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Proofs of Twin Prime Number Conjecture and First Hardy-Littlewood Conjecture

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Abstract: The twin prime conjecture was proposed by Alfonse de Polignac in 1849 and has not been proven for nearly 300 years. Since there is no mathematical model for prime numbers that can be completely and accurately represented, prime numbers are randomly distributed on the number axis, and twin prime numbers are also randomly distributed. In this paper, the twin prime conjecture is proved by probability and statistics, the twin prime distribution theorem and prime pair distribution theorem are found, and the First Hardy-Littlewood conjecture is further proved.

Keywords: twin prime number conjecture, probability and statistics method, twin prime number distribution theorem, prime number pair distribution theorem, First Hardy-Littlewood conjecture

Twin prime numbers refer to pairs of prime numbers that differ by 2. In 1849, Alphonse de Polignac proposed a general conjecture: let p be a prime number, for any natural number k, there are infinite pairs of prime numbers (p, p + 2k). In this paper, the twin prime conjecture is proved by probability and statistics, the twin prime distribution theorem and prime pair distribution theorem are found, and the First Hardy-Littlewood conjecture is further proved. The method is introduced as follows.

1. Proof method of twin prime number conjecture

Since there is no mathematical model for prime numbers that can be completely and accurately represented, prime numbers are randomly distributed on the number axis, and the distribution of twin prime numbers is also the same. It is a natural and effective method to analyze using probability statistics. If for all natural numbers k, it can be judged that there are infinitely many pairs of prime twins, which proves the conjecture of prime twins.

According to big data statistics, it is found that the appearance of prime numbers within a certain range has the following laws.

When the integer x has a small value, the number of prime numbers in the interval [1, x] is more; when the value is large, the number of prime numbers in the same interval decreases. When the integer value gradually increases, the increase rate of the number of prime numbers in the interval gradually slows down, and the ratio of the prime numbers to natural numbers gradually decreases. According to the prime number theorem, the number of prime numbers in the first x integers is approximately[1]

$$\frac{x}{\ln x} \tag{1}$$

Let x be any natural number, the probability that two odd numbers (2x-1) and (2x+1) being both prime numbers is approximately:

$$\left(\frac{2x+2}{\ln(2x+2)} - \frac{2x}{\ln(2x)}\right)\cdot\left(\frac{2x}{\ln(2x)} - \frac{2x-2}{\ln(2x-2)}\right) \tag{2}$$

From the law of prime numbers, it can be conservatively calculated that the cumulative probability that adjacent odd numbers are both prime numbers in the interval less than the integer (2x+2) is approximately

$$x\cdot\left(\frac{2x+2}{\ln(2x+2)} - \frac{2x}{\ln(2x)}\right)\cdot\left(\frac{2x}{\ln(2x)} - \frac{2x-2}{\ln(2x-2)}\right) \tag{3}$$

Because the discriminant function (3) is similar to a straight with slight radian, showing a divergent shape, as shown in Figure 1 (x is greater than 4), as the natural number x increases, the calculated value gradually increases.

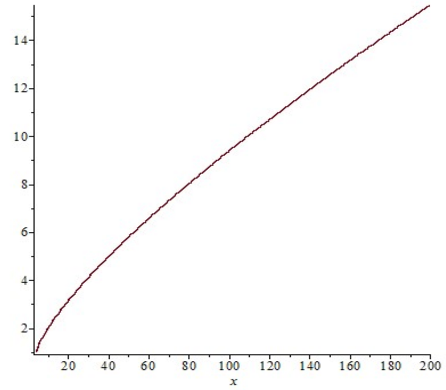

Fig. 1 Discriminant function.

It can be proved that the first derivative of the discriminant function (3) is always positive and takes the x-axis as the asymptote, as shown in Figure 2.

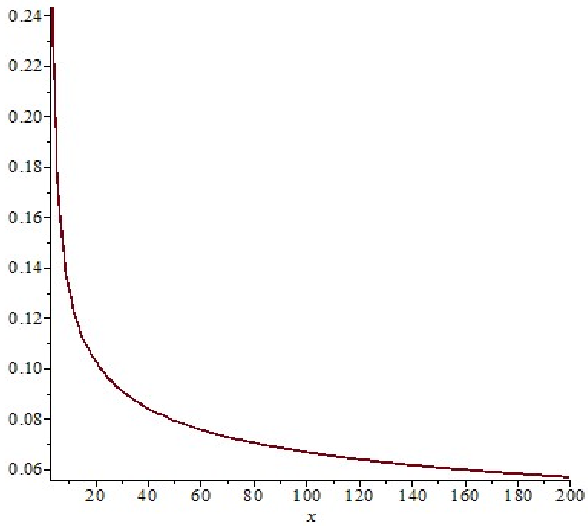

Fig. 2 The first derivative of the discriminant function.

The second derivative of the discriminant function (3) is always negative, as shown in Figure 3.

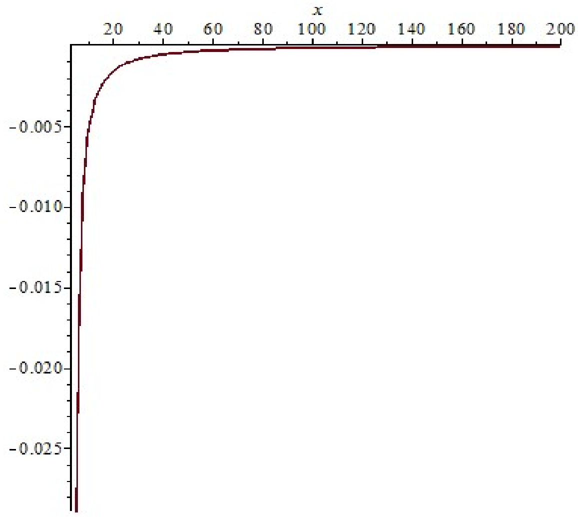

Fig. 3 The second derivative of the discriminant function.

Obviously, the value obtained by calculating the cumulative probability by the discriminant function (3), when x increases infinitely, it tends to infinity, indicating that there are infinitely many pairs of prime twins, which proves the prime twin conjecture.

2. Experimental verification

When the integer (2x+2) is a different value, it is less than the cumulative probability value of adjacent odd numbers in the integer interval that are both prime numbers. See Table 1 for details.

Table 1 The cumulative probability value of adjacent odd numbers being prime numbers

Integer （2x+2）	Probability of adjacent odd numbers being both prime numbers in the interval less than an even number (2x+2)	Actual number of adjacent odd numbers being both prime numbers
100	5.7	7
200	9.3	14
400	15.4	19
500	18.2	22
1000	30.6	32
2000	52.2	57
5000	107.4	119
10000	187.3	194

It can be seen from Table 1 that as the integer value (2x+2) gradually increases, the cumulative probability value of adjacent odd numbers that are both prime numbers also gradually increases, and the actual number of prime numbers where adjacent odd numbers are also prime numbers also gradually increases, both gradually close, and the actual number is slightly larger than the predicted number. With the increase of the integer value, although the growth rate of the probability value of adjacent odd numbers being both prime numbers gradually decreases, the cumulative probability value of the adjacent odd numbers being both prime numbers and the absolute value of the actual number still gradually increase.

And because natural numbers are infinite, as integers approach infinity, the calculated value is infinite, indicating that there are infinite pairs of prime twins, which proves the conjecture of prime twins.

Since the actual value of the prime number and the predicted value gradually approach as the integer becomes larger, the ratio limit is 1 [2]. Therefore, it can be shown that the cumulative probability value of adjacent odd numbers calculated by applying the prime number theorem is valid and reliable.

3. Proof of First Hardy-Littlewood conjecture

The First Hardy-Littlewood conjecture was proposed by two outstanding mathematicians Hardy and Littlewood [3]. The conjecture is that let τ (x) Is the number of prime numbers p ≤ x, and p + 2 is also a prime number, then there is a constant C (called twin prime number constant), so that

$$\tau(x) \sim 2C\frac{x}{\ln(x)^2} \tag{4}$$

Proof method: let x be an integer, and according to the judgment function (1), it can be deduced that the cumulative probability value (i.e. the number of twin prime pairs) that the adjacent prime numbers in the interval less than integer x are prime numbers is about:

$$\frac{\mathrm{x}}{2}\left(\frac{\mathrm{x}}{\ln(x)}-\frac{x-2}{\ln(x-2)}\right)\cdot\left(\frac{x-2}{\ln(x-2)}-\frac{x-4}{\ln(x-4)}\right) \tag{5}$$

Because when x approaches infinity, x-2 and x-4 approach the value of x, therefore, the above formula is simplified as

$$\frac{\mathrm{x}}{2}\left(\frac{\mathrm{x}}{\ln(x)}-\frac{x-2}{\ln(x)}\right)\cdot\left(\frac{x-2}{\ln(x)}-\frac{x-4}{\ln(x)}\right) \tag{6}$$

After simplification, it is:

$$2 \cdot \frac{x}{\ln(x)^2} \tag{7}$$

This proves the First Hardy-Littlewood conjecture:

$$\tau(x) \sim 2C \frac{x}{\ln(x)^2}$$

where the coefficient C is 1.

4. Conclusion

For all natural numbers k, because there are infinitely many natural numbers, the cumulative probability that adjacent odd numbers are both prime numbers is infinite, indicating that there are infinitely many pairs of prime twins, which proves the prime twin conjecture. At the same time, the First Hardy-Littlewood conjecture is further proved by the derivation of the mathematical formula of the number of pairs of twin prime numbers.

5. Corollaries

5.1 The theorem of twin prime number distribution: the number of twin prime number pairs smaller than the integer (2x+2) is approximately:

$$x \cdot \left(\frac{2x+2}{\ln(2x+2)} - \frac{2x}{\ln(2x)} \right) \cdot \left(\frac{2x}{\ln(2x)} - \frac{2x-2}{\ln(2x-2)} \right) \tag{8}$$

5.2 Prime number pair distribution theorem: k, x, and y are natural numbers, and k<x<y, the number of prime number pairs greater than an integer (2x+2k) and less than an integer (2y+2k) is approximately:

$$y \cdot \left(\frac{2y+2k}{\ln(2y+2k)} - \frac{2y}{\ln(2y)} \right) \cdot \left(\frac{2y}{\ln(2y)} - \frac{2y-2k}{\ln(2y-2k)} \right)$$
$$-x \cdot \left(\frac{2x+2k}{\ln(2x+2k)} - \frac{2x}{\ln(2x)} \right) \cdot \left(\frac{2x}{\ln(2x)} - \frac{2x-2k}{\ln(2x-2k)} \right) \tag{9}$$

5.3 The same method can be used to find the number of groups of triplet prime numbers and quadruplet prime numbers less than integer 2x+2.

5.4 The same method can prove the prime number pair, triplet prime number and quadruplet prime number conjecture.

6. References

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