Lerch's Φ and the Polylogarithm at the Negative Integers

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Abstract

At the negative integers, there is a simple relation between the Lerch Φ function and the polylogarithm. The literature has a formula for the polylogarithm at the negative integers, which utilizes the Stirling numbers of the second kind. Starting from that formula, we can deduce a simple closed formula for the Lerch Φ function at the negative integers, where the Stirling numbers are not needed. Leveraging that finding, we also produce alternative formulae for the k-th derivatives of the cotangent and cosecant (ditto, tangent and secant), as simple functions of the negative polylogarithm and Lerch Φ, respectively, which is evidence of the importance of these functions (they are less exotic than they seem). Lastly, we present a new formula for the Hurwitz zeta function at the positive integers using this novelty.

1 Introduction

As seen in reference [4], it's possible to derive a formula for the Hurwitz zeta function at the positive integers with results from both [2] and [3]. To greatly summarize the reasoning presented there, if k is a positive integer greater than one, then:

$$\int_0^1 u^k(1-\cos 2\pi\, n\, u)\cot \pi u\, du \sim -\frac{H(n)}{\pi} - \frac{k!}{\pi}\sum_{j=1}^{\lfloor (k-1)/2\rfloor} \frac{(-1)^j(2\pi)^{-2j}\zeta(2j+1)}{(k-2j)!},$$

which implies the following approximation (with the equality only valid for integer k):

$$\int_0^1 (u^k-u)(1-\cos 2\pi\, n\, u)\cot \pi u\, du \sim -\frac{k!}{\pi}\sum_{j=1}^{\lfloor (k-1)/2\rfloor} \frac{(-1)^j(2\pi)^{-2j}\zeta(2j+1)}{(k-2j)!} = \int_0^1 (u^k-u)\cot \pi u\, du,$$

which in turn justifies the formula[1] shown next.

[1] The formulae that can be derived with this method are not unique and the one shown may be the simplest.

For every integer k greater than one and every non-integer complex b:

$$\zeta(k,b) = \frac{1}{2b^k} + \frac{(2\pi\boldsymbol{i})^k}{4}\left(\frac{\mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)}{(k-1)!} + e^{-2\pi\boldsymbol{i}\,b}\sum_{j=1}^{k}\frac{\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi\boldsymbol{i}\,b})}{(j-1)!(k-j)!}\right)$$
$$-\frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k}{2}\int_0^1\sum_{j=1}^{k}\frac{\left(\delta_{1j}+\mathrm{Li}_{-j+1}(e^{-2\pi\boldsymbol{i}\,b})\right)\left(u^{k-j}e^{-2\pi\boldsymbol{i}\,b\,u}-e^{-2\pi\boldsymbol{i}\,b}\right)}{(j-1)!(k-j)!}\cot\pi u\,du$$

The discovery of a new, possibly first, closed formula for the Lerch transcendent function at the negative integers was made possible through the analysis of the above formula.

The big breakthrough is really in the following straightforward identity, which only works for the analytic continuation of the Lerch Φ and the polylogarithm functions at the negative integers, and makes it possible to obtain the former as a sum of the latter:

$$\sum_{j=0}^{k}\frac{\mathrm{Li}_{-j}\left(e^b\right)u^{k-j}}{j!(k-j)!} = \frac{e^b}{k!}\Phi(e^b,-k,u+1) \qquad (1)$$

In fact, this identity also applies to the sum of Lerches:

$$\sum_{j=0}^{k}\frac{\Phi\left(e^b,-j,v\right)u^{k-j}}{j!(k-j)!} = \frac{1}{k!}\Phi(e^b,-k,u+v) \qquad (2)$$

Last but not least, from two known facts from the literature, namely, a recurrence for the Bernoulli polynomials and a relation between them and the Hurwitz zeta function:

$$B_k(u+v) = k!\sum_{j=0}^{k}\frac{B_j(v)\,u^{k-j}}{j!(k-j)!} \text{ and } B_j(v) = -j\,\zeta(1-j,v),$$

one may conclude that,

$$\zeta(-k,u+v) = -\frac{u^{k+1}}{k+1} + k!\sum_{j=0}^{k}\frac{\zeta(-j,v)\,u^{k-j}}{j!(k-j)!},$$

and from this recurrence, one can obtain a natural expression relating the Hurwitz and the Riemann zeta functions, which completes the picture:

$$\zeta(-k,u+1) = -\frac{u^{k+1}}{k+1} + k!\sum_{j=0}^{k}\frac{\zeta(-j)\,u^{k-j}}{j!(k-j)!} \qquad (3)$$

Now, back to the Hurwitz formula, from relation (1) we obtain,

$$\sum_{j=1}^{k}\frac{\mathrm{Li}_{-j+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)u^{k-j}}{(j-1)!(k-j)!} = \frac{e^{-2\pi\boldsymbol{i}\,b}}{(k-1)!}\Phi(e^{-2\pi\boldsymbol{i}\,b},-k+1,u+1),$$

which leads to the below analytic continuation for the Hurwitz zeta, which then holds for $\Re(k) > 1$:

$$\zeta(k,b) = \frac{1}{2b^k} + \frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\left(e^{-2\pi\boldsymbol{i}\,b} + e^{-4\pi\boldsymbol{i}\,b}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, 2) + \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right)$$
$$- \frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k}{2(k-1)!}\int_0^1 \Big(u^{k-1}e^{-2\pi\boldsymbol{i}\,b\,u} - e^{-2\pi\boldsymbol{i}\,b}$$
$$+ e^{-2\pi\boldsymbol{i}\,b(u+1)}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, u+1) - e^{-4\pi\boldsymbol{i}\,b}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, 2)\Big)\cot\pi u\,du \quad (4)$$

Since this formula depends on the Lerch's Φ analytic continuation at the negative integers, we can rewrite (4), if can we figure out Lerch's Φ at the negative integers.

Relation (1) may be the counterpart to the functional equation that relates the Lerch Φ to the Hurwitz zeta at the positive integers, as demonstrated in a previous paper[5]. This new relation therefore presupposes the existence of another relation between the polylogarithm and the Hurwitz zeta at the negative integers.

2 Stirling numbers of the second kind

The existing formula for the polylogarithm $\mathrm{Li}_{-j+1}(z)$, available in the literature, makes use of the Stirling numbers of the second kind (the $S(j,q)$ in the curly brackets):

$$\mathrm{Li}_{-j+1}(z) = \sum_{q=1}^{j}(q-1)!\left\{ {j \atop q} \right\}\left(\frac{z}{1-z}\right)^q \quad (5)$$

Let's see how to obtain the Lerch Φ from it. Making $z = e^{-2\pi\boldsymbol{i}\,b}$, we have:

$$\frac{z}{1-z} = -\frac{1+\boldsymbol{i}\cot\pi\,b}{2}$$

Therefore, back to equation (1):

$$\sum_{j=1}^{k}\frac{\mathrm{Li}_{-j+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)u^{-j}}{(j-1)!(k-j)!} = \sum_{j=1}^{k}\sum_{q=1}^{j}(q-1)!\left\{ {j \atop q} \right\}\left(-\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q \Rightarrow$$

$$\sum_{j=1}^{k}\sum_{q=1}^{j}\frac{(q-1)!u^{-j}}{(j-1)!(k-j)!}\left\{ {j \atop q} \right\}\left(-\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q \Rightarrow$$

$$\sum_{q=1}^{k}(q-1)!\left(-\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q\sum_{j=q}^{k}\frac{u^{-j}}{(j-1)!(k-j)!}\left\{ {j \atop q} \right\} \quad (6)$$

Now we need to study the second sum from equation (6) and see if it's possible to rewrite it. An expression for a similar sum exists in the literature,

$$\sum_{j=q}^{k} \frac{1}{j!(k-j)!} \left\{ {j \atop q} \right\} = \frac{1}{k!} \left\{ {k+1 \atop q+1} \right\},$$

but this new one is much more complicated.

3 Binomial formula for Stirling numbers

Let's try and rewrite the below finite sum,

$$\sum_{j=q}^{k} \frac{u^{-j}}{(j-1)!(k-j)!} \left\{ {j \atop q} \right\} \tag{7}$$

First off, the literature provides us with the below relation:

$$\sum_{j=q}^{k} \left\{ {j \atop q} \right\} \frac{x^j}{j!} = \frac{(e^x-1)^q}{q!},$$

therefore,

$$\sum_{k=q}^{\infty} \sum_{j=q}^{k} \left\{ {j \atop q} \right\} \frac{x^j}{j!} \frac{y^{k-j}}{(k-j)!} = \frac{e^y (e^x-1)^q}{q!},$$

and differentiating with respect to x,

$$\sum_{k=q}^{\infty} \sum_{j=q}^{k} \left\{ {j \atop q} \right\} \frac{x^{j-1}}{(j-1)!} \frac{y^{k-j}}{(k-j)!} = \frac{e^y e^x (e^x-1)^{q-1}}{(q-1)!}$$

Now, making $x = u\,z$ and $y = z$, we have:

$$\sum_{k=q}^{\infty} z^{k-1} \sum_{j=q}^{k} \left\{ {j \atop q} \right\} \frac{u^{j-1}}{(j-1)!(k-j)!} = \frac{e^{(u+1)z}(e^{u\,z}-1)^{q-1}}{(q-1)!} = \sum_{j=0}^{q-1} \frac{(-1)^{q-1-j} e^{(u+1+j\,u)z}}{j!(q-1-j)!},$$

and hence differentiating $k-1$ times with respect to z, we conclude that,

$$\sum_{j=q}^{k} \frac{u^j}{(j-1)!(k-j)!} \left\{ {j \atop q} \right\} = \frac{u}{(k-1)!} \sum_{j=1}^{q} \frac{(-1)^{q-j}(j\,u+1)^{k-1}}{(j-1)!(q-j)!}, \tag{8}$$

To obtain a more proper version of this formula, we integrate as below,

$$\int_0^u \sum_{j=q}^{k} \frac{x^{j-1}}{(j-1)!(k-j)!} \left\{ {j \atop q} \right\} dx = \frac{1}{(k-1)!} \int_0^u \sum_{j=1}^{q} \frac{(-1)^{q-j}(j\,x+1)^{k-1}}{(j-1)!(q-j)!}\, dx,$$

which gives us the neat expression below, which holds for every non-negative integer q and every u:

$$\sum_{j=q}^{k} \frac{u^j}{j!(k-j)!} \left\{ {j \atop q} \right\} = \frac{1}{k!} \sum_{j=0}^{q} \frac{(-1)^{q-j}(j\,u+1)^k}{j!(q-j)!} \tag{9}$$

Though it's not going to be used here, another pattern similar to the binomial theorem emerges in the factorial power of the sum of two numbers, x and y:

$$(x+y)^{(k)} = k! \sum_{j=0}^{k} \frac{x^{(j)}\, y^{(k-j)}}{j!(k-j)!}, \text{ where } x^{(j)} = \frac{x!}{(x-j)!}$$

4 Lerch's Φ at the negative integers

Now, if we combine equations (1), (6) and (8), we obtain:

$$\Phi(e^{-2\pi \boldsymbol{i}\, b}, -k+1, u+1) = e^{2\pi \boldsymbol{i}\, b} \sum_{q=1}^{k} (q-1)! \left(\frac{1 + \boldsymbol{i} \cot \pi\, b}{2} \right)^q \sum_{j=1}^{q} \frac{(-1)^j (j+u)^{k-1}}{(j-1)!(q-j)!} \tag{10}$$

For integer b the formula is not defined as the cotangent is infinity, so we can't extract the Hurwitz zeta at the negative integers from it. But from the relation,

$$\Phi(e^{-2\pi \boldsymbol{i}\, b}, -k+1, 1) = e^{2\pi \boldsymbol{i}\, b}\, \mathrm{Li}_{-k+1}\left(e^{-2\pi \boldsymbol{i}\, b}\right),$$

we can derive the polylogarithm, which however is just a rewrite of equation (5) with an expression for $S(k,q)$ known from the literature, which nonetheless confirms equation (10):

$$\mathrm{Li}_{-k+1}\left(e^{-2\pi \boldsymbol{i}\, b}\right) = \sum_{q=1}^{k} (q-1)! \left(\frac{1 + \boldsymbol{i} \cot \pi\, b}{2} \right)^q \sum_{j=1}^{q} \frac{(-1)^j\, j^{k-1}}{(j-1)!(q-j)!} \tag{11}$$

Looking at (10) and (11) now, it might look simple to go from the latter straight to the former without having to figure (8), but that's misleading.

Finally, we can turn (10) into another form, which holds for every non-negative integer k:

$$\Phi(z, -k,\, u) = -\frac{1}{z-1} \sum_{q=0}^{k} q! \left(\frac{z}{z-1} \right)^q \sum_{j=0}^{q} \frac{(-1)^j (j+u)^k}{j!(q-j)!} \tag{12}$$

It's interesting to note how much simpler the analytic continuation of the Lerch Φ at the negative integers is than the proper function at the positive integers. And also how strikingly similar it is to the power series for the Lerch Φ at the positive integers available in the literature:

$$\Phi(z, k,\, u) = -\frac{1}{z-1} \sum_{q=0}^{\infty} q! \left(\frac{z}{z-1} \right)^q \sum_{j=0}^{q} \frac{(-1)^j (j+u)^{-k}}{j!(q-j)!}$$

5 Derivatives of trigonometric functions

In his paper *On the Hurwitz function for rational arguments*[1], Victor Adamchik provides the first ever formula for the intricate patterns of the k-th derivatives of the cotangent. It looks like this:

$$\frac{\mathrm{d}^k(\cot a\,x)}{\mathrm{d}\,x^k} = (2\,\boldsymbol{i}\,a)^k(-\boldsymbol{i}+\cot a\,x)\sum_{q=1}^{k} q!\left\{ {k \atop q} \right\}\left(\frac{-1+\boldsymbol{i}\cot a\,x}{2}\right)^q$$

It's possible to express this formula as a simple function of the polylogarithm. First, we rewrite it as:

$$\frac{\mathrm{d}^k(\cot a\,x)}{\mathrm{d}\,x^k} = (2\,\boldsymbol{i}\,a)^k(-\boldsymbol{i}+\cot a\,x)\sum_{q=1}^{k} q!\left(\frac{1-\boldsymbol{i}\cot a\,x}{2}\right)^q \sum_{j=1}^{q}\frac{(-1)^j\,j^{k-1}}{(j-1)!(q-j)!} \tag{13}$$

Secondly, we note how similar it looks to the polylogarithm from (11):

$$\mathrm{Li}_{-k+1}\left(e^{2\,\boldsymbol{i}\,a\,x}\right) = \sum_{q=1}^{k}(q-1)!\left(\frac{1-\boldsymbol{i}\cot a\,x}{2}\right)^q \sum_{j=1}^{q}\frac{(-1)^j\,j^{k-1}}{(j-1)!(q-j)!}$$

If the above polylog is differentiated once with respect to x and transformed,

$$\mathrm{Li}_{-k}\left(e^{2\,\boldsymbol{i}\,a\,x}\right) = \frac{1}{1-e^{2\,\boldsymbol{i}\,a\,x}}\sum_{q=1}^{k} q!\left(\frac{1-\boldsymbol{i}\cot a\,x}{2}\right)^q \sum_{j=1}^{q}\frac{(-1)^j\,j^{k-1}}{(j-1)!(q-j)!}, \tag{14}$$

an alternative expression is obtained for the polylogarithm of order k, which, however, is not exactly equal to form (11). That stems from a property of polylogs, that when differentiated they yield the next order polylog. Finally, comparing the two expressions, (13) and (14), we conclude that:

$$\frac{\mathrm{d}^k(\cot a\,x)}{\mathrm{d}\,x^k} = -\boldsymbol{i}\,\sigma_{0\,k} - 2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,\mathrm{Li}_{-k}\left(e^{2\,\boldsymbol{i}\,a\,x}\right),\ \text{where } \sigma_{0\,k}=1 \text{ iff } k=0 \tag{15}$$

To obtain the cosecant, we can resort to a simple logic:

$$\frac{\cos a\,x+\boldsymbol{i}\sin a\,x}{\sin a\,x} = \frac{e^{\boldsymbol{i}\,a\,x}}{\sin a\,x} = \boldsymbol{i}+\cot a\,x \Rightarrow \frac{1}{\sin a\,x} = e^{-\boldsymbol{i}\,a\,x}(\boldsymbol{i}+\cot a\,x)$$

Then the Leibniz rule for the derivative of a product of two functions leads to:

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{1}{\sin a\,x}\right) = -2\,\boldsymbol{i}\,e^{-\boldsymbol{i}\,a\,x}\sum_{q=0}^{k}\frac{k!\,(-\boldsymbol{i}\,a)^{k-q}}{q!(k-q)!}(2\,\boldsymbol{i}\,a)^q\mathrm{Li}_{-q}\left(e^{2\,\boldsymbol{i}\,a\,x}\right)$$

Lastly, formula (1) allows us to rewrite the above expression as:

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{1}{\sin a\,x}\right) = -2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,e^{\boldsymbol{i}\,a\,x}\,\Phi\left(e^{2\,\boldsymbol{i}\,a\,x}, -k, \frac{1}{2}\right), \tag{16}$$

which holds for every non-negative integer k.

5.1 Tangent and secant

To be able to obtain the tangent and secant, first we need to produce a formula for the cotangent and cosecant of a translated arc. Adamchik's formula[1] becomes:

$$\frac{\mathrm{d}^k\left(\cot\left(a\,x+b\right)\right)}{\mathrm{d}\,x^k}=\left(2\,\boldsymbol{i}\,a\right)^k\left(-\boldsymbol{i}+\cot\left(a\,x+b\right)\right)\sum_{q=1}^{k}q!\left(\frac{1-\boldsymbol{i}\cot\left(a\,x+b\right)}{2}\right)^q\sum_{j=1}^{q}\frac{(-1)^j\,j^{k-1}}{(j-1)!(q-j)!}$$

The polylog formula then changes to:

$$\mathrm{Li}_{-k}\left(e^{2\,\boldsymbol{i}(a\,x+b)}\right)=\frac{1}{1-e^{2\,\boldsymbol{i}(a\,x+b)}}\sum_{q=1}^{k}q!\left(\frac{1-\boldsymbol{i}\cot\left(a\,x+b\right)}{2}\right)^q\sum_{j=1}^{q}\frac{(-1)^j\,j^{k-1}}{(j-1)!(q-j)!},$$

and then the final formula is not different from the simple case:

$$\frac{\mathrm{d}^k\left(\cot\left(a\,x+b\right)\right)}{\mathrm{d}\,x^k}=-\boldsymbol{i}\,\sigma_{0\,k}-2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,\mathrm{Li}_{-k}\left(e^{2\,\boldsymbol{i}(a\,x+b)}\right),\ \text{where }\sigma_{0\,k}=1\text{ iff }k=0$$

Similarly, the cosecant for a translated arc is:

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{1}{\sin\left(a\,x+b\right)}\right)=-2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,e^{\boldsymbol{i}(a\,x+b)}\,\Phi\left(e^{2\,\boldsymbol{i}(a\,x+b)},-k,\frac{1}{2}\right)$$

Finally, to obtain the tangent and secant, we just need to set b to $\pi/2$. And since the formulae for the translated arc are not very different from $b=0$, for the tangent we have:

$$\frac{\mathrm{d}^k\left(\tan\left(a\,x+b\right)\right)}{\mathrm{d}\,x^k}=\boldsymbol{i}\,\sigma_{0\,k}+2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,\mathrm{Li}_{-k}\left(-e^{2\,\boldsymbol{i}(a\,x+b)}\right),\ \text{where }\sigma_{0\,k}=1\text{ iff }k=0,\tag{17}$$

and for the secant:

$$\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{1}{\cos\left(a\,x+b\right)}\right)=2\,\boldsymbol{i}(2\,\boldsymbol{i}\,a)^k\,e^{\boldsymbol{i}(a\,x+b)}\,\Phi\left(-e^{2\,\boldsymbol{i}(a\,x+b)},-k,\frac{1}{2}\right)\tag{18}$$

It's surprising that these derivatives can be expressed by means of negative Lerch and polylogs. For example, the negative polylog is known to yield the derivatives of a simple exponential function at a point, but not the derivative itself:

$$\left.\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(\frac{x}{e^{a\,x+b}-1}\right)\right|_{x=0}=-k\left(\sigma_{1\,k}+\mathrm{Li}_{-k+1}\left(e^b\right)\right)a^{k-1}$$

6 Hurwitz zeta at the positive integers

Now that fomula (10) is known, the Hurwitz zeta formula at the positive integers greater than one can be rewritten with only elementary functions references:

$$\zeta(k,b)=\frac{1}{2b^k}+\frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\left(e^{-2\pi\boldsymbol{i}\,b}+\sum_{q=1}^{k}(q-1)!\left(\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q\sum_{j=1}^{q}\frac{(-1)^j\left(e^{-2\pi\boldsymbol{i}\,b}(j+1)^{k-1}+j^{k-1}\right)}{(j-1)!(q-j)!}\right)$$
$$-\frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k}{2(k-1)!}\int_0^1\left(u^{k-1}e^{-2\pi\boldsymbol{i}\,b\,u}-e^{-2\pi\boldsymbol{i}\,b}+\right.$$
$$\left.\sum_{q=1}^{k}(q-1)!\left(\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q\sum_{j=1}^{q}\frac{(-1)^j\left(e^{-2\pi\boldsymbol{i}\,b\,u}(j+u)^{k-1}-e^{-2\pi\boldsymbol{i}\,b}(j+1)^{k-1}\right)}{(j-1)!(q-j)!}\right)\cot\pi u\,du$$

The pattern of the above formula becomes clearer and looks better if we include the outside terms into the summation symbol, assuming the convention that $(-1)!/(-1)!=1$:

$$\zeta(k,b)=\frac{1}{2b^k}+\frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\sum_{q=0}^{k}(q-1)!\left(\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q\sum_{j=0}^{q}\frac{(-1)^j\left(e^{-2\pi\boldsymbol{i}\,b}(j+1)^{k-1}+j^{k-1}\right)}{(j-1)!(q-j)!}$$
$$-\frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k}{2(k-1)!}\int_0^1\sum_{q=0}^{k}(q-1)!\left(\frac{1+\boldsymbol{i}\cot\pi\,b}{2}\right)^q\sum_{j=0}^{q}\frac{(-1)^j\left(e^{-2\pi\boldsymbol{i}\,b\,u}(j+u)^{k-1}-e^{-2\pi\boldsymbol{i}\,b}(j+1)^{k-1}\right)}{(j-1)!(q-j)!}\cot\pi u\,du$$

One of the advantages of this new formula is the fact it allows one to get rid of its non-real parts more easily, though the resulting formula is inevitably more complicated.

6.1 When the parameter b is a half-integer

The below result stems from $e^{-2\pi\boldsymbol{i}\,b}=-1$ and $\cot\pi\,b=0$ when b is a half-integer:

$$\zeta(k,b)=\frac{1}{2b^k}-\frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\left(1+\sum_{q=1}^{k}(q-1)!\left(\frac{1}{2}\right)^q\sum_{j=1}^{q}\frac{(-1)^j\left((j+1)^{k-1}-j^{k-1}\right)}{(j-1)!(q-j)!}\right)$$
$$-\frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k}{2(k-1)!}\int_0^1\left(u^{k-1}e^{-2\pi\boldsymbol{i}\,b\,u}+1+\sum_{q=1}^{k}(q-1)!\left(\frac{1}{2}\right)^q\sum_{j=1}^{q}\frac{(-1)^j\left(e^{-2\pi\boldsymbol{i}\,b\,u}(j+u)^{k-1}+(j+1)^{k-1}\right)}{(j-1)!(q-j)!}\right)\cot\pi u\,du$$

7 A new formula for the Hurwitz zeta

In [4], we had created a generating function for the Hurwitz zeta function, $f(x)$. When b is not a half-integer, the expression is:

$$f(x)=\sum_{k=2}^{\infty}x^k\zeta(k,b)=-\frac{x^2}{2b(x-b)}-\frac{1}{2\sin\pi b}\frac{\pi x\sin\pi x}{\sin\pi(x-b)}$$
$$-\pi x\int_0^1\left(\frac{\sin 2\pi u(x-b)}{\sin 2\pi(x-b)}-\frac{\sin 2\pi bu}{\sin 2\pi b}\right)\cot\pi u\,du \quad (19)$$

The k-th derivative of $f(x)$ yields the Hurwitz zeta function of order k:

$$\zeta(k,b) = \frac{f^{(k)}(0)}{k!}$$

And now that we know how to differentiate the cosecant successively, it's possible to produce an explicit formula from $f(x)$, again through the Leibniz rule. However, to make this process simpler, we resort to two artifices. First, to get rid of the extra x factor in the integral, we divide $f(x)$ by x and take the $(k-1)$-th derivative instead of the k-th. Second, to avoid the complications of differentiating the sine, we replace it with an equivalent sum of exponential functions.

The first and second parts of (19) are straightforward, they coincide with the terms outside of the integral from (4), that is:

$$\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left(-\frac{1}{2\sin\pi b}\frac{\pi x\sin\pi x}{\sin\pi(x-b)}\right)\Bigg|_{x=0} = \frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\left(e^{-2\pi\boldsymbol{i}\,b} + e^{-4\pi\boldsymbol{i}\,b}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, 2) + \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right)$$

The same isn't true for the third part of (19), since $f(x)$ was created using a different process than (4) (see [4] for details). The integrals evaluate to the same number, but the integrands are not the same.

After all is put together, the final formula holds for any positive integer k greater than one, and any $2b$ that's not an integer:

$$\zeta(k,b) = \frac{1}{2b^k} + \frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\left(e^{-2\pi\boldsymbol{i}\,b} + e^{-4\pi\boldsymbol{i}\,b}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, 2) + \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right) + \\ -\frac{\boldsymbol{i}(2\pi\boldsymbol{i})^k\,e^{-2\pi\boldsymbol{i}\,b}}{4}\int_0^1\sum_{j=1}^{k}\frac{2^j u^{k-j}\left(e^{-2\pi\boldsymbol{i}\,b\,u} - (-1)^{k-j}e^{2\pi\boldsymbol{i}\,b\,u}\right)}{(j-1)!(k-j)!}\Phi\left(e^{-4\pi\boldsymbol{i}\,b}, -j+1, \frac{1}{2}\right)\cot\pi u\,du \quad (20)$$

Finally, by using the relation (2), an analytic continuation can be produced:

$$\zeta(k,b) = \frac{1}{2b^k} + \frac{(2\pi\boldsymbol{i})^k}{4(k-1)!}\left(e^{-2\pi\boldsymbol{i}\,b} + e^{-4\pi\boldsymbol{i}\,b}\,\Phi(e^{-2\pi\boldsymbol{i}\,b}, -k+1, 2) + \mathrm{Li}_{-k+1}\left(e^{-2\pi\boldsymbol{i}\,b}\right)\right) + \\ -\frac{\boldsymbol{i}(4\pi\boldsymbol{i})^k\,e^{-2\pi\boldsymbol{i}\,b}}{4(k-1)!}\int_0^1\left(e^{-2\pi\boldsymbol{i}\,b\,u}\Phi\left(e^{-4\pi\boldsymbol{i}\,b}, -k+1, \frac{u+1}{2}\right) - e^{2\pi\boldsymbol{i}\,b\,u}\Phi\left(e^{-4\pi\boldsymbol{i}\,b}, -k+1, \frac{-u+1}{2}\right)\right)\cot\pi u\,du$$

References

[1] Adamchik, Victor S. *On the Hurwitz function for rational arguments, Applied Mathematics and Computation, Volume 187, Issue 1,* 2007, Pages 3–12.

[2] Risomar Sousa, Jose *Generalized Harmonic Numbers, eprint arXiv:1810.07877,* 2018.

[3] Risomar Sousa, Jose *Generalized Harmonic Progression Part II, eprint arXiv:1902.01008,* 2019.

[4] Risomar Sousa, Jose *The Hurwitz Zeta Function at the Positive Integers, eprint arXiv:1902.06885,* 2019.

[5] Risomar Sousa, Jose *Lerch's* Φ *and the Polylogarithm at the Positive Integers, eprint arXiv:2006.08406,* 2020.