf{X}-\mathbf{X}_0\|} = 0. \quad (14.73)$$

Theorem 14.34. If f is differentiable at at $\mathbf{X}_0 = (x_{10}, x_{20}, \ldots, x_{n0})$, then $f'^{\#}_{x_{i0}}(\mathbf{X}_0)$, $1 \leq i \leq n$, exist and the constants $m_i, 1 \leq i \leq n$, in Eq.(14.73) are given by

$$m_i = f'^{\#}_{x_{i0}}(\mathbf{X}_0). \quad (14.74)$$

Theorem 14.35.If f is #-differentiable at $\mathbf{X}_0$, then f is #-continuous at $\mathbf{X}_0$.

Definition 14.32. A linear function $L : \mathbb{R}_c^{\#n} \to \mathbb{R}_c^{\#}$ is a $\mathbb{R}_c^{\#}$-valued function of the form

$$L(\mathbf{X}) = Ext\text{-}\sum_{i=1}^{n} m_i x_i, \quad (14.75)$$

where $m_i, 1 \leq i \leq n$ are constants. From Definition 14.31, f is #-differentiable at $\mathbf{X}_0$ if and only if there is a linear function L such that $f(\mathbf{X}) - f(\mathbf{X}_0)$ can be approximated so well near $\mathbf{X}_0$ by

$$f(\mathbf{X}) - f(\mathbf{X}_0) = L(\mathbf{X}-\mathbf{X}_0) + E(\mathbf{X})\|\mathbf{X}-\mathbf{X}_0\|, \quad (14.76)$$

where

$$\#\text{-}\lim_{\|\mathbf{X}-\mathbf{X}_0\|\to_\# 0} E(\mathbf{X}) = 0. \tag{14.77}$$

Remark 14.3.Theorem 14.34 implies that if f is #-differentiable at $\mathbf{X}_0$, then there is exactly one linear function L that satisfies (14.76) and (14.77).
This function is called the #-differential of f at $\mathbf{X}_0$. We will denote it by $d'^{\#}_{\mathbf{X}_0} f$ and its value by $(d'^{\#}_{\mathbf{X}_0} f)(\mathbf{X})$; thus,

$$(d'^{\#}_{\mathbf{X}_0} f)(\mathbf{X}) = Ext\text{-}\sum_{i=1}^{n} f'^{\#}_{x_{i0}}(\mathbf{X}_0) x_i. \tag{14.78}$$

For convenience in writing $d'^{\#}_{\mathbf{X}_0} f$, and to conform with standard notation, we introduce the function $d'^{\#} x_i : \mathbb{R}_c^{\#n} \to \mathbb{R}_c^{\#}$ defined by $dx_i(\mathbf{X}) = x_i$. That is, $d'^{\#} x_i$ is the function whose value at a point in $\mathbb{R}_c^{\#n}$ is the i-th coordinate of the point. It is the #-differential of the function $g_i(X) = x_i$. From Eq.(14.78)

$$d'^{\#}_{\mathbf{X}_0} f = Ext\text{-}\sum_{i=1}^{n} f'^{\#}_{x_{i0}}(\mathbf{X}_0) d'^{\#} x_i. \tag{14.78}$$

15.#-Analytic functions $f : \mathbb{C}_c^{\#} \to \mathbb{C}_c^{\#}$.

15.1.$\mathbb{C}_c^{\#}$-valued #-analytic functions $f : \mathbb{C}_c^{\#} \to \mathbb{C}_c^{\#}$.

The class of #-analytic functions is formed by the complex functions of a complex variable $z \in \mathbb{C}_c^{\#} = \mathbb{R}_c^{\#} + i\mathbb{R}_c^{\#}$ which possess a #-derivative wherever the function is defined. The term #-holomorphic function is used with identical meaning. For the purpose of this preliminary investigation the reader may think primarily of functions which are defined in the whole plane $\mathbb{C}_c^{\#}$.
The definition of the #-derivative can be written in the form

$$f'^{\#}(z) = \#\text{-}\lim_{h\to_\# 0} \frac{f(z+h) - f(z)}{h} \tag{15.1}$$

As a first obvious consequence $f(z)$ is necessarily #-continuous. Indeed, from $f(z+h) - f(z) = h \times (f(z+h) - f(z))/h$ one obtains $\#\text{-}\lim_{h\to_\# 0}(f(z+h) - f(z)) = 0 \times f'^{\#}(z) = 0$. If we write $f(z) = u(z) + iv(z)$ it follows, moreover, that $u(z)$ and $v(z)$ are both #-continuous.

Remark 15.1. When we consider the #-derivative of a $\mathbb{C}_c^{\#}$-valued function, defined on a set A$\subset \mathbb{C}_c^{\#}$ in the complex plane $\mathbb{C}_c^{\#}$, it is
of course understood that $z \in A$ and that the limit is with respect to values h such that $z + h \in A$. The existence of the #-derivative will therefore have a different meaning depending on whether z is an interior point or a #-boundary point of A. The way to avoid this is to insist that all #-analytic functions be defined on open sets.

Definition 15.1. A $\mathbb{C}_c^{\#}$-valued function $f(z)$, defined on an open set Ω, is said to be $\mathbb{C}_c^{\#}$-analytic in Ω if it has a #-derivative at each point of Ω. And more explicitly that $f(z)$ is #-analytic function. A commonly used synonym is #-holomorphic function.

Definition 15.2.A function $f(z)$ is #-analytic on an arbitrary point set A if it is the restriction to A of a function which is #-analytic in some open set containing A.

Remark 15.2. Note that the real and imaginary parts of an #-analytic function in Ω satisfy the generalized Cauchy-Riemann equations

$$\frac{\partial^{\#} u}{\partial^{\#} x} = \frac{\partial^{\#} v}{\partial^{\#} y}; \frac{\partial^{\#} u}{\partial^{\#} y} = -\frac{\partial^{\#} v}{\partial^{\#} x}. \tag{15.2}$$

Conversely, if u and v satisfy these equations in Ω, and if the partial #-derivatives are #-continuous, then $u+iv$ is an #-analytic function in Ω.

Theorem 15.1. An #-analytic function f in a region Ω whose #-derivative vanishes identically must reduce to a constant. The same is true if either the real part, the imaginary part, the modulus, or the argument is constant.

15.2.The $\mathbb{C}_c^{\#}$-valued #-Exponential Ext-$\exp(z)$.

We define the #-exponential Ext-$\exp(z)$ function as the solution of the differential equation

$$f'^{\#}(z) = f(z), f(0) = 1. \quad (15.3)$$

We solve it by setting

$$f(z) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} a_n z^n, f'^{\#}(z) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} n a_n z^n. \quad (15.4)$$

If Eq.(15.4) is to be satisfied, we must have $a_{n-1} = na_n, n \in \mathbb{N}^{\#}$ and the initial condition gives $a_0 = 1$. It follows by hypee infinite induction that $a_n = 1/n!$.

Abbreviation 14.1. The solution of the Eq.(15.4) is denoted by $Ext\text{-}e^z$ or Ext-$\exp(z)$ or Ext-$\exp z$. Thus finally we obtain

$$Ext\text{-}\exp(z) = Ext\text{-}\sum_{n=0}^{\infty^{\#}} \frac{z^n}{n!}. \quad (15.5)$$

15.3.The $\mathbb{C}_c^{\#}$-valued Trigonometric Functions Ext-$\sin(z)$, Ext-$\cos(z)$.

The $\mathbb{C}_c^{\#}$-valued trigonometric functions Ext-$\sin(z)$, Ext-$\cos(z)$ are defined by

$$Ext\text{-}\sin(z) = \frac{1}{2}(Ext\text{-}\exp(iz) - Ext\text{-}\exp(-iz)) \quad (15.6)$$

and

$$Ext\text{-}\cos(z) = \frac{1}{2}(Ext\text{-}\exp(iz) + Ext\text{-}\exp(-iz)). \quad (15.7)$$

Substitution (14.)-(14.) in (14.) gives that

$$Ext\text{-}\sin(z) = \quad (15.8)$$

and

$$Ext\text{-}\cos(z) = \quad (15.9)$$

From (14) we obtain generalized Euler’s formula

$$Ext\text{-}\exp(iz) = Ext\text{-}\cos(z) + i(Ext\text{-}\sin(z)) \quad (15.10)$$

and as well as the identity

$$(Ext\text{-}\sin(z))^2 + (Ext\text{-}\cos(z))^2 = 1. \quad (15.11)$$

15.4. The periodicity of the #-exponential Ext-$\exp(iz)$.

Definition 15.4.We say that $f(z)$ has the period c if $f(z+c) = f(z)$ for all $z \in \mathbb{C}_c^{\#}$. Thus a period of $Ext\text{-}e^z$ satisfies $Ext\text{-}e^{z+c} = Ext\text{-}e^z$, or $Ext\text{-}e^c = 1$. It follows that $c = i\omega$ with real $\omega \in \mathbb{R}_c^{\#}$ we prefer to say that ω is a period of $Ext\text{-}e^{iz}$. We shall show that there are periods, and that they are all integral multiples of a positive period ω_0.From

$(Ext\text{-}\sin(y))'^{\#} = Ext\text{-}\cos(y) \leq 1$ and $Ext\text{-}\sin(0) = 0$ one obtains $Ext\text{-}\sin(y) < y$ for $y > 0$, either by integration or by use of the generalized mean-value theorem. In the same way $(Ext\text{-}\cos(y))'^{\#} = -Ext\text{-}\sin(y) > -y$ and $Ext\text{-}\cos(0) = 1$ gives $Ext\text{-}\cos(y) > 1 - y^2/2$, which in turn leads to $Ext\text{-}\sin(y) > y - y^3/6$ and finally to $Ext\text{-}\cos(y) < 1 - y^2/2 + y^4/24$. This inequality shows that $Ext\text{-}\cos\left(\sqrt{3}\right) < 0$, and therefore there is a y_0 such that $0 < y_0 < \sqrt{3}$ and $Ext\text{-}\cos(y_0) = 0$. Because $(Ext\text{-}\sin(y_0))^2 + (Ext\text{-}\cos(y_0))^2 = 1$, we have $Ext\text{-}\sin(y_0) = \pm 1$, that is, $Ext\text{-}e^{iy_0} = \pm i$, and hence $Ext\text{-}e^{4iy_0} = 1$. We have shown that $4y_0$ is a period. Actually, it is the smallest positive period. To see this, take $0 < y < y_0$.
Then $Ext\text{-}\sin(y) > y(1 - y^2/6) > y/2 > 0$, which shows that $Ext\text{-}\cos(y)$ is strictly decreasing. Because $Ext\text{-}\sin(y)$ is positive and $(Ext\text{-}\sin(y))^2 + (Ext\text{-}\cos(y))^2 = 1$ it follows that $Ext\text{-}\sin(y)$ is strictly increasing, and hence $Ext\text{-}\sin(y) < Ext\text{-}\sin(y_0) = 1$.
The double inequality $0 < Ext\text{-}\sin(y) < 1$ guarantees that $Ext\text{-}e^{iy}$ is neither ± 1 nor $\pm i$. Therefore $Ext\text{-}e^{4iy} \neq 1$, and $4y_0$ is indeed the smallest positive period.We denote it by ω_0.Consider now an arbitrary period ω_0. There exists an integer n such that $n\omega_0 \leq \omega < (n+1)\omega_0$. If w were not equal to $n\omega_0$, then ω - $n\omega_0$ would be a positive period $< \omega_0$. Since this is not possible, every period must be an integral multiple of ω_0.
Abbreviation 15.2.The smallest positive period of $Ext\text{-}e^{iz}$ is denoted by $2\pi_{\#}$.
Remark 15.3.Note that $\mathbf{st}(\pi_{\#}) = \pi \in \mathbb{R}$.

15.5.The $\mathbb{C}_c^{\#}$-valued Logarithm.

Together with the exponential function $Ext\text{-}e^{iz}$ we must also introduce its inverse function, the $\mathbb{C}_c^{\#}$-valued logarithm. By definition, $z = Ext\text{-}\log w$ is a root of the equation $Ext\text{-}e^{iz} = w$. First of all, since $Ext\text{-}e^{iz}$ is always $\neq 0$, the number 0 has no logarithm. For $w \neq 0$ the equation $Ext\text{-}e^{x+iv} = w$ is equivalent to

$$Ext\text{-}e^{iz} = |w|, Ext\text{-}e^{iy} = \frac{w}{|w|}. \tag{15.12}$$

The first equation has a unique solution $x = Ext\text{-}\log|w|$, the $\mathbb{R}_c^{\#}$-valued logarithm of the positive number $|w| \in \mathbb{R}_c^{\#}$. The right-hand member of the second equation (15.12) is a complex number in $\mathbb{C}_c^{\#}$ of absolute value 1. Therefore, as we have just seen, it has one and only one solution in the interval $0 \leq y < 2\pi_{\#}$. In addition, it is also satisfied by all y that differ from this solution by an integral multiple of $2\pi_{\#}$. We see that every complex number other than 0 has hyper infinitely many logarithms which differ from each other by multiples of $2\pi_{\#}i$.
The imaginary part of $Ext\text{-}\log w$ is also called the argument of w, $Ext\text{-}\arg w$, and it is interpreted geometrically as the angle, measured in radians, between the positive real axis and the half line from 0 through the point w. According to this definition the argument has hyper infinitely many values which differ by multiples of $2\pi_{\#}$, and

$$Ext\text{-}\log w = Ext\text{-}\log|w| + i\arg w. \tag{15.13}$$

Remark 15.4.The addition property of the exponential function $Ext\text{-}e^{iz}$ implies

$$\begin{aligned} Ext\text{-}\log(z_1 \times z_2) &= Ext\text{-}\log z_1 + Ext\text{-}\log z_2, \\ Ext\text{-}\arg(z_1 \times z_2) &= Ext\text{-}\arg z_1 + Ext\text{-}\arg z_2, \end{aligned} \tag{15.14}$$

but only in the sense that both sides represent the same hyper infinite set of complex numbers.The inverse of $Ext\text{-}\cos(z)$ is obtained by solving the equation

$$Ext\text{-}\cos(z) = \frac{1}{2}(Ext\text{-}e^{iz} + Ext\text{-}e^{-iz}) = w. \quad (15.15)$$

This is a quadratic equation in $Ext\text{-}e^{iz}$ with the roots

$$Ext\text{-}e^{iz} = w \pm \sqrt{w^2 - 1} \quad (15.16)$$

and therefore

$$z = Ext\text{-}\arccos(w) = -i\left(Ext\text{-}\log\left(w \pm \sqrt{w^2 + 1}\right)\right), \quad (15.17)$$

or in the form

$$Ext\text{-}\arccos(w) = \pm i\left(Ext\text{-}\log\left(w + \sqrt{w^2 + 1}\right)\right) \quad (15.18)$$

The hyper infinitely many values of $Ext\text{-}\arccos(w)$ reflect the evenness and periodicity of $Ext\text{-}\cos(w)$. The inverse sine is most easily defined by formula

$$Ext\text{-}\arcsin(w) = \frac{\pi_{\#}}{2} - (Ext\text{-}\arccos(w)). \quad (15.19)$$

16.Complex Integration of the $\mathbb{C}_c^{\#}$-valued function $f(t)$.

16.1.Definition and basic properties of the complex integral.

If $f(t) = u(t) + iv(t)$ is a #-continuous function, defined in an interval (a,b), we set by definition

$$Ext\text{-}\int_a^b f(t)d^{\#}t = Ext\text{-}\int_a^b u(t)d^{\#}t + i\left(Ext\text{-}\int_a^b v(t)d^{\#}t\right). \quad (16.1)$$

This integral has most of the properties of the real integral. In particular, if $c = \alpha + i\beta$ is a complex constant we obtain

$$Ext\text{-}\int_a^b cf(t)d^{\#}t = c\left(Ext\text{-}\int_a^b f(t)d^{\#}t\right). \quad (16.2)$$

The fundamental inequality

$$\left|Ext\text{-}\int_a^b f(t)d^{\#}t\right| \leq Ext\text{-}\int_a^b |f(t)|d^{\#}t. \quad (16.3)$$

holds for arbitrary $\mathbb{C}_c^{\#}$-valued function $f(t)$.

We consider now a piecewise #-differentiable arc γ with the equation $z = z(t), a \leq t \leq b.$

If the function $f(z)$ is defined and #-continuous on γ, then $f(z(t))$ is also #-continuous and we can set

$$v \quad Ext\text{-}\int_\gamma f(z)d^{\#}z = Ext\text{-}\int_a^b f(z(t))z'^{\#}(t)d^{\#}t. \quad (16.4)$$

The most important property of the integral (16.4) is its invariance under a change of parameter. A change of parameter is determined by an increasing function $t = t(\tau)$ which maps an interval $\alpha \leq \tau \leq \beta$ onto $a \leq t \leq b$; we assume that $t(\tau)$ is piecewise

#-differentiable. By the rule for changing the variable of integration we get

$$Ext\text{-}\int_a^b f(z(t))z'^{\#}(t)d^{\#}t = Ext\text{-}\int_\alpha^\beta f(z(t(\tau)))z'^{\#}(t(\tau))t'^{\#}(\tau)d^{\#}\tau. \quad (16.5)$$

We defined the opposite arc $-\gamma$ by the equation $z = z(-t), -b \leq t \leq -a$. We have thus

$$Ext\text{-}\int_{-\gamma} f(z)d^{\#}z = -\left(Ext\text{-}\int_\gamma f(z)d^{\#}z\right). \quad (16.6)$$

The integral (16.4) has also a very obvious additive property. It is clear what is meant by subdividing an arc γ into a finite or hyperfinite number of subarcs. A subdivision can be indicated by a symbolic equation: $\gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_n, n \in \mathbb{N}^{\#}$, and the corresponding integrals satisfy the relation

$$Ext\text{-}\int_{\gamma_1+\gamma_2+\ldots+\gamma_n} f(z)d^{\#}z = Ext\text{-}\sum_{i=1}^{n}\left(Ext\text{-}\int_{\gamma_i} f(z)d^{\#}z\right). \quad (16.7)$$

Finally, the integral over a closed curve is also invariant under a shift of parameter. The old and the new initial point determine two subarcs γ_1, γ_2, and the invariance follows from the fact that the integral over $\gamma_1 + \gamma_2$ is equal to the integral over $\gamma_2 + \gamma_1$
In addition to integrals of the form (16.4) we can also consider line integrals with respect to $\bar{z}$. The most convenient definition is by double conjugation

$$Ext\text{-}\int_\gamma f(z)\overline{d^{\#}z} = \overline{Ext\text{-}\int_\gamma f(z)d^{\#}z}. \quad (16.8)$$

Using notation (16.7), line integrals with respect to x or y can be introduced by

$$\begin{aligned} Ext\text{-}\int_\gamma f(z)d^{\#}x &= \frac{1}{2}\left(Ext\text{-}\int_\gamma f(z)d^{\#}z + Ext\text{-}\int_\gamma f(z)\overline{d^{\#}z}\right), \\ Ext\text{-}\int_\gamma f(z)d^{\#}y &= \frac{1}{2i}\left(Ext\text{-}\int_\gamma f(z)d^{\#}z - Ext\text{-}\int_\gamma f(z)\overline{d^{\#}z}\right). \end{aligned} \quad (16.9)$$

With $f = u + iv$ we find that the integral (16.4) can be written in the form

$$Ext\text{-}\int_\gamma (ud^{\#}x - vd^{\#}y) + i\left(Ext\text{-}\int_\gamma (ud^{\#}y + vd^{\#}x)\right). \quad (16.10)$$

Of course we could just as well have started by defining integrals of the form

$$Ext\text{-}\int_\gamma (pd^{\#}x + qd^{\#}y), \quad (16.11)$$

in which case formula (16.10) would serve as definition of the integral (16.4).
An essentially different line integral is obtained by integration with
respect to arc length. Two notations are in common use, and the definition is

$$Ext\text{-}\int_{\gamma} fd^{\#}s = Ext\text{-}\int_{\gamma} f(z)|d^{\#}z| = Ext\text{-}\int_{\gamma} f(z(t))|z'^{\#}(t)|d^{\#}t. \quad (16.12)$$

This integral is again independent of the choice of parameter. In contrast to (16.6) we get

$$Ext\text{-}\int_{-\gamma} f(z)|d^{\#}z| = Ext\text{-}\int_{\gamma} f(z)|d^{\#}z|, \quad (16.13)$$

while (16.7) remains valid in the same form. The inequality

$$\left|Ext\text{-}\int_{\gamma} f(z)d^{\#}z\right| \leq Ext\text{-}\int_{\gamma}|f(z)||d^{\#}z| \quad (16.14)$$

is a consequence of (16.3).

Remark 16.1.For $f \equiv 1$ the integral (16.12) reduces to $\int_{\gamma} |dz|$ which is by definition the length of γ. As an example we compute the length of a circle. From the parametric equation $z = z(t) = a + \rho(Ext\text{-}e^{it}),\ 0 \leq t \leq 2\pi_{\#}r,$ of a full circle we obtain $z'^{\#}(t) = i\rho(Ext\text{-}e^{it})$ and hence

$$\int_{0}^{2\pi_{\#}} |z'^{\#}(t)|d^{\#}t = \int_{0}^{2\pi_{\#}} \rho d^{\#}t = 2\pi_{\#}\rho \quad (16.15)$$

as expected.

16.2.Line Integrals as Functions of Arcs.

Remind that the length of an arc can also be defined as the least upper bound of all hyperfinite sums

$$Ext\text{-}\sum_{i=1}^{n}|z(t_i) - z(t_{i-1})|, \quad (16.16)$$

$n \in \mathbb{N}^{\#}/\mathbb{N},$ where $a = t_0 < t_1 < \ldots < t_n = b.$ If this least upper bound is finite or hyperfinite we say that the arc is rectifiable. It is quite easy to show that piecewise #-differentiable arcs are rectifiable, and that the two definitions of length coincide.
It is clear that the sums (16.6) and the corresponding sums

$$Ext\text{-}\sum_{i=1}^{n}|x(t_i) - x(t_{i-1})|; Ext\text{-}\sum_{i=1}^{n}|y(t_i) - y(t_{i-1})|, \quad (16.17)$$

where $z(t) = x(t) + iy(t),$ are bounded or hyperbounded at the same time. When the latter sums are bounded (or hyperbounded), one says that the functions $x(t)$ and $y(t)$ are of bounded (or hyperbounded) variation. An arc $z = z(t)$ is rectifiable if and only if the real and imaginary parts of $z(t)$ are of bounded (or hyperbounded) variation.
If γ is rectifiable and $f(z)$ #-continuous on γ it is possible to define integrals of type (16.12) as a #-limit

$$Ext\text{-}\int_{\gamma} fd^{\#}s = \#\text{-}\lim_{n\to\infty^{\#}}\left(Ext\text{-}\sum_{k=1}^{n} f(z(t_k))|z(t_i) - z(t_{i-1})|\right). \quad (16.18)$$

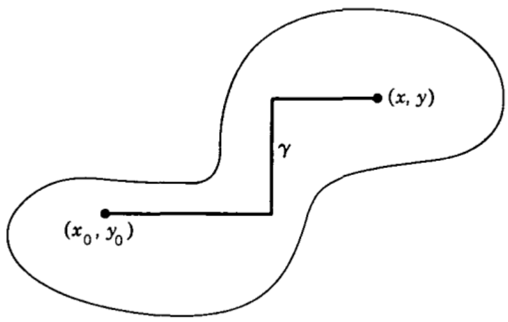
General line integral of the form $Ext\text{-}\int_{\gamma}(pd^{\#}x + qd^{\#}y)$ can be considered as functional of the arc γ. It is then assumed that p and q are defined and #-continuous in a region Ω and that γ is free to vary in Ω. An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. In other words, if γ_1 and γ_2 have the same initial point and the same end point, we require that

$$Ext\text{-}\int_{\gamma_1}(pd^{\#}x + qd^{\#}y) = Ext\text{-}\int_{\gamma_2}(pd^{\#}x + qd^{\#}y). \tag{16.19}$$

To say that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if γ is a closed curve, then γ and $-\gamma$ have the same end points, and if the integral depends only on the end points, we obtain

$$Ext\text{-}\int_{\gamma}(pd^{\#}x + qd^{\#}y) = Ext\text{-}\int_{-\gamma}(pd^{\#}x + qd^{\#}y) = -\left(Ext\text{-}\int_{\gamma}(pd^{\#}x + qd^{\#}y)\right) \tag{16.20}$$

and consequently $\int_{\gamma}(pd^{\#}x + qd^{\#}y) = 0$. Conversely, if γ_1 and γ_2 have the same end points, then $\gamma_1 - \gamma_2$ is a closed curve, and if the integral over any closed curve vanishes, it follows that $Ext\text{-}\int_{\gamma_1}(pd^{\#}x + qd^{\#}y) = Ext\text{-}\int_{\gamma_2}(pd^{\#}x + qd^{\#}y)$.



Pic. 1.

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem 16.1.The line integral $Ext\text{-}\int_{\gamma}(pd^{\#}x + qd^{\#}y)$, defined in Ω, depends only on the end points of γ if und only if there exists a function $U(x,y)$ in Ω with the partial #-derivatives $\partial^{\#}u/\partial^{\#}x = p, \partial^{\#}u/\partial^{\#}y = q$.

The sufficiency follows at once, for if the condition is fulfilled we can write, with the usual notations,

$$\begin{gathered}Ext\text{-}\int_{\gamma}(pd^{\#}x + qd^{\#}y) = Ext\text{-}\int_{a}^{b}\left[\frac{\partial^{\#}U}{\partial^{\#}x}x'^{\#}(t) + \frac{\partial^{\#}U}{\partial^{\#}y}y'^{\#}(t)\right]d^{\#}t = \\ Ext\text{-}\int_{a}^{b}\frac{d^{\#}}{d^{\#}t}U(x(t),y(t))d^{\#}t = U(x(b),y(b)) - U(x(a),y(a)).\end{gathered} \tag{16.21}$$

and the value of this difference depends only on the end points. To prove the necessity we choose a fixed point $(x_0,y_0) \in \Omega$, join it to (x,y) by a polygon γ, contained in Ω, whose sides are parallel to the coordinate axes (Pic.1) and define a function $U(x,y)$ by

$$U(x,y) = Ext\text{-}\int_\gamma (pd^\#x + qd^\#y). \tag{16.22}$$

Since the integral depends only on the end points, the function is well defined. Moreover, if we choose the last segment of γ horizontal, we can keep y constant and let x vary without changing the other segments. On the last segment we can choose x for parameter and obtain

$$U(x,y) = Ext\text{-}\int^x p(x,y)d^\#x + const., \tag{16.23}$$

the lower limit of the integral being irrelevant. From Eq.(16.23) it follows at once that $\frac{\partial^\# U}{\partial^\# x} = p.$ In the same way, by choosing the last segment vertical, we can show that $\frac{\partial^\# U}{\partial^\# y} = q.$ It is customary to write $d^\#U = (\partial^\# U/\partial^\# x)d^\#x + (\partial^\# U/\partial^\# y)d^\#y$ and to say that an expression $pd^\#x + qd^\#y$ which can be written in this form is an exact #-differential. Thus an integral depends only on the end points if and only if the integrand is an exact differential. Observe that p,q and U can be either real or complex. The function $U,$ if it exists, is uniquely determined up to an additive constant, for if two functions have the same partial #-derivatives their #-difference must be constant.
When is $f(z)d^\#z = f(z)d^\#x + if(z)d^\#y$ an exact #-differential? According to the definition there must exist a function $F(z)$ in Ω with the partial #-derivatives

$$\frac{\partial^\# F(z)}{\partial^\# x} = f(z), \frac{\partial^\# F(z)}{\partial^\# y} = if(z). \tag{16.24}$$

If this is so, $F(z)$ fulfills the generalized Cauchy-Riemann equation

$$\frac{\partial^\# F(z)}{\partial^\# x} = i\frac{\partial^\# F(z)}{\partial^\# y}, \tag{16.25}$$

since $f(z)$ is by assumption #-continuous $F(z)$ is #-analytic with the #-derivative $f(z)$. The integral $Ext\text{-}\int_\gamma fd^\#z,$ with #-continuous $f,$ depends only on the end points of γ if and only iff is the derivative of an analytic function in Ω. Under these circumstances we shall prove later that $f(z)$ is itself #-analytic.
As an immediate application of the above result we find that

$$\int_\gamma (z-a)^n d^\#z = 0 \tag{16.26}$$

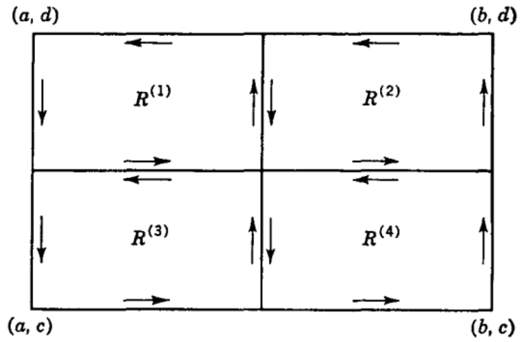
for all closed curves γ, provided that the integer $n \in \mathbb{N}^\#$ is $\geq 0.$ In fact, $(z-a)^n$ is the #-derivative of $(z-a)^{n+1}/(n+1),$ a function which is #-analytic in the whole plane $\mathbb{C}_c^\#.$ If n is negative, but $\neq -1,$ the same result holds for all closed curves which do not pass through $a,$ for in the complementary region of the point a the indefinite integral is still #-analytic and single-valued. For $n = -1,$ Eq.(16.26) does not always hold. Consider a circle C with the center $a,$ represented by the equation $z = a + \rho(Ext\text{-}e^{it}),$ $0 \leq t \leq 2\pi_\#.$ We obtain

$$\int_{\gamma} \frac{d^{\#}z}{(z-a)} = \int_{0}^{2\pi_{\#}} id^{\#}t = 2\pi_{\#}i. \quad (16.27)$$

This result shows that it is impossible to define a single-valued branch of $Ext\text{-}\log(z-a)$ in an annulus $\rho_1 < |z-a| < \rho_2$. On the other hand, if the closed curve γ is contained in a half plane which does not contain a, the integral vanishes, for in such a half plane a single-valued and #-analytic branch of $Ext\text{-}\log\,(z-a)$ can be defined.

16.3.Generalized Cauchy's Theorem for a Rectangle.

We consider, specifically, a rectangle $R \subset \mathbb{C}_c^{\#}$ defined by inequalities $a \leq x \leq b$, $c \leq y \leq d$. Its perimeter can be considered as a simple closed curve consisting of four line segments whose direction we choose so that R lies to the left of the directed segments. The order of the vertices is thus $(a,c),(b,c),(b,d),(a,d)$. We refer to this closed curve as the boundary curve or contour of R, and we denote it by $\partial^{\#}R$



Pic.2. Bisection of rectangle.

Theorem 16.2. If the function $f(z)$ is #-analytic on R, then

$$Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z = 0. \quad (16.28)$$

Proof. The proof is based on the method of bisection. Let us introduce the notation

$$\eta(R) = Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z\,. \quad (16.29)$$

If R is divided into four congruent rectangles $R^{(1)},R^{(2)},R^{(3)},R^{(4)}$, we get

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}). \quad (16.30)$$

for the integrals over the common sides cancel each other,see Pic.1.It follows from Eq.(16.) that at least one of the rectangles $R^{(k)}, k = 1,2,3,4,$ must satisfy the condition $|\eta(R^{(k)})| \geq |\eta(R)|/4$. This process can be repeated inductively by hyper infinite induction, and we obtain a hyper infinite sequence of nested rectangles $R \supset R_1 \supset R_2 \ldots \supset$ $\supset R_n \ldots \supset \ldots$ with the property $|\eta(R_n)| \geq 4^{-n}|\eta(R_{n-1})|, n \in \mathbb{N}^{\#}$. Thus

$$|\eta(R_n)| \geq 4^{-n}|\eta(R)|. \quad (16.31)$$

The rectangles R_n converge to a point $z^* \in R$ in the sense that R_n will be contained in a prescribed neighborhood $|z-z^*| < \delta$ as soon as $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ is sufficiently large.

First of all, we choose δ so small that $f(z)$ is defined and #-analytic in $|z - z^*| < \delta$, $\delta \approx 0$. Secondly, if $\varepsilon > 0, \varepsilon \approx 0$ is given, we can choose δ such that

$$\left|\frac{f(z) - f(z^*)}{z - z^*} - f'^{\#}(z^*)\right| < \varepsilon, \tag{16.32}$$

and therefore

$$\left|f(z) - f(z^*) - (z - z^*)f'^{\#}(z^*)\right| < \varepsilon|z - z^*|. \tag{16.33}$$

for $|z - z^*| < \delta$. We assume that δ satisfies both conditions and that R_n is contained in $|z - z^*| < \delta$. We make now the observation that

$$Ext\text{-}\int_{\partial^{\#}R_n} d^{\#}z = 0, Ext\text{-}\int_{\partial^{\#}R_n} zd^{\#}z = 0 \tag{16.34}$$

By virtue of the equations (16.34) we are able to write

$$|\eta(R_n)| = Ext\text{-}\int_{\partial^{\#}R_n} \left|f(z) - f(z^*) - (z - z^*)f'^{\#}(z^*)\right|d^{\#}z \tag{16.35}$$

and it follows by (16.33) that

$$|\eta(R_n)| \leq \varepsilon\left(Ext\text{-}\int_{\partial^{\#}R_n} |z - z^*| \times |d^{\#}z|\right). \tag{16.36}$$

In the last integral $|z - z^*|$ is at most equal to the length d_n of the diagonal of R_n. If L_n denotes the length of the perimeter of R_n, the integral is hence $\leq d_nL_n$. But if d and L are the corresponding quantities for the original rectangle R, it is clear that $d_n = 2^{-n}d$ and $L_n = 2^{-n}L$. By (16.36) we have hence

$$|\eta(R_n)| \leq 4^{-n}dL\varepsilon \tag{16.37}$$

and comparison with (16.31) yields

$$|\eta(R)| \leq dL\varepsilon. \tag{16.38}$$

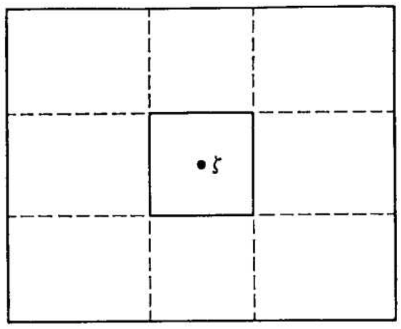
Since $\varepsilon \approx 0$ is arbitrary, we can only have $\eta(R) \equiv 0$, and the theorem is proved.

Theorem 16.3.Let $f(z)$ be #-analytic on the set R' obtained from a rectangle R by omitting a finite or hyperfinite number of interior points ζ_j If it is true that #-$\lim_{z \to_{\#} \zeta_j}(z - \zeta_j)f(z) = 0$ for all $j \in \mathbb{N}^{\#}$, then $Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z = 0$.

Proof. It is sufficient to consider the case of a single exceptional point ζ, for evidently R can be divided into smaller rectangles which contain at most one ζ_j. We divide now R into nine rectangles, as shown in Pic.2, and apply Theorem 16.2 to all but the rectangle R_0 in the center. If the corresponding equations (16.28) are added, we obtain,

after cancellations,

$$Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z = Ext\text{-}\int_{\partial^{\#}R_0} f(z)d^{\#}z \tag{16.39}$$



Pic. 3.

If $\varepsilon > 0, \varepsilon \approx 0$ we can choose the rectangle R_0 so infinite small that $|f(z)| \leq \varepsilon|z-\zeta|$ on $\partial^{\#}R_0$. By (16.39) we have thus

$$\left| Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z \right| = \varepsilon\left(Ext\text{-}\int_{\partial^{\#}R_0} \frac{|d^{\#}z|}{|z-\zeta|} \right) \quad (16.40)$$

If we assume, as we may, that R_0 is a square of center ζ, elementary estimates show that

$$Ext\text{-}\int_{\partial^{\#}R_0} \frac{|d^{\#}z|}{|z-\zeta|} < 8. \quad (16.41)$$

Thus finally we obtain

$$\left| Ext\text{-}\int_{\partial^{\#}R} f(z)d^{\#}z \right| < 8\varepsilon. \quad (16.42)$$

and since ε is arbitrary the theorem follows.We conclude that the hypothesis of the theorem is certainly fulfilled if $f(z)$ is #-analytic and bounded or hyperbounded on R'.

16.4.Generalized Cauchy's Theorem in a Disk.

It is not true that the integral of an #-analytic function over a closed curve is always zero. For example

$$\int_{C} \frac{d^{\#}z}{|z-a|} = 2i\pi_{\#}. \quad (16.43)$$

Theorem 16.4.If $f(z)$ is #-analytic in an open disk Δ, then

$$Ext\text{-}\int_{\gamma} f(z)d^{\#}z = 0 \quad (16.44)$$

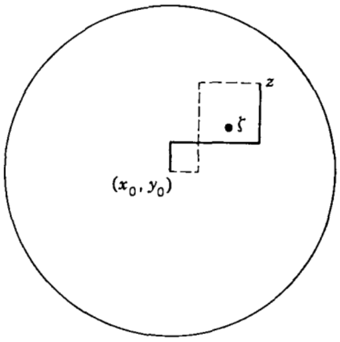
for every closed curve $\gamma \subset \Delta$.

Proof. We define a function $F(z)$ by

$$F(z) = Ext\text{-}\int_{\sigma} f(z)d^{\#}z, \quad (16.45)$$

where σ consists of the horizontal line segment from the center (x_0,y_0) to (x,y_0) and the vertical segment from (x,y_0) to (x,y); it is immediately seen that $\partial^{\#}F/\partial^{\#}y = if(z)$. On the other hand, by Theorem 16.2 σ can be replaced by a path consisting of a vertical segment followed by a horizontal segment. This choice defines the same function $F(z)$, and we obtain $\partial^{\#}F/\partial^{\#}x = f(z)$. Hence $F(z)$ is #-analytic in Δ. with the #-derivative $f(z)$, and $f(z)d^{\#}z$ is an exact #-differential.

Theorem 16.5. Let $f(z)$ be #-analytic in the region Δ' obtained by omitting a finite or hyperfinite number of points ζ_j from an open disk Δ. If $f(z)$ satisfies the condition #-$\lim_{z\to_{\#}\zeta_j}(z-\zeta_j)f(z) = 0$ for all j, then (16.44) holds for any closed curve $\gamma \subset \Delta'$.



Pic.4.

The proof must be modified, for we cannot let rr pass through the exceptional points. Assume first that no ζ_j lies on the lines $x = x_0$ and $y = y_0$. It is then possible to avoid the exceptional points by letting σ consist of three segments (Pic.4). By an obvious application of Theorem 16.3 we find that the value of $F(z)$ in (16.44) is independent of the choice of the middle segment; moreover, the last segment can be either vertical or horizontal. We conclude as before that $F(z)$ is an indefinite integral of $f(z)$, and the theorem follows..

16.5.Generalized Cauchy's integral formula.

Through a very simple application of the generalized Cauchy's theorem it becomes possible to represent an #-analytic function $f(z)$ as a line integral in which the variable $z \in \mathbb{R}_c^{\#}$ enters as a parameter. This representation, known in classical case as Cauchy's integral formula,has numerous important applications. Above all, it enables us to study the local properties of an #-analytic function in full detail.

Lemma 16.1. If the piecewise #-differentiable closed curve γ does not pass through the point a, then the value of the integral

$$\int_{\gamma} \frac{d^{\#}z}{|z-a|}. \tag{16.46}$$

is a multiple of $2i\pi_{\#}$.

Definition 16.1.We define the index of the point a with respect to the curve γ by the equation

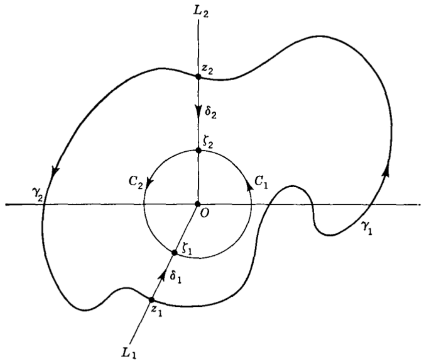
$$n(\gamma,a) = \frac{1}{2\pi_{\#}i}\int_{\gamma}\frac{d^{\#}z}{z-a}. \tag{16.47}$$

The index (16.47) is also called the winding number of γ with respect to a. It is clear that $n(-\gamma,a) = -n(\gamma,a)$.The following property is an immediate consequence of

Theorem 16.4.
(i) If γ lies inside of a circle, then $n(\gamma, a) = 0$ for all points a outside of the same circle.
As a point set γ is #-closed and bounded (or hyperbounded). Its complement is #-open and can be represented as a union of disjoint regions, the components of the complement. We shall say, for short, that γ determines these regions.
If the complementary regions are considered in the extended plane, there is exactly one which contains the point at infinity. Consequently, γ determines one and only one unbounded region.
(ii) As a function of a the index $n(\gamma, a)$ is constant in each of the regions determined by γ, and zero in the unbounded region.
Any two points in the same region determined by γ can be joined by a polygon which does not meet γ. For this reason it is sufficient to prove that $n(\gamma, a) = n(\gamma, b)$ if γ does not meet the line segment from a to b. Outside of this segment the function $(z-a)/(z-b)$ is never real and ≤ 0. For this reason the principal branch of $Ext\text{-}\log[(z-a)/(z-b)]$ is #-analytic in the complement of the segment. Its derivative is equal to $(z-a)^{-1} - (z-b)^{-1}$, and if γ does not meet the segment we get

$$Ext\text{-}\int_{\gamma}\left(\frac{1}{z-a} - \frac{1}{z-b}\right)d^{\#}z = 0; \qquad (16.48)$$

hence $n(\gamma, a) = n(\gamma, b)$. If laI is sufficiently large, γ is contained in a disk $|z| < \rho < |a|$ and we conclude by (i) that $n(\gamma, a) = 0$. This proves that $n(\gamma, a) = 0$ in the unbounded region.
We shall find the case $n(\gamma, a) = 1$ particularly important, and it is desirable to formulate a geometric condition which leads to this consequence.
For simplicity we take $a = 0$.
Lemma 16.2. Let z_1, z_2 be two points on a closed curve γ which does not pass through the origin. Denote the subarc from z_1 to z_2 in the direction of the curve by γ_1, and the subarc from z_2 to z_1 by γ_2. Suppose that z_1 lies in the lower half plane and z_2 in the upper half plane. If γ_1 does not meet the negative real axis and γ_2 does not meet the positive real axis, then $n(\gamma, 0) = 1$.
For the proof we draw the half lines L_1 and L_2 from the origin through z_1 and z_2 (Pic. 4-5). Let s_1, s_2 be the points in which L_1, L_2 intersect a circle C about the origin. If C is described in the positive sense, the arc C_1 from s_1 to s_2 does not intersect the negative axis, and the arc C_2 from s_2 to s_1 does not intersect the positive axis. Denote the directed line segments from z_1 to s_1 and from z_2 to s_2 by δ_1, δ_2. Introducing the closed curves $\sigma_1 = \gamma_1 + \delta_2 - C_1 - \delta_1$, $\sigma_2 = \gamma_2 + \delta_1 - C_2 - \delta_2$ we get that $n(\gamma, 0) = n(C, 0) + n(\sigma_1, 0) + n(\sigma_2, 0)$ because of cancellations. But σ_1 does not meet the negative axis. Hence the origin belongs to the unbounded region determined by σ_1, and we obtain $n(\sigma_1, 0) = 0$. For a similar reason $n(\sigma_2, 0) = 0$, and we conclude that $n(\gamma, 0) = n(C, 0) = 1$.



Pic.5

Let $f(z)$ be #-analytic in an open disk Δ. Consider a closed curve $\gamma \subset \Delta$. and a point $a \in \Delta$
which does not lie on γ. We apply Cauchy's theorem to the function

$$F(z) = \frac{f(z) - f(a)}{z - a}. \tag{16.49}$$

This function is analytic for $z \neq a$. For $z = a$ it is not defined, but it satisfies the condition $\#\text{-}\lim_{z \to_\# a}[(z-a)F(z)] = \#\text{-}\lim_{z \to_\# a}[f(z) - f(a)] = 0,$ which is the condition of Theorem 16.5. We conclude that

$$Ext\text{-}\int_\gamma \frac{f(z) - f(a)}{z - a} d^\# z = 0. \tag{16.50}$$

This equation can be rewritten in the form

$$Ext\text{-}\int_\gamma \frac{f(z) d^\# z}{z - a} = f(a)\left(Ext\text{-}\int_\gamma \frac{d^\# z}{z - a} \right), \tag{16.51}$$

and we observe that the integral in the right-hand member is by definition $2\pi_\# i n(\gamma, a)$.
Theorem 16.6. Suppose that $f(z)$ is #-analytic in an open disk Δ, and let γ be a closed curve in Δ. For any point a not on γ

$$n(\gamma, a) \times f(a) = \frac{1}{2\pi_\# i}\left(Ext\text{-}\int_\gamma \frac{f(z) d^\# z}{z - a} \right), \tag{16.52}$$

where $n(\gamma, a)$ is the index of a with respect to γ.
In this statement we have suppressed the requirement that a be a point in Δ. We have done so in view of the obvious interpretation of the formula (16.) for the case that a is not in Δ. Indeed, in this case $n(\gamma, a)$ and the integral in the right-hand member are both zero.
It is clear that Theorem 6 remains valid for any region Ω to which Theorem 16.5 can be applied. The presence of exceptional points ζ_j is permitted, provided none of them coincides with a.
The most common application is to the case where $n(\gamma, a) = 1$. We have then

$$f(a) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{\gamma} \frac{f(z)d^{\#}z}{z-a} \right) \quad (16.)$$

and this we interpret as a representation formula. Indeed, it permits us to compute $f(a)$ as soon as the values of $f(z)$ on γ are given, together with the fact that $f(z)$ is #-analytic in Δ. In (16.) we may let a take different values, provided that the order of a with respect to γ remains equal to 1. We may thus treat a as a variable, and it is convenient to change the notation and rewrite (16.) in the form

$$f(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{\gamma} \frac{f(\zeta)d^{\#}\zeta}{\zeta-z} \right) \quad (16.)$$

It is this formula which is usually referred to as Cauchy's integral formula. We must remember thflt it is valid only when $n(\gamma,z) = 1$, and that we have proved it only when $f(z)$ is #-analytic in a disk.

The representation formula (22) gives us a tool for the study of the local properties of #-analytic functions. In particular we can now show that an #-analytic function has #-derivatives of all orders $n \in \mathbb{N}^{\#}$, which are then also #-analytic.

We consider a function $f(z)$ which is #-analytic in an arbitrary region Ω.

To a point $a \in \Omega$ we determine a δ-neighborhood $\Delta \subset \Omega$, and in Δ a circle C about a. Theorem 6 can be applied to $f(z)$ in Δ. Since $n(C,a) = 1$ we have $n(C,z) = 1$ for all points z inside of C. For such z we obtain by (22)

$$f(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{C} \frac{f(\zeta)d^{\#}\zeta}{\zeta-z} \right) \quad (16.)$$

Provided that the integral in(16.) can be #-differentiated under the sign of integration we find

$$f'^{\#}(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{C} \frac{f(\zeta)d^{\#}\zeta}{(\zeta-z)^2} \right) \quad (16.)$$

and

$$f^{(n)\#}(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{C} \frac{f(\zeta)d^{\#}\zeta}{(\zeta-z)^n} \right) \quad (16.)$$

If the #-differentiations can be justified, we shall have proved the existence of all #-derivatives at the points inside of C. Since every point in Ω lies inside of some such circle, the existence will be proved in the whole

region Ω.
Lemma 16.3. Suppose that $\varphi(\zeta)$ is #-continuous on the arc γ. Then the function

$$F_n(z) = \frac{1}{2\pi_{\#}i}\left(Ext\text{-}\int_{\gamma} \frac{\varphi(\zeta)d^{\#}\zeta}{(\zeta - z)^n} \right) \quad (16.)$$

is analytic in each of the regions determined by γ, and its #-derivative is $F_n'^{\#}(z) = nF_{n+1}(z)$.
It is clear that Lemma 16.3 is just what is needed in order to deduce (23) and (24) in a rigorous way. We have thus proved that an analytic function has derivatives of all orders which are #-analytic and can be represented by the formula (24).
Theorem 16.7. (Generalized Morera's theorem) If $f(z)$ is defined and #-continuous in a region Ω, and if $Ext\text{-}\int_{\gamma} f(z)d^{\#}z = 0$ for all closed curves γ in Ω, then $f(z)$ is #-analytic in Ω.

16.6.Generalized Liouville's theorem.

Theorem 16.8. (Generalized Liouville's theorem) A function $f(z)$ which is #-analytic and bounded in the whole plane $\mathbb{C}_c^{\#}$ must reduce to a constant.
Proof. We make use of a simple estimate derived from (24). Let the radius of C be r, and assume that $|f(z)| \leq M$ on C. If we apply (24) with $z = a$, we obtain

$$\left| f^{(n)\#}(a) \right| \leq Mn!r^{-n} \quad (16.)$$

We need only the case $n = 1$. The hypothesis means that $|f(z)| \leq M$ on all circles. Hence we can let r tend to $\infty^{\#}$, and (25) leads to $f'^{\#}(a) = 0$ for all a. We conclude that the function is constant.

16.7.Generalized fundamental theorem of algebra.

Liouville's theorem leads to an almost trivial proof of the generalized fundamental theorem of algebra.
Theorem 16.9. (Generalized fundamental theorem of algebra) Suppose that $P(z)$ is external polynomial of degree $n \in \mathbb{N}^{\#}$. The equation $P(z) = 0$ must have a root $\xi \in \mathbb{C}_c^{\#}$.
Proof. Suppose that $P(z)$ is a polynomial of degree $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. If $P(z)$ were never zero, the function $1/P(z)$ would be #-analytic in the whole plane $\mathbb{C}_c^{\#}$. We know that $P(z) \to \infty^{\#}$, and therefore $1/P(z)$ tends to zero. This implies boundedness (the absolute value is #-continuous on the Riemann sphere and has thus a finite or hyperfinite maximum), and by Liouville's theorem $1/P(z)$ would be constant. Since this is not so, the equation $P(z) = 0$ has a root.

17.The local properties of #-analytic function.

17.1.Removable Singnlarities. Taylor's Theorem.

Theorem 17.1. Suppose that $f(z)$ is #-analytic in the region Δ' obtained by omitting a point a from a region Δ. A necessary and sufficient condition that there exist an #-analytic function in Δ which coincides with $f(z)$ in Δ' is that #-$\lim_{z \to_{\#} a}(z - a)f(z) = 0$. The extended function is uniquely determined.
Proof. The necessity and the uniqueness are trivial since the extended function

must be #-continuous at a. To prove the sufficiency we draw a circle C about a so that C and its inside are contained in Δ. Cauchy's formula is valid, and therefore we have

$$f(z) = \frac{1}{2\pi_{\#}i}\int_{C}\frac{f(\zeta)d^{\#}z}{\zeta - z} \tag{17.1}$$

for all $z \neq a$ inside of C. But the integral in the right-hand member represents an #-analytic function of z throughout the inside of C. Consequently, the function which is equal to $f(z)$ for $z \neq a$ and which has the value

$$\frac{1}{2\pi_{\#}i}\int_{C}\frac{f(\zeta)d^{\#}z}{\zeta - z}. \tag{17.2}$$

for $z = a$ is #-analytic in Δ. It is natural to denote the extended function by $f(z)$ and the value (17.2) by $f(a)$. We apply this result to the function $F(z) = [f(z) - f(a)]/(z - a)$. It is not defined for $z = a$, but it satisfies the condition $\#\text{-}\lim_{z\to_{\#} a}(z - a)F(z) = 0$. The #-limit of $F(z)$ as $z \to_{\#} a$ is $f'^{\#}(a)$. Hence there exists an #-analytic function which is equal to $F(z)$ for $z = a$ and equal to $f'^{\#}(a)$ for $z = a$. Let us denote this function by $f_1(z)$. Repeating the process we can define an #-analytic function $f_2(z)$ which equals $[f_1(z) - f_2(a)]/(z - a)$ for $z \neq a$ and $f_1'^{\#}(a)$ for $z = a$, and so on.
The recursive scheme by which $f_n(z)$ is defined reads

$$\begin{aligned} f(z) &= f(a) + (z - a)f_1(z) \\ f_1(z) &= f_1(a) + (z - a)f_2(z) \\ &\cdots\cdots\cdots\cdots \\ f_{n-1}(z) &= f_{n-1}(a) + (z - a)f_n(z). \end{aligned} \tag{17.3}$$

From these equations which are trivially valid also for $z = a$ we obtain

$$f(z) = f(a) + Ext\text{-}\sum_{i=1}^{n}(z - a)^i f_i(z), n \in \mathbb{N}^{\#}. \tag{17.4}$$

Differentiating n times and setting $z = a$ we get

$$f^{(n)\prime\#}(a) = n! f_n(a). \tag{17.5}$$

This determines the coefficients $f_n(a)$, and we obtain the following form of Taylor's theorem.
Theorem 17.2. If $f(z)$ is #-analytic in a region Δ, containing a, then

$$f(z) = f(a) + Ext\text{-}\sum_{i=1}^{n-1}\frac{f^{(i)\prime\#}(a)}{i!}(z - a)^i + f_n(z)(z - a)^n, \tag{17.6}$$

where $n \in \mathbb{N}^{\#}$ and $f_n(z)$ is #-analytic in Ω.

17.2. Zeros and Poles of #-Analytical Functions.

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17.3.The Generalized Maximum Principle.

Theorem 17.. (Generalized maximum principle.) If $f(z)$ is #-analytic and nonconstant in a region Ω, then its absolute value $|f(z)|$ has no maximum in Ω.

Proof. If $w_0 = f(z_0)$ is any value taken in Ω, there exists a neighborhood $|w - w_0| < \varepsilon$ contained in the image of Ω. In this #-neighborhood there are points of modulus $> w_0$, and hence $|f(z_0)|$ is not the maximum of $|f(z)|$.

Theorem 17..If $f(z)$ is defined and #-continuous on a #-closed bounded set E and #-analytic on the interior of $E,$ then the maximum of $|f(z)|$ on E is assumed on the boundary of E.

Proof.Since E is #-compact, $|f(z)|$ has a maximum on $E.$ Suppose that it is assumed at z_0. If z_0 is on the boundary, there is nothing to prove. If z_0 is an interior point, then $|f(z_0)|$ is also the maximum of $|f(z)|$ in a disk $|z - z_0| < \delta$ contained in E. But this is not possible unless $f(z)$ is constant in the component of the interior of E which contains z_0. It follows by #-continuity that $|f(z)|$ is equal to its maximum on the whole #-boundary of that component. This boundary is not empty and it is contained in the #-boundary of $E.$ Thus the maximum is always assumed at a #-boundary point.

Remark 17.**1**.The generalized maximum principle can also be proved analytically, as a consequence of generalized Cauchy's integral formula. If the formula (22) is specialized to the case where γ is a circle of center z_0 and radius $r,$ we can write $\zeta = z_0 + r(Ext\text{-}e^{i\theta})$, $d^{\#}\zeta = ir(Ext\text{-}e^{i\theta})$ and obtain for $z = z_0$

$$f(z_0) = \frac{1}{2\pi_{\#}i}\int_0^{2\pi_{\#}} f(z_0 + r(Ext\text{-}e^{i\theta}))d^{\#}\theta. \qquad (17.)$$

The formula (17.) shows that the value of an #-analytic function at the center of a circle is equal to the arithmetic mean of its values on the circle, subject to the condition
that the #-closed disk $|z - z_0| \leq r$ is contained in the region of #-analyticity. From (34) we get the inequality

$$|f(z_0)| \leq \frac{1}{2\pi_{\#}i}\int_0^{2\pi_{\#}} |f(z_0 + r(Ext\text{-}e^{i\theta}))|d^{\#}\theta. \qquad (17.)$$

Suppose that $|f(z_0)|$ were a maximum. Then we would have $|f(z_0 + r(Ext\text{-}e^{i\theta}))| \leq |f(z_0)|$, and if the strict inequality held for a single value of θ it would hold, by #-continuity, on a whole arc. But then the mean value of $|f(z_0 + r(Ext\text{-}e^{i\theta}))|$ would be strictly less than $|f(z_0)|$, and (35) would lead to the contradiction $|f(z_0)| < |f(z_0)|$. Thus $|f(z)|$ must be constantly equal to $|f(z_0)|$ on all sufficiently small circles $|z - z_0| = r$ and, hence, in a neighborhood of z_0. It follows easily that $f(z)$ must reduce to a constant.

This reasoning provides a second proof of the generalized maximum principle.
Consider now the case of a function $f(z)$ which is #-analytic in the #-open disk $|z| < R$ and #-continuous on the #-closed disk $|z| \leq R$. If it is known that $|f(z)| \leq M$ on $|z| = R$, then $|f(z)| \leq M$ in the whole disk. The equality can hold only if $f(z)$ is a constant of absolute value M. Therefore, if it is known that $f(z)$ takes some value of modulus $< M$, it may be expected that a better estimate can be given.

Theorem 17..If $f(z)$ is #-analytic for $|z| < 1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'^{\#}(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$, or if $|f'^{\#}(0)| = 1$, then $f(z) = cz$ with a constant $c \in \mathbb{C}_c^{\#}$ of absolute value 1.

Proof.

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Part II. $\mathbb{R}_c^{\#}$-Valued Lebesgue Integral

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1.External $\mathbb{R}_c^{\#}$-Valued Lebesgue Measure

Let us consider a bounded interval $I \subset \mathbb{R}_c^{\#}$ with endpoints a and b $(a < b)$.The length of this bounded interval I is defined by $l(I) = b - a$. In contrast,the length of an unbounded interval, such as $(a, \infty^{\#}), (-\infty^{\#}, b)$ or $(-\infty^{\#}, \infty)$, is defined to be gyperinfinite. Obviously, the length of a line segment is easy to quantify.However, what should we do if we want to measure an arbitrary subset of $\mathbb{R}_c^{\#}$? Given a set $E \subset \mathbb{R}_c^{\#}$ of gyperreal numbers, we denote the Lebesgue measure of set E by $\mu(E)$. To correspond with the length of a line segment, the measure of a set $A \subset \mathbb{R}_c^{\#}$ should keep the following properties:

(1) If A is an interval, then $\mu(A) = l(A)$.

(2) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(3) Given $A \subseteq \mathbb{R}_c^{\#}$ and $x_0 \in \mathbb{R}_c^{\#}$, define $A + x_0 = \{x + x_0 : x \in A\}$. Then $\mu(A) = \mu(A + x_0)$.

(4) If A and B are disjoint sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}_{i \in \mathbb{N}^{\#}}$ is a hyperinfinite sequence of disjoint sets, then $\mu(\bigcup_{i \in \mathbb{N}^{\#}} A_i) = \sum_{i=1}^{\infty^{\#}} \mu(A_i)$.

1.External $\mathbb{R}_c^{\#}$-Valued Lebesgue outer measure

Definition 1.1. Let E be a subset of $\mathbb{R}_c^{\#}$. Let $\{I_k\} \triangleq \{I_k\}_{k \in \mathbb{N}^{\#}}$ be a hyperinfinite sequence

of open intervals such that $E \subseteq \bigcup_{k \in \mathbb{N}^{\#}} A_k$ and let Σ be a set of the all such hyperinfinite sequences. The external Lebesgue outer measure of E is defined by

$$\mu^*(E) = \inf_{\{I_k\} \in \Sigma} \left\{ \sum_{k=1}^{\infty^{\#}} l(I_k) \right\}. \tag{1.1}$$

Note that $0 \leq \mu^*(E) \leq \infty^{\#}$.

Definition 1.2. A set E is #-countable if there exists an injective function f from E to the

gypernatural numbers $\mathbb{N}^{\#}$. If such an f can be found that is also surjective (and therefore

bijective), then E is called #-countably infinite or gyperinfinite, i.e. a set is #-countably infinite if it has one-to-one correspondence with the set $\mathbb{N}^{\#}$.

Theorem 1.1.The external Lebesgue outer measure has the following properties:

(a) If $E_1 \subseteq E_2$, then $\mu^*(E_1) \leq \mu^*(E_2)$.

(b) The external Lebesgue outer measure of any #-countable set is zero.

(c) The external Lebesgue outer measure of the empty set is zero.

(d) The external Lebesgue outer measure is invariant under translation, that is, $\mu^*(E + x_0) = \mu^*(E)$.

(e) Lebesgue outer measure is #-countably sub-additive, that is,

$$\mu^*\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right) \leq \sum_{i=1}^{\infty^{\#}} \mu^*(E_i). \qquad (14.2)$$

(f) For any interval $I, \mu^*(I) = l(I)$.

Proof. Part (a) is trivial.

For part (b) and (c), let $E = \{x_k | k \in \mathbb{Z}_+^{\#}\}$ be a #-countably hyper infinite set.

Let $\varepsilon > 0, \varepsilon \approx 0$ and let ε_k be a hyper infinite sequence of positive numbers such that $\sum_{k=1}^{\infty^{\#}} \varepsilon_k = \varepsilon/2$. Since $E \subseteq \cup_{k=1}^{\infty^{\#}} (x_k - \varepsilon_k, x_k + \varepsilon_k)$, it follows that $\mu^*(E) \leq \varepsilon$. Hence, $\mu^*(E) = 0$. Since $\varnothing \subseteq E$, then $\mu^*(\varnothing) = 0$.

For part (d), since each cover of E by open intervals can generate a cover of $E + x_0$ by open intervals with the same length, then $\mu^*(E + x_0) \leq \mu^*(E)$. Similarly, $\mu^*(E + x_0) \geq \mu^*(E)$, since $E + x_0$ is a translation of E Therefore, $\mu^*(E + x_0) = \mu^*(E)$.

For part (e), if $\sum_{i=1}^{\infty^{\#}} \mu^*(E_i) = \infty^{\#}$, then the statement is trivial. Suppose that the sum is hyperfinite and let $\varepsilon > 0, \varepsilon \approx 0$. For each $i \in \mathbb{N}^{\#}$, there exists a hyperinfinite sequence $\{I_i^k\}$ of open intervals such that $E_i \subseteq \cup_{k=1}^{\infty^{\#}}$ and $\sum_{k=1}^{\infty^{\#}} l(I_i^k) < \mu^*(E_i) + \varepsilon/2^i$.

Now $\{I_i^k\}$ is a double-indexed sequence of open intervals such that $\cup_{i=1}^{\infty^{\#}} E_i \subseteq \cup_{i=1}^{\infty^{\#}} \cup_{k=1}^{\infty^{\#}} I_i^k$ and

$\sum_{i=1}^{\infty^{\#}} \sum_{k=1}^{\infty^{\#}} l(I_i^k) \leq \sum_{i=1}^{\infty^{\#}} (\mu^*(E_i) + \varepsilon/2^i) = \sum_{i=1}^{\infty^{\#}} \mu^*(E_i) + \varepsilon$.

Therefore, $\mu^*\left(\cup_{i=1}^{\infty^{\#}} E_i\right) \leq \sum_{i=1}^{\infty^{\#}} \mu^*(E_i) + \varepsilon$. The result follows since $\varepsilon > 0, \varepsilon \approx 0$ was arbitrary.

For part (f), we need to prove $\mu^*(I) \leq l(I)$ and $\mu^*(I) \geq l(I)$ respectively.

We can assume that $I = [a,b]$ where $a,b \in \mathbb{R}_c^{\#}$.

First, we want to prove $\mu^*(I) \leq l(I)$. Let $\varepsilon > 0, \varepsilon \approx 0$, we have $I \subseteq (a,b) \cup (a - \varepsilon, a + \varepsilon) \cup (b - \varepsilon, b + \varepsilon)$.

Thus,$\mu^*(I) \leq l(a,b) + l(a - \varepsilon, a + \varepsilon) + l(b - \varepsilon, b + \varepsilon) =$
$= (b - a) + 2\varepsilon + 2\varepsilon = b - a + 4\varepsilon$.

As $\varepsilon > 0, \varepsilon \approx 0$ is arbitrary, we conclude that $\mu^*(I) \leq b - a = l(I)$.

Then, we want to prove that $\mu^*(I) \geq l(I)$. Let $\{I_k\}$ be any sequence of open intervals that covers I. Since I is compact, by the generalized Heine-Borel theorem, there is a gyperfinite subcollection $\{J_i | 1 \leq i \leq n\}, n \in \mathbb{N}^{\#}$ of I_k that still covers I. By reordering and deleting if necessary, we can assume that $a \in J_1 = (a_1, b_1), b_1 \in J_2 = (a_2, b_2), \ldots, b_{n-1} \in J_n = (a_n, b_n)$, where $b_{n-1} \leq b < b_n$.

We then can compute that

$$b - a < b_n - a_1 = Ext\text{-}\sum_{i=2}^{n} (b_i - b_{i-1}) + (b_1 - a_1) < Ext\text{-}\sum_{i=1}^{n} l(J_i) \leq Ext\text{-}\sum_{i=1}^{\infty^{\#}} l(I_k).$$

Therefore, $l(I) \leq \mu^*(I)$. We can now conclude that $\mu^*(I) = l(I)$. This proves the result for closed and bounded intervals.

Suppose that $I = (a,b)$ is an open and bounded interval. Then, $\mu^*(I) \leq l(I)$ as above and $b - a = \mu^*([a,b]) \ \leq \mu^*((a,b)) \ + \mu^*(a) + \mu^*(b) = \mu^*((a,b))$. Hence $l(I) \leq \mu^*(I)$. The proof for half-open intervals is similar.
Finally, suppose that I is an hyper infinite interval and let $M > 0$. There exists a bounded interval $J \subseteq I$ such that $\mu^*(J) = l(J) = M$ and it follows that $\mu^*(I) \geq \mu^*(J) = M$. Since $M > 0$ was arbitrary, $\mu^*(I) = \infty^\# = l(I)$.
This completes the proof.

14.2.External Lebesgue inner measure

In previous subsection, we have discussed external Lebesgue outer measure. There is another external measure named external Lebesgue inner measure. Let's define the external inner measure and see some basic properties.

Definition 14.2. Let E be a subset of $\mathbb{R}^\#$. The external inner measure of E is defined by

$$\mu^*(E) = \sup\{\mu(K) : K \subseteq E \text{ and } K \text{ is \#-closed}\} \qquad (14.3)$$

iff supremum in RHS of the (14.3) exists.
Recall that external Lebesgue outer measure of a set E uses an infimum of the union of a sequence open sets that cover the set E, while external Lebesgue inner measure of a set E uses a supremum of a set inside the set E. Then, it is obvious that

$$\mu_*(E) \leq \mu^*(E) \qquad (14.4)$$

for any set E. Also, for $A \subseteq B, \mu_*(A) \leq \mu^*(B)$.

Theorem 14.2. Let A and E be subsets of $\mathbb{R}_c^\#$.
(i) Suppose that $\mu^*(E) < \infty^\#$. Then E is measurable if and only if $\mu_*(E) = \mu^*(E)$.
(ii) If E is measurable and $A \subseteq E$, then $\mu(E) = \mu_*(A) + \mu^*(E \setminus A)$.

Proof. For part (i), suppose that E is a measurable set and let $\varepsilon > 0, \varepsilon \approx 0$. According to Theorem 2.9, there exists a #-closed set K such that $K \subseteq E$ and $\mu(E \setminus K) < \varepsilon$.
Thus,$\mu^*(E) \geq \mu_*(E) \geq \mu(K) > \mu(E) - \varepsilon = \mu^*(E) - \varepsilon,$which implies that the external inner measure and external outer measure of E are equal. Now let's prove the reverse direction. Suppose that $\mu_*(E) = \mu^*(E)$. Let $\varepsilon > 0, \varepsilon \approx 0$. Then there exists a #-closed set K and an #-open set G such that $K \subseteq E \subseteq G$ and $\mu(K) > \mu_*(E) - \varepsilon/2$ and $\mu(G) < \mu^*(E) + \varepsilon/2$.Then we find that
$\mu^*(G \setminus E) \leq \mu^*(G \setminus K) = \mu(G \setminus K) = \mu(G) - \mu(K) < \varepsilon$.
According to Theorem 2.9, the set E is measurable.
For part (2), let $\varepsilon > 0, \varepsilon \approx 0$. There exists a #-closed set $K \subseteq A$ such that $\mu(K) > \mu_*(A) - \varepsilon$. Then, $\mu(E) = \mu(K) + \mu(E \setminus K) > \mu_*(A) - \varepsilon + \mu^*(E \setminus A)$ and it follows that $\mu(E) \geq \mu_*(A) + \mu^*(E \setminus A)$. According to Theorem 2.9, there exists a measurable set B such that $E \setminus A \subseteq B \subseteq E$ and $\mu(B) = \mu^*(E \setminus A)$.
Since $E \setminus B \subseteq A$, it follows that $\mu_*(E \setminus B) \geq \mu_*(A)$. Thus,
$\mu(E) = \mu(B) + \mu(E \setminus B) = \mu^*(E \setminus A) + \mu(E \setminus B) \leq \mu^*(E \setminus A) + \mu_*(A)$.
By combining these two inequalities, we can obtain $\mu(E) = \mu_*(A) + \mu^*(E \setminus A)$.

14.3.External Lebesgue measure

Definition 14.3. A set $E \subseteq \mathbb{R}_c^\#$ is Lebesgue measurable if for each set $A \subseteq \mathbb{R}_c^\#$, the equality $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement})$ is satisfied. If E is a Lebesgue measurable set, then the external Lebesgue measure of E is its external Lebesgue outer measure and will be written as $\mu(E)$.

Since the external Lebesgue outer measure satisfies the property of subadditivity, then we always have $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*\left(A \cap E^{\complement}\right)$,$E^{\complement} = \mathbb{R}_c^{\#} \setminus E$ and we only need to check the reverse inequality.

Note that there is always a set E that can divide A into two mutually exclusive sets, $A \cap E$
and $A \cap E^{\complement}$. But only when $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement})$ holds, the set E is Lebesgue
measurable. The latter theorem will show some properties of measurable sets.

Theorem 14.3. The collection of measurable sets defined on $\mathbb{R}_c^{\#}$ has the following properties:

(a) Both $\varnothing$ and $\mathbb{R}_c^{\#}$ are measurable.

(b) If E is measurable, then $E^{\complement}$ is measurable,where $E^{\complement} = \mathbb{R}_c^{\#} \setminus E$.

(c) If $\mu^*(E) = 0$, then E is measurable.

(d) If E_1 and E_2 are measurable, then $E_1 \cup E_2$ and $E_2 \cap E_2$ are measurable.

(e) If E is measurable, then $E + x_0$ is measurable.

Proof. For part (a), let $A \subseteq \mathbb{R}_c^{\#}$.Then

$\mu^*(A \cap \varnothing) + \mu^*(A \cap \varnothing^{\complement}) = \mu^*(\varnothing) + \mu^*(A) = 0 + \mu^*(A) = \mu * (A)$,

$\mu * (A \cap \mathbb{R}_c^{\#}) + \mu * (A \cap \mathbb{R}_c^{\#\complement}) = \mu^*(A) + \mu^*(\varnothing) = \mu^*(A) + 0 = \mu^*(A)$.

For part (b), if E is measurable, then for every set $A \subseteq \mathbb{R}_c^{\#}$, such that

$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^{\complement})$. Then,

$\mu^*(A \cap E^{\complement}) + \mu^*(A \cap (E^{\complement}) = \mu^*(A \cap E^{\complement}) + \mu^*(A \cap E) = \mu^*(A)$.

For part (c), let $A \subseteq \mathbb{R}_c^{\#}$. Since $\mu^*(E) = 0$ and $A \cap E \subseteq E$, then $\mu^*(A \cap E) = 0$.

We can obtain that $\mu^*(A) \geq \mu^*(A \cap E) = \mu^*(A \cap E) + \mu^*(A \cap E)$,which implies that $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E)$ by Theorem 14.1 part (e).

For part (d), let $A \subseteq \mathbb{R}_c^{\#}$. Note that

$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2) = (A \cap E_1) \cup (A \cap E_1 \cap E_2)$

Then, by De Morgan Law and Theorem 14.1 part (e), we know that

$\mu^*(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1) =$

$= \mu^*(A \cap E_1) + \mu^*(A \cap E_1 \cap E_2) + \mu^*(A \cap E_1 \cap E_2) \geq \mu^*(A \cap (E_1 \cup E_2)) +$

$+\mu^*(A \cap (E_1 \cup E_2))$,

showing that $E_1 \cup E_2$ is measurable. Since $E_1 \cap E_2 = (E_1 \cup E_2)$, then the set $E_1 \cap E_2$ is measurable by Theorem 14.1 part (b).

For part (e), let $A \subseteq \mathbb{R}_c^{\#}$. Then,

$\mu^*(A) = \mu^*(A - x_0) = \mu^*((A - x_0) \cap E) + \mu^*((A - x_0) \cap E^{\complement}) =$

$\mu^*(((A - x_0) \cap E) + x_0) + \mu^*(((A - x_0) \cap E^{\complement}) + x_0) =$

$\mu^*(A \cap (E + x_0)) + \mu^*(A \cap (E^{\complement} + x_0))$.

Therefore, $E + x_0$ is measurable.

Lemma 14.1. Let $E_i : 1 \leq i \leq n \in \mathbb{N}^{\#}$ be a gyperfinite collection of disjoint measurable sets. If $A \subseteq \mathbb{R}_c^{\#}$, then

$\mu^*\left(\bigcup_{i=1}^n (A \cap E_i)\right) = \mu^*\left(A \cap \left(\bigcup_{i=1}^n E_i\right)\right) = Ext\text{-}\sum_{i=1}^n \mu^*(A \cap E_i)$.

Proof. We will prove this by the principle of mathematical induction. When $n = 1$, the equality holds. Suppose that the statement is valid for $n - 1$ disjoint measurable sets when $n > 1$. Then, when there are n disjoint measurable sets,

$\mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) =$
$= \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n\right) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n^{\complement}\right) =$
$= \mu^*(A \cap E_n) + \mu^*\left(A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right) =$
$= \mu^*(A \cap E_n) + Ext\text{-}\sum_{i=1}^{n-1} \mu^*(A \cap E_i) = Ext\text{-}\sum_{i=1}^{n} \mu^*(A \cap E_i).$

Note that when $A = \mathbb{R}_c^{\#}$, $\mu\left(\bigcup_{i=1}^{n} E_i\right) = Ext\text{-}\sum_{i=1}^{n} \mu(E_i)$.

Theorem 14.4. If $\{E_i\}_{i=1}^{\infty^{\#}}$ is a hyper infinite sequence of disjoint measurable sets, then

$$\mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right) = Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i). \quad (14.5)$$

Proof. According to Lemma 14.1, $Ext\text{-}\sum_{i=1}^{n} \mu(E_i) = \mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$

for each positive integer $n \in \mathbb{N}^{\#}$, which implies that $Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i) \leq \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$.

By #-countably subadditive property,$Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i) \geq \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$.

Therefore, $Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(E_i) = \mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right)$.

The previous theorem shows that if A and B are disjoint measurable sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}_{i \in \mathbb{N}^{\#}}$ is a hyper infinite sequence of disjoint measurable sets,then $\mu\left(\bigcup_{i=1}^{\infty^{\#}} E_i\right) = Ext\text{-}\sum_{i=1}^{\infty^{\#}} \mu(A_i)$. As so far, we have already seen that when the sets are measurable, Lebesgue measure satisfies property (1),(2),(3) and (4). But what kinds of sets are measurable? Certainly every interval is measurable.

Theorem 14.5. Every interval $[a,b] \subset \mathbb{R}_c^{\#}$ is measurable.

Theorem **14.6**. If $\{E_i\}_{i \in \mathbb{N}^{\#}}$ is a hyper infinite sequence of measurable sets, then

∞
S
i=1
E i and
∞
T
i=1
E i
are measurable sets.

Definition 14.4. Let *f* be a function from $E \subset \mathbb{R}_c^{\#}$ into $\mathbb{R}_c^{\#} \cup (-\infty^{\#}, \infty^{\#})$. The function *f* is (Lebesgue) measurable if

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15.External Lebesgue Integral

Let $(\mathbb{R}, B, \mu)$ be the standard Lebesgue space on $\mathbb{R}$. Our internal starting point could be the internal measure space $({}^*\mathbb{R}, {}^*B, {}^*\mu)$. By transfer we can write down internal Lebesgue integrals

$$\mathcal{L}[{}^*f(t)] = \int_A {}^*f(t)\,d^*\mu(t),$$

where $A \in {}^*B$ and $f : \mathbb{R} \to \mathbb{R}$.

15.1.Lebesgue Integral of a $\mathbb{R}_c^\#$-valued external function $f(x)$.

First, in particular, we need external function that can help us distinguish whether a given value x is in the measurable set A_i. We call this function the characteristic function. The following statement is the formal definition of characteristic function and introduces the simple function.

Definition 15.1. For any set A, the function

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise} \end{cases} \tag{15.1}$$

is called the characteristic function of set A. A linear combination of characteristic functions,

$$\phi(x) = \sum_{i=1}^n a_i \chi_{A_i}(x) \tag{15.2}$$

is called a simple function if the sets A_i are measurable.

For a function $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$ defined on a measurable set A that takes no more than gyper finitely many distinct values $a_1, \ldots, a_n, n \in \mathbb{N}^\#$ the function f can always be written as a simple function

$$f(x) = \sum_{i=1}^n a_i \chi_{A_i}(x), \tag{15.3}$$

where $A_i = \{x \in A |\, f(x) = a_i\}$. That is a simple function of the first kind.

Therefore, simple functions can be thought of as dividing

the range of f, where resulting sets A_i may or may not be intervals.

Let us pause for a second. We want to ask ourselves: is the simple function $\phi(x)$ unique? The answer is no. Because we might define different disjoint sets that have a same function value. The simplest expression is

$$\phi(x) = \sum_{i=1}^n a_i \chi_{A_i}(x) \tag{15.4}$$

where $A_i = \{x \in A | \phi(x) = a_i\}$. At this case, the constants a_i are distinct, the sets A_i are disjoint and we call that representation the canonical representation of φ.

Then, for simple functions, we define the Lebesgue integral as follows:

Definition 15.2. If $\phi(x) = \sum_{i=1}^n a_i \chi_{A_i}(x)$ is a simple function and $\mu(A_i)$ is gyperfinite for all i, then the Lebesgue integral of $\phi(x)$ is defined as

$$\int_E \phi(x) = \sum_{i=1}^n a_i \chi_{A_i}(x). \tag{15.5}$$

Definition 15.3. Suppose $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$ is a bounded function defined on a measurable set E with giperfinite measure. We define the upper and lower Lebesgue integrals if exist, respectively, as

$$I_L^\#(f) = \int_E \inf\{\phi(x)|\phi \text{ is simple and } \phi \geq f\} \tag{15.6}$$

and

$$I_{\#L}(f) = \int_E \sup\{\phi(x)|\phi \text{ is simple and } \phi \leq f\}. \tag{15.7}$$

If (i) the quantity $I_L^\#(f)$ and $I_{\#L}(f)$ exist and (ii) $I_L^\#(f) = I_{\#L}(f)$, then the function f is called Lebesgue integrable over set E and the external Lebesgue integral of f over set E is denoted by $I_L(f) = \int_E f dx$.

The Lebesgue Integral for Simple Functions of the second kind

Let $\varphi(x)$ be some simple external function of the second kind which takes on the gyperinfinitely many distinct values $y_1,\ldots,y_n,\ldots,n \in \mathbb{N}^\#, y_i \neq y_j$ for $i \neq j$.

It is natural to define the integral of the function $\varphi(x)$ over the set E by the equation

$$\int_E \varphi(x) d^\# \mu = \sum_{n\in\mathbb{N}^\#} y_n \mu\{x|x \in E, \varphi(x) = y_n\}. \tag{15.8}$$

Definition 15.4.The simple function $\varphi(x)$ of the second kind is called integrable (with respect to the measure μ) over the set E if the gyperinfinite series (15.8) #-converges absolutely.

If $\varphi(x)$ is #-integrable, then the sum of the series (15.8) is called the integral of $\varphi(x)$ over the set E.

Remark 15.1. Note that in definition 15.4 we assume that all the y_n are different. One can, however, represent the value of the integral of a simple function as a sum of products of the form $c_k\mu(B_k)$ and not assume that all the c_k are different.

Lemma 15.1. Let $A = \cup_k B_k, B_i \cap B_j = \varnothing$ for$i \neq j$, and assume that on each set B_k the function $f(x)$ takes on only one value c_k. Then

$$\int_A \varphi(x) d^\# \mu = \sum_{k\in\mathbb{N}^\#} c_k \mu(B_k). \tag{15.9}$$

moreover, the function $f(x)$ is integrable over A if and only if the gyper infinite series (15.9) #-converges absolutely.

Proof. It is easy to see that every set $A_n = \{x|x \in A, f(x) = y_n\}$ is the union of those B_k for which $c_k = y_n$. Therefore $\sum_{n\in\mathbb{N}^\#} y_n\mu(A_n) = \sum_{n\in\mathbb{N}^\#} y_n \sum_{c_k=y_k} \mu(B_k) = \sum_{k\in\mathbb{N}^\#} c_k\mu(B_k)$. Since the measure is non-negative, $\sum_{n\in\mathbb{N}^\#} |y_n|\mu(A_n) = \sum_{n\in\mathbb{N}^\#} |y_n| \sum_{c_k=y_k} \mu(B_k) = \sum_{k\in\mathbb{N}^\#} |c_k|\mu(B_k)$.

i.e., the series $\sum_{n\in\mathbb{N}^\#} y_n\mu(A_n)$ and $\sum_{k\in\mathbb{N}^\#} |c_k|\mu(B_k)$ both either #-converge absolutely or #-diverge.

Let us consider some properties of the external Lebesgue integral for simple external functions:

$$\int_A f(x) d^\# \mu + \int_A g(x) d^\# \mu = \int_A [f(x) + g(x)] d^\# \mu \tag{15.10}$$

moreover, from the existence of the integrals on the left-hand side it follows that the integrals on the right-hand side exist.

To prove this assume that $f(x)$ takes on the values f_i, on the sets $F_i \subseteq A$, and $g(x)$ the values g_i, on the sets $G_i \subseteq A$, since

$$J_1 = \int_A f(x) d^\# \mu = \sum_{i\in\mathbb{N}^\#} f_i \mu(F_i) \tag{15.11}$$

and

$$J_2 = \int_A g(x)d^\# \mu = \sum_{i\in\mathbb{N}^\#} g_i \mu(G_j). \quad (15.12)$$

Then, by the Lemma 14.1 we get

$$J = \int_A [f(x) + g(x)]d^\# \mu = \sum_{i\in\mathbb{N}^\#} \sum_{j\in\mathbb{N}^\#} [f_i + g_j]\mu(F_i \cap G_j), \quad (15.13)$$

where

$$\mu(F_i) = \sum_{j\in\mathbb{N}^\#} \mu(F_i \cap G_j), \mu(G_j) = \sum_{i\in\mathbb{N}^\#} \mu(F_i \cap G_j). \quad (15.14)$$

From the absolute #-convergence of the series (15.11)-(15.12) it follows the absolute #-convergence of the series (15.13); here $J = J_1 + J_2$.

For any constant $k \in \mathbb{R}_c^\#$

$$k\int_A f(x)d^\# \mu = \int_A [kf(x)]d^\# \mu \quad (15.15)$$

moreover, the existence of the integral on the left-hand side implies the existence of the

integral on the right.A simple function $f(x)$ which is bounded on the set $A \subset \mathbb{R}_c^\#$ is #-integrable over A; moreover, if $|f(x)| \leq M \in \mathbb{R}_c^\#$ on A, then

$$\left|\int_A f(x)d^\# \mu\right| \leq M\mu(A). \quad (15.16)$$

15.General Definition and Basic Properties of the external Lebesgue Integral.

Definition.15.1. We shall say that the function $f(z)$ is #-integrable over the set $A \subset \mathbb{R}_c^\#$, if

there exists a hyper infinite sequence of simple functions $f_n(z), n \in \mathbb{N}^\#$ which are #-integrable over A and #-converge uniformly to $f(x)$. We shall denote the #-limit

$$J = \#\text{-}\mathbf{lim}_{n\to\infty^\#} \int_A f_n(x)d^\# \mu \quad (15.1)$$

by

$$\int_A f(x)d^\# \mu. \quad (15.2)$$

and call it the integral of the external function $f : \mathbb{R}_c^\# \to \mathbb{R}_c^\#$ over the set A.

This definition 15.1 is correct if the following conditions are satisfied:

1.The #-limit (15.1) for any uniformly #-convergent hyperinfinite sequence of simple functions which are #-integrable over A exists.

2.This #-limit for a given function f(x) does not depend on the choice of the hyperinfinite sequence $\{f_n(x)\}_{n\in\mathbb{N}^\#}$.

3.For simple functions the definitions of #-integrability and #-integral are equivalent to those given in section 14.

Notice that all these conditions are indeed satisfied.

To prove the first it suffices to note that by properties for #-integrals of simple functions,

$$\left|\int_A f_n(x)d^\# \mu - \int_A f_m(x)d^\# \mu\right| \leq \mu(A)\sup_{x\in A}|f_n(x) - f_m(x)|. \quad (15.3)$$

To prove the second condition, we must consider the two hyperinfinite sequences $\{f_n(x)\}_{n\in\mathbb{N}^\#}$ and $\{f_n'(x)\}_{n\in\mathbb{N}^\#}$, and use the inequality

$$\left|\int_A f_n(x)d^\#\mu - \int_A f_n'(x)d^\#\mu\right| \leq \mu(A)\left[\sup_{x\in A}|f_n(x)-f(x)| + \sup_{x\in A}\left|f_n'(x)-f(x)\right|\right]. \qquad (15.4)$$

Finally, to prove the third condition it suffices to consider the hyperinfinite sequence $f_n(x) = f(x)$.

The basic properties of the external Lebesgue #-**integral**.

Theorem 15.1.

$$\int_A 1 \cdot d^\#\mu = \mu(A). \qquad (15.5)$$

Proof. Immediately from the definition of the #-integral.

Theorem 15.2.For any constant $k \in \mathbb{R}_c^\#$

$$k\int_A f(x)d^\#\mu = \int_A [kf(x)]d^\#\mu \qquad (15.6)$$

where the existence of the #-integral on the left-hand side implies the existence of the #-integral on the right.

Proof. The proof is obtained from property (8.15) by proceeding to the #-limit for an #-integral of simple functions.

Theorem 15.3. Assume that $f(x)$ and $g(x)$ are #-integrable over A then $f(x)+g(x)$ #-integrable over A and

$$\int_A f(x)d^\#\mu + \int_A g(x)d^\#\mu = \int_A [f(x)+g(x)]d^\#\mu \qquad (15.7)$$

Let $\{f_i(x)\}_{i=1}^n, n \in \mathbb{N}^\#$ be a hyperfinite sequence such that any $f_i(x)$ is #-integrable over A

then $\sum_{i=1}^n f_i(x)$ is #-integrable over A and

$$\sum_{i=1}^n \int_A f_i(x)d^\#\mu = \int_A \left[\sum_{i=1}^n f_i(x)\right]d^\#\mu \qquad (15.8)$$

where the existence of the #-integrals on the left implies the existence of the #-integral on the right.

Proof. The proof of (15.7) is obtained from property A) by proceeding to the #-limit for an
#-integral of simple functions.

Theorem 15.4. A function $f : A \to \mathbb{R}_c^\#$ which is hyperbounded on the set A is #-integrable
over A.

Proof. The proof is obtained from property C) by proceeding to the limit for an integral of
simple functions.

Theorem 15.5. If $f(x) \geq 0$, then

$$\int_A f(x)d^\#\mu \geq 0 \qquad (15.9)$$

assuming that the #-integral exists.

Proof. For simple functions this follows immediately from the definition; for the general case the proof is based on the possibility of approximating non-negative functions by non-negative simple functions

Corollary 15.1. If $f(x) \geq g(x)$, then

$$\int_A f(x)d^{\#}\mu \geq \int_A g(x)d^{\#}\mu. \qquad (15.10)$$

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Appendix A. Bivalent Hyper Infinitary first-order logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.Generalized Deduction Theorem.

Hyper infinitary language $L^{\#}_{\infty^{\#}}$ are defined according to the length of hyper infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\kappa < \aleph_0^{\#} = card(\mathbb{N}^{\#})$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernaturals indexed hyper infinite sequence $\{A_\delta\}_{\delta\in\mathbb{N}^{\#}}$ of formulas has length less than κ, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than λ, one can introduce one of the quantifiers $\forall$ or $\exists$ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of any well-formed formula is less than $\aleph_0^{\#}$ itself.

The syntax of bivalent hyper infinitary first-order logics ${}^{2}L^{\#}_{\infty^{\#}}$ consists of a (ordered) set of sorts and a set of function and relation symbols, these latter together with the corresponding type, which is a subset with less than $\aleph_0^{\#} = card(\mathbb{N}^{\#})$ many sorts. Therefore, we assume that our signature may contain relation and function symbols on $\gamma < \aleph_0^{\#}$ many variables, and we suppose there is a supply of $\kappa < \aleph_0^{\#}$ many fresh variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules.

If $\phi, \psi, \{\phi_\alpha : \alpha < \gamma\}$ (for each $\gamma < \kappa$) are formulas of $L^{\#}_{\infty^{\#}}$, the following are also formulas:

(i) $\bigwedge_{\alpha<\gamma} \phi_\alpha, \bigwedge_{\alpha\leq\gamma} \phi_\alpha,$

(ii) $\bigvee_{\alpha<\gamma} \phi_\alpha, \bigvee_{\alpha\leq\gamma} \phi_\alpha,$

(iii) $\phi \to \psi, \phi \wedge \psi, \phi \vee \psi, \neg\phi$

(iv) $\forall_{\alpha<\gamma} x_\alpha \phi$ (also written $\forall \mathbf{x}_\gamma \phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),

(v) $\exists_{\alpha<\gamma} x_\alpha \phi$ (also written $\exists \mathbf{x}_\gamma \phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$),

(vi) the statement $\bigwedge_{\alpha<\gamma} \phi_\alpha$ holds if and only if for any α such that $\alpha < \gamma$ the statement holds ϕ_α,

(vii) the statement $\bigvee_{\alpha<\gamma} \phi_\alpha$ holds if and only if there exist α such that $\alpha < \gamma$ the statement holds ϕ_α.

Definition 1.[7]. A valuation of a syntactic system is a function that as signs $\top$ (true) to some of its sentences, and/or $\perp$ (false) to some of its sentences.Precisely, a valuation maps a nonempty subset of the set of sentences into the set $\{\top, \perp\}$. We call a valuation bivalent iff it maps all the sentences into $\{\top, \perp\}$.

Definition 2.[7]. L is a bivalent propositional language iff its admissible valuations are the functions v such that for all sentences A, B of L,

(a) $v(A) \in \{\top, \perp\}$

(b) $v(\neg A) = \top$ iff $v(A) = \perp$

(c) $v(A \wedge B) = \top$ iff $v(A) = v(B) = \top$.

(d) by definition of the implication $A \Rightarrow B$ the following truth table holds

	$v(A)$	$v(B)$	$v(A \Rightarrow B)$
(1)	$\top$	$\top$	$\top$
(2)	$\top$	$\bot$	$\bot$
(3)	$\bot$	$\top$	$\top$
(4)	$\bot$	$\bot$	$\top$

Truth table 1.

Remark 1. Note that in the case (4) on a truth table 1
In this case we call implication $A \Rightarrow B$ a weak implication and abbreviate

$$A \Rightarrow_w B \qquad (1)$$

We call a statement (1) as a weak statement and often abbreviate $v(A \Rightarrow B) = \top_w$ instead (1).

Definition 3.[7-8]. A is a valid (logically valid) sentence (in symbols, $\Vdash A$) in L iff every admissible valuation of L satisfies A.

The axioms of hyper infinitary first-order logic ${}^2L^{\#}_{\infty^{\#}}$ consist of the following schemata:

I. Logical axiom

A 1. $A \to [B \to A]$

A 2. $[A \to [B \to C] \to [[A \to B] \to [A \to C]]]$

A 3. $[\neg B \to \neg A] \to [A \to B]$

A 4. $[\bigwedge_{i<\alpha}[A \to A_i]] \ \to [A \to \bigwedge_{i<\alpha} A_i], \alpha \in \mathbb{N}^{\#}$

A 5. $[\bigwedge_{i<\alpha} A_i] \to A_j, \alpha \in \mathbb{N}^{\#}$

A 6. $[\forall \mathbf{x}[A \to B] \to [A \to \forall \mathbf{x} B]]$

provided no variable in $\mathbf{x}$ occurs free in A;

A 7. $\forall \mathbf{x} A(\mathbf{x}) \to S_f(A),$

where $S_f(A)$ is a substitution based on a function f from $\mathbf{x}$ to the terms of the language; in particular:

A 7′. $\forall x_i[A(x_i)] \Rightarrow A(\mathbf{t})$ is a wff of ${}^2L^{\#}_{\infty^{\#}}$ and $\mathbf{t}$ is a term of ${}^2L^{\#}_{\infty^{\#}}$ that is free for x_i in $A(x_i)$. Note here that $\mathbf{t}$ may be identical with x_i; so that all wffs $\forall x_i A \Rightarrow A$ are axioms by virtue of axiom (7),see [8].

A 8.Gen (Generalization).

$\forall x_i B$ follows from B.

II.Restricted rules of conclusion.

Let $\mathcal{F}_{\mathbf{wff}}$ be a set of the all closed wffs of $L^{\#}_{\infty^{\#}}$.

R1.RMP (Restricted Modus Ponens).

There exist subsets $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.

From A and $A \Rightarrow B$, conclude B iff $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subset \mathcal{F}_{\mathbf{wff}}$.

In particular for any $A, B \in \mathcal{F}_{\mathbf{wff}} : A \Rightarrow_w B \in \Delta_2$.

If $A \notin \Delta_1$ and $(A \Rightarrow B) \notin \Delta_2$ we also abbraviate by $A, A \Rightarrow B \vdash_{RMP} B$.

R2.RMT (Restricted Modus Tollens)

There exist subsets $\Delta_1', \Delta_2' \subset \mathcal{F}_{\mathbf{wff}}$ such that the following rules are satisfied.

$P \Rightarrow Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \Rightarrow Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subset \mathcal{F}_{\mathbf{wff}}$.

In particular for any $P, Q \in \mathcal{F}_{\mathbf{wff}} : P \Rightarrow_w Q \in \Delta_2$.

Remark 2. Note that RMP and RMT easily prevent any paradoxes of naive Cantor set theory (NC), see [1],[9].

III.Additional derived rule of conclusion.

Particularization rule (RPR)
Remind that canonical unrestricted particularization rule (UPR) reads
UPR: If $\mathbf{t}$ is free for x in $B(x)$, then $\forall x[B(x)] \vdash B(\mathbf{t})$, see [8].
Proof.From $\forall x[B(x)]$ and the instance $\forall x[B(x)] \Rightarrow B(\mathbf{t})$ of axiom (A7), we obtain $B(\mathbf{t})$ by unrestricted modus ponens rule.Since x is free for x in $B(x)$, a special case of unrestricted particularization rule is:$\forall xB \vdash B$.
Definition 4.Any formal theory L with a hyper infinitary lenguage $L^{\#}_{\infty^{\#}}$ is defined when the following conditions are satisfied:
1. A hyper infinite set of symbols is given as the symbols of L. A finite or hyperfinite sequence of symbols of L is called an expression of L.
2. There is a subset of the set of expressions of L called the set of well formed formulas (wffs) of L. There is usually an effective procedure to determine whether a given expression is a wff.
3. There is a set of wfs called the set of axioms of L. Most often, one can effectively decide whether a given wff is an axiom; in such a case, L is called an axiomatic theory.
4. There is a finite set $R_1,\ldots,R_n$, of relations among wffs, called rules of conclusion. For each R_i, there is a unique positive integer j such that, for every set of j wfs and each wff B, one can effectively decide whether the given j wffs are in the relation R_i to B, and, if so, B is said to follow from or to be a direct consequence of the given wffs by virtue of R_j.
Definition 5.A proof in L is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that for each i, either B_i is an axiom of L or B_i is a direct consequence of some of the preceding wffs in the sequence by virtue of one of the rules of inference of L.
Definition 6. A theorem of L is a wff B of Y such that B is the last wff of some proof in L. Such a proof is called a proof of B in L.
Definition 7. A wff E is said to be a consequence in L of a set of Γ of wffs if and only if there is a finite or hyperfinite sequence $B_1,\ldots,B_k, k \in \mathbb{N}^{\#}$ of wffs such that E is B_k and, for each i, either B_i is an axiom or B_i is in Γ, or B_i is a direct consequence by some rule of inference of some of the preceding wffs in the sequence. Such a sequence is colled a proof (or deduction) E from Γ. The members of Γ are called the hypotheses or premisses of the proof.
We use $\Gamma \vdash E$ as an abbreviation for E as a consequence of Γ.
In order to avoid confusion when dealing with more than one theory, we write $\Gamma \vdash_L E$, adding the subscript L to indicate the theory in question.
If Γ is a finite or hyperfinite set $\{H_i\}_{1\leq i\leq m}, m \in \mathbb{N}^{\#}$ we write $H_1,\ldots,H_m \vdash E$ instead of $\{H_i\}_{1\leq i\leq m} \vdash E$.
Lemma 1.[8]. $\vdash B \Rightarrow B$ for all wffs B.
Theorem 1.(Generalized Deduction Theorem1). If Γ is a set of wffs and B and E are wffs, and $\Gamma, B \vdash E$, then $\Gamma \vdash B \Rightarrow_s E$. In pticular, if $B \vdash E$ then $\vdash B \Rightarrow E$.
Proof. Let $E_1,\ldots,E_n, n \in \mathbb{N}^{\#}$ be a proof of E form $\Gamma \cup \{B\}$, where E_n is E.
Let us prove, by hyperfinite induction on j, that $\Gamma \vdash B \Rightarrow_s E_j$ for $1 \leq j \leq n$.
First of all, E_1 must be either in Γ or an axiom of L or B itself.
By axiom schema A1, $E_1 \Rightarrow_s (B \Rightarrow_s E_1)$ is an axiom. Hence, in the first two cases,

by MP, $\Gamma \vdash B \Rightarrow_s E_1$ For the third case, when E_1 is B, we have $\vdash B \Rightarrow_s E_1$ by Lemma 1, and, therefore, $\Gamma \vdash B \Rightarrow_s E_1$. This takes care of the case $j = 1$.
Assume now that: $\vdash B \Rightarrow_s E_k$ for all $k < j, j \in \mathbb{N}^\#$. Either E_j is an axiom, or E_j is in Γ, or E_j is B, or E_j follows by modus ponens from some E_l and E_m where $l < j$, $m < j$, and E_m has the form $E_l \Rightarrow_s E_j$. In the first three cases, $\Gamma \vdash B \Rightarrow_s E_j$ as in the case $j = 1$ above. In the last case, we have, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s E_l$ and $\Gamma \vdash B \Rightarrow_s (E_l \Rightarrow_s E_j)$ But, by axiom schema (A2),
$\vdash B \Rightarrow_s (E_l \Rightarrow_s E_j) \Rightarrow_s ((B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j))$
Hence, by MP, $\Gamma \vdash (B \Rightarrow_s E_l) \Rightarrow_s (B \Rightarrow_s E_j)$ and, again by MP, $\Gamma \vdash B \Rightarrow_s E_j$.
Thus, the proof by hyperfinite induction is complete.
The case $j = n \in \mathbb{N}^\#$ is the desired result. Notice that, given a deduction of E from Γ and B, the proof just given enables us to construct a deduction of $B \Rightarrow_s E$ from Γ. Also note that axiom schema A3 was not used in proving the generalized deduction theorem.
Remark 3.For the remainder of the chapter, unless something is said to the contrary, we shall omit the subscript L in $\vdash_L$. In addition, we shall use $\Gamma, B \vdash E$ to stand for $\Gamma \cup \{B\} \vdash E$. In general, we let $\Gamma, B_1, \ldots, B_n \vdash E$ stand for $\Gamma \cup \{B_i\}_{1 \leq i \leq n} \vdash E$.
Remark 4.We shall use the terminology proof, theorem, consequence, axiomatic, etc. and notation $\Gamma \vdash E$ introduced above.
Proposition 1. Every wff B of K that is an instance of a tautology is a theorem of K, and it may be proved using only axioms A1-A3 and MP.
Proposition 2.If E does not depend upon B in a deduction showing that $\Gamma, B \vdash E$, then $\Gamma \vdash E$.
Proof.Let $D_1, \ldots, D_n$ be a deduction of E from Γ and B, in which E does not depend upon B. In this deduction, D_n is E. As an inductive hypothesis, let us assume that the proposition is true for all deductions of length less than $n \in \mathbb{N}^\#$
If E belongs to Γ or is an axiom, then $\Gamma \vdash E$. If E is a direct consequence of one or two preceding wffs by Gen or MP, then, since E does not depend upon B, neither do these preceding wfs. By the inductive hypothesis, these preceding wfs are deducible from Γ alone. Consequently, so is E .
Theorem 2.(Generalized Deduction Theorem 2).Assume that, in some deduction showing that $\Gamma, B \vdash E$, no application of Gen to a wff that depends upon B has as its quantified variable a free variable of B. Then $\Gamma \vdash B \Rightarrow_s E$.
Proof.Let $D_1, \ldots, D_n$ be a deduction of E from Γ and B satisfying the assumption of this theorem. In this deduction, D_n is E. Let us show by hyperfinite induction that $\Gamma \vdash B \Rightarrow_s D_i$ for each $i \leq n \in \mathbb{N}^\#$. If D_i is an axiom or belongs to Γ, then $\Gamma \vdash B \Rightarrow_s D_i$, since $D_i \Rightarrow_s (B \Rightarrow_s D_i)$ is an axiom. If D_i is B, then $\Gamma \vdash B \Rightarrow_s D_i$, since, by Proposition 1, $\vdash B \Rightarrow_s B$. $\cdot$ If there exist j and k less than i such that D_k is $\vdash D_j \Rightarrow_s D_i$, then, by inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$ and $\Gamma \vdash B \Rightarrow_s (D_j \Rightarrow_s D_i)$. Now, by axiom A2,
$\vdash B \Rightarrow_s (D_j \Rightarrow_s D_i) \Rightarrow_s ((B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s D_i))$. Hence, by MP twice, $\Gamma \vdash B \Rightarrow_s D_i$. Finally, suppose that there is some $j < i$ such that D_i is $\forall x_k D_j$.
By the inductive hypothesis, $\Gamma \vdash B \Rightarrow_s D_j$, and, by the hypothesis of the theorem, either D_j does not depend upon B or x_k is not a free variable of B. If D_j does not depend upon B, then, by Proposition 2, $\Gamma \vdash D_j$ and, consequently, by Gen, $\Gamma \vdash \forall x_k D_j$. Thus, $\Gamma \vdash D_i$. Now, by axiom A1, $\vdash D_i \Rightarrow_s (B \Rightarrow_s D_i)$.

So, $\Gamma \vdash B \Rightarrow_s D_i$ by MP. If, on the other hand, x_k is not a free variable of B, then, by axiom A5, $\vdash \forall x_k(B \Rightarrow_s D_j) \Rightarrow_s (B \Rightarrow_s \forall x_k D_j)$ Since $\Gamma \vdash B \Rightarrow_s D_j$, we have, by Gen,$\Gamma \vdash \forall x_k(B \Rightarrow_s D_j)$, and so, by MP,$\Gamma \vdash B \Rightarrow_s \forall x_k D_j$ that is, $\Gamma \vdash B \Rightarrow_s D_i$. This completes the induction, and our proposition is just the special case $i = n$.

Appendix B.The Generalized Recursion Theorem.

Theorem 1. Let S be a set, $c \in S$ and $G : S \to S$ is any function with $\mathbf{dom}(G) = S$ and $\mathbf{range}(G) \subseteqq S$. Let $W[G] \in \mathbb{N}^\# \times S$ be a binary relation such that:
(a) $(1,c) \in W[G]$ and
(b) if $(x,y) \in W[G]$ then $(\mathbf{Sc}(x), G(y)) \in W[G]$.
Then there exists a function $\mathcal{F} : \mathbb{N}^\# \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^\#$and $\mathbf{range}(\mathcal{F}) \subseteqq S$;
(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^\#, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x))$.
1.The desired function $\mathcal{F}$ is a certain relation $\mathbf{W} \subseteqq \mathbb{N}^\# \times S$. It is to have the properties:
(ii′) $(1,c) \in \mathbf{W}$;
(iii′) if $(x,y) \in \mathbf{W}$ then $(\mathbf{Sc}(x), G(y)) \in \mathbf{W}$.
Remark 1. The latter is just another way of expressing (iii), that if

$$\mathcal{F}(x) = y \tag{B.1}$$

then

$$\mathcal{F}(\mathbf{Sc}(x)) = G(y). \tag{B.2}$$

Remark.**2**.Note that any relation $\mathbf{W}$ mentioned above is hyper inductiverelation since the hyper inductivity conditions (ii′)-(iii′) are satisfied.
However there are many hyper inductive relations which satisfy the conditions (ii′)-(iii′); on such is $\mathbb{N}^\# \times S$. What distinguishes the desired function from all these other relations is that we want (a,b) to be on it only as required by (ii′) and (iii′). In other words, it is to be the smallest relation satisfying (ii′)-(iii′). This can be expressed precisely as follows:
(1) Let $\mathbf{M}$ be a set of the relations $\mathbf{W}$ satisfying the conditions (ii′) and (iii′); then we define

$$\mathcal{F} = \bigcap_{\mathbf{W} \in \mathbf{M}} \mathbf{W}.$$

Hence
(2) whenever $\mathbf{W} \in \mathbf{M}$ then $\mathcal{F} \subseteq \mathbf{W}$.
We shall now show that we can derived from (1) that $\mathcal{F}$ is also one relation in $\mathbf{M}$.
(3) $(1,c) \in \mathcal{F}$.
This follows immediately from the definition of $\bigcap_{\mathbf{W} \in \mathbf{M}}$ and the fact that $(1,c) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$.
(4) If $(x,y) \in \mathcal{F}$ then $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$.
For if $(x,y) \in \mathcal{F}$ then $(x,y) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$; hence by (iii′) $(\mathbf{Sc}(x), G(y)) \in \mathbf{W}$ for all $\mathbf{W} \in \mathbf{M}$ so that $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$ by (1).
We must now verify that $\mathcal{F}$ ís actually a function, i,e., we wish to show

that for any $x, z_1, z_2 \in \mathbb{N}^{\#}$, if $(x, z_1) \in \mathcal{F}$ and $(x, z_2) \in \mathcal{F}$, then $z_1 = z_2$.
We shall prove this by hyper infinite induction ( ) on x. Let
(5) $A = \{x | x \in \mathbb{N}^{\#}$ and for all $z_1, z_2 \in \mathbb{N}^{\#}$, if $(x, z_1) \in \mathcal{F}$ and $(x, z_2) \in \mathcal{F}$
then $z_1 = z_2\}$.
We shall show $A = \mathbb{N}^{\#}$ by applying hyper infinite induction ( ). First we have
(6) $1 \in A$.
To prove (6), it suffices to show that for any z, if $(1, z) \in \mathcal{F}$ then $z = c$.
We prove this by contradiction; in other words, suppose to tbe contrary that there is some z with $(1, z) \in \mathcal{F}$ but $z \neq c$. Consider the relation $W = \mathcal{F} \setminus \{(1, z)\}$. Since $(1, c) \in \mathcal{F}$ and $(1, c) \neq (1, z)$, it follows that $(1, c) \in W$. Moreover, whenever $(u, y) \in W$ then $(u, y) \in \mathcal{F}$ and hence $(\mathbf{Sc}(u), G(y)) \in \mathcal{F}$ but $\mathbf{Sc}(u) \neq 1$, so $(\mathbf{Sc}(u), G(y)) \neq (1, z)$, and hence $(\mathbf{Sc}(u), G(y)) \in W$. Thus W satisfies both conditions (ii′) and (iii′); in other words, $\mathbf{W} \in \mathbf{M}$. But then it follows from (2) that $\mathcal{F} \subseteq \mathbf{W}$ however this is elearly false sinee $(1, z) \in \mathcal{F}$ and $(1, z) \notin \mathbf{W}$. Tbus our hypothesis has led us to a contradiction, and henee (6) is proved. Next we show that
(7) for any $x \in \mathbb{N}^{\#}$ if $x \in A$ then $\mathbf{Sc}(x) \in A$.
Suppose that $x \in A$, so that whenever $(x, z_1) \in \mathcal{F}$ and $(x, z_2) \in \mathcal{F}$ then $z_1 = z_2$. We must show that whenever $(\mathbf{Sc}(x), w_1) \in \mathcal{F}$ and $(\mathbf{Sc}(x), w_2) \in \mathcal{F}$ then $w_1 = w_2$. To prove this, it suffices to show that
(8) whenever $(\mathbf{Sc}(x), w) \in \mathcal{F}$ then there exists a z with $w = G(z)$ and $(x, z) \in \mathcal{F}$.
For if (8) ia true, we would have for the given w_1, w_2 some $z_1 = z_2$ with $w_1 = G(z_1)$, $w_2 = G(z_2)$, $(x, z_1) \in \mathcal{F}$ and $(x, z_2) \in \mathcal{F}$. Then, since $x \in A, z_1 = z_2$ and henee $G(z_1) = G(z_2)$, that is, $w_1 = w_2$.
Now to prove (8) suppose, to the contrary, that it is not true; in other words, suppose that we have some w with $(\mathbf{Sc}(x), w) \in \mathcal{F}$ but such that for all z which $(x, z) \in \mathcal{F}$ we have $w \neq G(z)$. Consider the relation $\mathbf{W} = \mathcal{F} \setminus \{(\mathbf{Sc}(x), w)\}$.
We shall show that $\mathbf{W} \in \mathbf{M}$. First of all $(1, c) \in \mathcal{F}$ and $(1, c) \neq (\mathbf{Sc}(x), w)$; hence $(1, c) \in \mathbf{W}$. Suppose tbat $(u, y) \in \mathbf{W}$; then $(u, y) \in \mathcal{F}$ and $(\mathbf{Sc}(u), G(y)) \in \mathcal{F}$.
Clearly if $u \neq x$ then $(\mathbf{Sc}(u), G(y)) \neq (\mathbf{Sc}(x), w)$, so that in this case $(Sc(u), G(y)) \in \mathbf{W}$.
On the other hand, if $u = x$ and $(\mathbf{Sc}(u), G(y)) = (\mathbf{Sc}(x), w)$, then $w = G(y)$, where $(x, y) \in \mathcal{F}$, contrary to the choice of w henee $(\mathbf{Sc}(u), G(y)) \neq (\mathbf{Sc}(x), w))$, so again $(\mathbf{Sc}(u), G(y)) \in \mathbf{W}$. Thus whenever $(u, y) \in \mathbf{W}$, also $(\mathbf{Sc}(u), G(y)) \in \mathbf{W}$. Now that we have shown $\mathbf{W} \in \mathbf{M}$ we see by (2) that $\mathcal{F} \subseteq \mathbf{W}$ but this is false since $(\mathbf{Sc}(x), w) \in \mathcal{F}$ and $(\mathbf{Sc}(x), w) \notin \mathbf{W}$. Thus our hypothesis that (8) is incorrect has led to a contradiction, and now (8) is proved. Sinee (7) follows from (8), we have by hyper infinite induction from (6) that $A = \mathbb{N}^{\#}$. Hence
(9) $\mathcal{F}$ is a function.
We have still to prove that $\mathcal{F}$ satisfies,condition (i); we must show that for each $x \in \mathbb{N}^{\#}$ there· is a y with $(x, y) \in \mathcal{F}$. Since $\mathcal{F} \subseteq \mathbb{N}^{\#} \times S$, it will then follow that $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$ and $\mathbf{range}(\mathcal{F}) \subseteq S$. Let $B = \mathbf{dom}(\mathcal{F})$, that is,
(10) $B = \{x | x \in \mathbb{N}^{\#}$ and for some $y, (x, y) \in \mathcal{F}\}$.
We prove now by hyper infinite induction that $B = \mathbb{N}^{\#}$. First, $1 \in B$, sínce $(1, c) \in \mathcal{F}$ by (3). Next, if $x \in B$, pick some y with $(x, y) \in \mathcal{F}$; then by (4), $(\mathbf{Sc}(x), G(y)) \in \mathcal{F}$, and henee $\mathbf{Sc}(x) \in B$.
Thus concludes the first part of the proof, that there is at least one function $\mathcal{F}$ satisfying conditions (i)-(iii).

Part 2. We prove that there cannot be more than one such function.
Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ both satisfy the conditions (i)-(iii) we wish to show $\mathcal{F}_1 = \mathcal{F}_2$, i.e., that for all $x \in \mathbb{N}^{\#}, \mathcal{F}_1(x) = \mathcal{F}_2(x)$. Thus
is proved by hyper infinite induction on X. By (ii), $\mathcal{F}_1(1) = c$ and $\mathcal{F}_2(1) = c$, so $\mathcal{F}_1(1) = \mathcal{F}_2(1)$. Suppose that $\mathcal{F}_1(x) = \mathcal{F}_2(x)$; then $\mathcal{F}_1(\mathbf{Sc}(x)) = G(\mathcal{F}_1(x))$ and $\mathcal{F}_2(\mathbf{Sc}(x)) = G(\mathcal{F}_2(x))$, so $\mathcal{F}_1(\mathbf{Sc}(x)) = \mathcal{F}_2(\mathbf{Sc}(x))$.
Theorem 2. Let S be a set, $c \in S$ and $G : S \times \mathbb{N}^{\#} \to S$ is a binary funetion with $\mathbf{dom}(G) = S \times \mathbb{N}^{\#}$ and $\mathbf{range}(G) \subseteqq S$.
Then there exists a function $\mathcal{F} : \mathbb{N}^{\#} \to S$ such that:
(i) $\mathbf{dom}(\mathcal{F}) = \mathbb{N}^{\#}$and $\mathbf{range}(\mathcal{F}) \subseteqq S$;
(ii) $\mathcal{F}(1) = c$;
(iii) for all $x \in \mathbb{N}^{\#}, \mathcal{F}(\mathbf{Sc}(x)) = G(\mathcal{F}(x), x)$.
We omit the proof of the Theorem 2 since it can be given by simple modification of the proof to Theorem 1.

Appendix C.General associatíve and commutative laws.

Definition 1. Let $\langle x_1, \ldots, x_n \rangle, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^{\#}$.
Then $Ext\text{-}\sum_{k=m}^{n} x_k$ and $Ext\text{-}\prod_{k=m}^{n} x_k$ are defined for any $n, m \in \mathbb{N}^{\#}$ by the recursions

(i) $Ext\text{-}\sum_{k=m}^{n} x_k = 0$ and $Ext\text{-}\prod_{k=m}^{n} x_k = 1$ if $n < m$;
(ii) $Ext\text{-}\sum_{k=m}^{n} x_k = \left(Ext\text{-}\sum_{k=m}^{n-1} x_k \right) + x_n$ and
(iii) $Ext\text{-}\prod_{k=m}^{n} x_k = x_n \times \left(Ext\text{-}\prod_{k=m}^{n} x_k \right)$ if $m < n$.
The condition (ii) of the above definitíon is justified by recursive definition, see Appendix B.
Definition 2. Let $\langle x_1, \ldots, x_j, \ldots \rangle, j \in \mathbb{N}$ be a countable sequence of elements of $\mathbb{R}_c^{\#}$.
Then ω-sum $Ext\text{-}\sum_{j=m}^{\omega} x_k$ and ω-product $Ext\text{-}\prod_{j=m}^{\omega} x_k$ are defined for any $m \in \mathbb{N}$ by
(iv) $Ext\text{-}\sum_{j=m}^{\omega} x_j \triangleq Ext\text{-}\sum_{j=m}^{n} y_j$, where $\langle y_1, \ldots, y_j, \ldots, y_n \rangle, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ is a hyperfinite sequence such that $x_j = y_j$ for all $j \in \mathbb{N}$ and $y_j = 0$ for all $j \in \mathbb{N}^{\#} \backslash \mathbb{N}$;
(v) $Ext\text{-}\prod_{j=m}^{\omega} x_j \triangleq Ext\text{-}\prod_{j=m}^{\omega} y_j$, where $\langle y_1, \ldots, y_j, \ldots, y_n \rangle, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ is a hyperfinite sequence such that $x_j = y_j$ for all $j \in \mathbb{N}$ and $y_j = 1$ for all $j \in \mathbb{N}^{\#} \backslash \mathbb{N}$.
Theorem 1.Let $\langle x_1, \ldots, x_n \rangle, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^{\#}$.
Then we have

$$Ext\text{-}\sum_{k=m}^{n} x_k = Ext\text{-}\sum_{k=m}^{n-m+q} x_{k+m-q} \tag{1}$$

and

$$z \times \left(Ext\text{-}\sum_{k=m}^{n} x_k \right) = Ext\text{-}\sum_{k=m}^{n} z \times x_k, \tag{2}$$

$z \in \mathbb{R}_c^{\#}$.
Let $\langle x_1,\ldots,x_n \rangle, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ be an hyperfinite sequence of elements of $\mathbb{R}_c^{\#}$. Consider now any infinite nonnegative integers $n_1, n_2, .., n_i, \ldots, n_{\mathbf{t}}, n_i \in \mathbb{N}^{\#} \backslash \mathbb{N}, 1 \leq i \leq \mathbf{t},$ and set

$$n = n_1 + n_2 + \ldots + n_{\mathbf{t}}. \tag{3}$$

Given $x_1,\ldots,x_n$, we can group these as:

$$x_1,\ldots,x_{n_1};\ \ x_{n_1+1},\ldots,x_{n_1+n_2};\ x_{n_1+n_2+1},\ldots,x_{n_1+n_2+n_3};\ldots x_{n_1+n_2+\ldots n_i+1},\ldots,x_{n_1+n_2+\ldots n_{i+1}};\ldots \tag{4}$$

Here, if $n_i = 0$, the corresponding subsequence is regarded as being empty.
Theorem 1. Let $\langle x_1,\ldots,x_k,\ldots \rangle$ be an hyper infiniie sequence of elements of $\mathbb{R}_c^{\#}$.
Let $\langle n_1,\ldots,n_{\mathbf{t}} \rangle$ be a sequence of nonnegalive integers. For each $i = 1,\ldots,\mathbf{t} \in \mathbb{N}^{\#}$, let $m_i = \sum_{j=1}^{i-1} n_j$ and let $n = m_t + n_t$. Then

$$Ext\text{-}\sum_{k=1}^{n} x_k = \sum_{i=1}^{\mathbf{t}} \left(Ext\text{-}\sum_{k=1}^{n_i} x_{m_i+k} \right) \tag{5}$$

and

$$Ext\text{-}\prod_{k=1}^{n} x_k = \prod_{i=1}^{\mathbf{t}} \left(Ext\text{-}\prod_{k=1}^{n_i} x_{m_i+k} \right). \tag{6}$$

Definition 3. A function F is said to be a permutation of a set S if it is one-to-one and $\mathbf{dom}(F) = \mathbf{range}(F) = S$.
Definition 4. Let $[1,n]$ a set $\{k | k \in \mathbb{N}^{\#} \wedge (1 \leq k \leq n)\}$
Theorem 2.Let $\langle x_1,\ldots,x_n \rangle, n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ be an hyperfinite external sequence of elements of $\mathbb{R}_c^{\#}$. Then for any $n \in \mathbb{N}^{\#}$ and any permutalion $\mathbf{F}$ of $[1,n]$ following holds

$$Ext\text{-}\sum_{k=1}^{n} x_k = Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)}. \qquad (\quad)$$

The same holds if we replace $Ext\text{-}\sum$ by $Ext\text{-}\prod$.
Proof. The proof is by hyper infinite induction on $n \in \mathbb{N}^{\#}$. For $n = 1$ it is trivial. Suppose that it is true for n. Let $\mathbf{G}$ be a permutation of $[1,n+1]$.Then $G(m) = n+1$ for a unique m, such that $1 \leq m \leq n+1$. Then by Eq.()

$$Ext\text{-}\sum_{k=1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + x_{n+1} + Ext\text{-}\sum_{k=m+1}^{n+1} x_{\mathbf{G}(k)} \quad (\quad)$$

and by Eq.()

$$Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + x_{n+1} + Ext\text{-}\sum_{k=m+1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{G}(k+1)} + x_{n+1}. \quad (\quad)$$

Thus by Eq.()-Eq.() we obtain

$$Ext\text{-}\sum_{k=1}^{n+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{G}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{G}(k+1)} + x_{n+1}. \qquad (\quad)$$

To reduce this to the inductive hypothesis, we wish to rewrite the external sum of the first

two terms as $Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)}$ for suitable $\mathbf{F}$. Define $\mathbf{F}$ by

$$\mathbf{F}(k) = \begin{cases} \mathbf{G}(k) & \text{if} \quad 1 \le k < m \\ \mathbf{G}(k+1) & \text{if} \quad m \le k \le n \end{cases} \qquad (\quad)$$

Since all valucs of $\mathbf{G}$(k) for $k \neq m,$ we have for all $k \le n$

$$1 \le \mathbf{F}(k) \le n \qquad (\quad)$$

Now we claim that

$\mathbf{F}$ is a permutation of $[1, n]$. $\qquad (\quad)$

By (2), (3) we need only check that $\mathbf{F}$ is one-to one. Suppose that $\mathbf{F}(k_1) = \mathbf{F}(k_2)$. If both k_1, k_2 are $< m$ or both are $\ge m,$ it Iollows from (2) and the fact that $\mathbf{G}$ is a permutation that $k_1 = k_2$. If, say, $k_1 < m \le k_2$, we have $\mathbf{G}(k_1) = \mathbf{G}(k_2 + 1),$ hence $k_1 = k_2 + 1,$ which contradicts our assumption. Thus neither this case· nor, by symmetry, the case $k_2 < m \le k_1$ can occur. We have from (1) and (2) that

$$Ext\text{-}\sum_{k=1}^{m+1} x_{\mathbf{G}(k)} = Ext\text{-}\sum_{k=1}^{m-1} x_{\mathbf{F}(k)} + Ext\text{-}\sum_{k=m}^{n} x_{\mathbf{F}(k)} + x_{n+1} = Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)} + x_{n+1} \qquad (\quad)$$

by (4) and inductive hypothesis

$$Ext\text{-}\sum_{k=1}^{n} x_{\mathbf{F}(k)} + x_{n+1} = Ext\text{-}\sum_{k=1}^{n} x_k + x_{n+1} = Ext\text{-}\sum_{k=1}^{n+1} x_k \qquad (\quad)$$

This equality completes the inductive step and hence the proof of the theorem.