An Operator Theory Problem Book: Introduction and references

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Preface

Operator Theory is such a beautiful and elegant part of modern (pure and applied) Mathematics. It belongs to a larger domain which is Functional Analysis. It is also indispensable to Physics, in particular, Quantum Mechanics as well as some parts of Engineering and Statistics.

The main subject treated in this manuscript is linear operators on a Hilbert space (with a not negligible part on the foundations of Functional Analysis). The book is mainly intended for an undergraduate course: 3rd and 4th (even 5th in some cases) year students depending on each country and on each university. It is also a good source of interesting exercises, problems, examples and counterexamples for lectures and other researchers.

There are good books on Operator Theory and Functional Analysis in the existing literature but only a very few propose such a wide and varied range of exercises on many topics and with very detailed solutions. So, this is one main value of this book. Other features (among others) of this manuscript is the way positive operators and square roots are presented. So is the case with the absolute value of an operator. Also, a special treatment is given to the exponential of an operator.

In general, I have tried as much as possible to stick to the classical facts and notions of Operator Theory. At the same time, I have tried to keep all the material involved readily reachable by an undergraduate readership. This is why deeper notions on Spectral Theory or Operator Theory have not been included or have only been mentioned superficially. For instance, the spectral theorem and the spectral measures have been included but not too deeply. However, the way they are presented and the good amount of examples (such as the important properties of A^α, e^A and $\log(A)$ with some conditions on $A \in B(H)$) will hopefully help students and beginners to understand this beautiful and powerful theorem. Besides, due to the rarity of simple examples in famous books dealing with the spectral theorem, students and nonspecialists often find this theorem fairly hard to understand and they are usually struggling with applying it. In the beginning, they regularly wonder what is permissible? What is prohibited? This is not only my point of view. This opinion is shared by many other mathematicians including the "big names" as "The Halmos" who wrote in [**93**]: "*Another reason the spectral theorem is thought to be hard is that its proof is hard*".

Non-normal operators like subnormal, quasinormal and paranormal operators have not been included. Hyponormal operators, however, are presented in detail. Banach or C^*-algebras have not really been considered either. As regards unbounded operators, they have been covered without considering the concept of the adjoint. As a result, self-adjoint and normal unbounded operators have not been considered. There is, however, a good part on closed operators.

The prerequisites to use this book are basics of:

(1) Real Analysis.
(2) Linear Algebra.
(3) Metric Spaces.
(4) Basic Topology.
(5) Measure and Integration.
(6) Complex Analysis.

The topics covered in this book (split into 13 chapters) are:

(1) Banach Spaces.
(2) Hilbert Spaces.
(3) Bounded Linear Operators.
(4) Closed Operators.
(5) Spectral Theory.
(6) Spectral Theorem and Functional Calculi.
(7) Hyponormal Operators.

The manuscript is divided into two big parts: "Exercises et al." and "Solutions". It contains more than 720 problems (and just over 100 are given without solutions) which are classified in four types and the book is structured as follows (as in [**155**]):

(1) `Basics:` In this part, we recall the essential of notions and results which are needed for the exercises. Some proofs are given as solved exercises but not all of them. So, readers may wish to consult the following standard references for further reading: [**2**], [**12**], [**13**], [**25**], [**48**], [**63**], [**64**], [**85**], [**97**], [**113**], [**115**], [**125**], [**126**], [**127**], [**140**], [**180**], [**192**], [**193**], [**201**], [**212**] and [**218**].
(2) `True or False:` In this part (liked by many), some interesting questions are proposed to the reader. Sometimes, the questions contain traps. Some of these are common errors which

appear with different students almost every year. Thanks to this section, students should hopefully assimilate well the presented material and should avoid making many silly mistakes. Readers may even observe some redundancies in some cases, but this is mainly because it is meant to deepen their understanding. I am certainly not a big fan of pleonasm, however, I do believe that no detail is unnecessary!

(3) `Exercises with Solutions:` The major part and the core of each chapter where many exercises are given with detailed solutions, and as an illustration of this point, we notice that the solutions of Exercises **??** & **??** are over twenty page long!

The exercises are culled from different sources (some are classic, others are research papers and some are made by myself). I have cited most of the relevant references, and if there is some source which I have forgotten to mention, then I sincerely apologize for that. Finally, I have done my best to include some elegant proofs on many topics.

(4) `Tests:` This section usually contains short exercises given with just answers or simply hints.

(5) `More Exercises:` In this part, some exercises are proposed without solutions to the interested reader.

Writing a book in mathematics is not an easy task and most authors would certainly agree with me. I have written many papers and several books but this one has really taken much of my time and in many times I just wanted to abandon the whole project. Nonetheless, even with the hard times I have gone through, I can say that overall I did enjoy writing this book. I really hope that readers will enjoy reading it and that it will benefit them. To conclude, I can say that the day I thought would never come has finally and thankfully arrived! This manuscript was accomplished on January the 06^{th}, 2018 in the city of Oran.

I would like to warmly thank all the staff of World Scientific Publishing Company for their patience and help (in particular, Dr. Lim Swee Cheng and Ms Tan Rok Ting).

Lastly, even though I believe that I have done my best, as a human work, the book is certainly not perfect. So, I will be happy to hear from readers about any possible errors, typos, suggestions or omissions at my personal email address: **mhmortad@gmail.com.**

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