Application of Multipoints Summation Method to Nonlinear Differential Equations

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Abstract

I suggest a new approximate approach, the Multipoints Summation method, to solve non-linear differential equations analytically. The method connects several local asymptotic series. I present applications of the method to two examples of non-linear differential equations: saddle-node bifurcation and the non-linear differential equation of the pendulum. Explicit approximate solutions expressed in terms of elementary functions are obtained from an analysis of phase space. This approach may be also applied to other non-linear differential equations.

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1 Introduction

There are very few methods to solve nonlinear differential equations[1] exactly. The qualitative properties of the solution of a non-linear differential equation strongly depends on the initial conditions. Non-linear differential equations are classified by dimension and strength of nonlinearity. In the late 1800s, H. Poincaré studied nonlinear equations qualitatively rather than quantitatively. This approach pioneered the modern study of non-linear differential equations having many interesting applications such as chaos.

Another powerful tool to study non-linear differential equations is numerical simulation but this requires careful handling. One demerit of numerical methods is that we need to re-calculate the solution for each initial condition. On the other hand, analytical methods often have the demerit that the obtained solution may be too complicated or even that it may be impossible to express the solution of a non-linear differential equation analytically.

In my previous papers and books[2, 3, 4, 5], I and K. Slevin demonstrated a new kind of approximate re-summation method so called multipoints summation method. A first application of the multipoints summation method was the estimation of the critical exponent ν of the Anderson transition by connecting asymptotic behaviors at $d = 2$ and $d = \infty$. The multipoints summation method gave estimates of the critical exponent in reasonable agreement with the available numerical results for $d = 3, 4, 5, 6$ and $2 < d < 3$. This suggests the possibility to obtain the global behavior from asymptotic series at several points sufficiently separated from each other.

There are cases where the long time behavior of the solution of a non-linear differential equation may be obtained from an analysis of phase space. This suggests a new approximate approach to obtain the solution of a non-linear differential equation explicitly by the multipoints summation method using asymptotic behavior of several points including short time behavior for given initial conditions.

The motivation for my work is to demonstrate the possibility of multipoints summation method to obtain approximate solution of non-linear differential equation explicitly including initial conditions. For this purpose, it is effective to use phase space analysis to get several asymptotic behavior in addition to short time asymptotic behavior. The multipoints summation method is expected to be useful when phase space has fixed points.

In this paper, I discuss the application of the multipoint summation method to two examples of non-linear differential equations: saddle-node bifurcation and the non-linear differential equation of the pendulum. The paper is organized as follows. In Sect. 2, I discuss how a fixed point in phase space gives the asymptotic behavior of the solution of a non-linear differential equation. The multipoints summation method is shown to yield a global approximate solution of a non-linear differential equation. In Sect. 3, I use a two point Padé approximant to obtain an approximate solution of the equation of saddle-node bifurcation. In Sect. 4, I use a new multipoints summation method to obtain an approximate solution of the equation of the pendulum. In Sect. 5, I conclude.

2 Fixed point in phase space and basic idea of the multipoints summation method

Let's consider, a general non-linear differential equation.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x). \tag{1}$$

Here, f is a function of x. The fixed points $x = x^*$ of this non-linear differential equation is given by

$$f(x^*) = 0. \tag{2}$$

A fixed point is classified by its property, i.e. attractive, repulsive, or neutral. If x^* is an attractive fixed point and the initial value of $x = x_0$ is enough close to the attractive fixed point, the long time asymptotic behavior of solution of the non-linear differential equation is given by

$$x(t) \sim x^* + o(1) \quad (t \to \infty). \tag{3}$$

In addition to this, the early time asymptotic behavior makes it possible to apply the multipoints summation method to obtain an approximate analytic solution of the non-linear differential equation. The multipoints summation method guarantees that the approximate analytic solution obeys Eq.(3)

Let's consider a non-linear differential equation that has a constant parameter c.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x; c). \tag{4}$$

If we can solve this equation at any two points $c = c_1, c_2$. we may apply the multipoints summation method. Then we will obtain approximate analytic solution for any c which exactly matches the solutions for both $c = c_1$ and $c = c_2$.

3 Two points Padé approximant for the equation of saddle-node bifurcation

The non-linear differential equation of saddle-node bifurcation is given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r + x^2. \tag{5}$$

An attractive fixed point exists when r is negative.

$$x = x^* = -\sqrt{-r}. \tag{6}$$

The short time behavior of the equation is given by a Taylor expansion. In this paper, I truncate 6-th order terms and higher to approximate the solution.

$$x(t) \equiv f_0(t) \sim x_0 + x_1 t + x_2 t^2 + x_3 t^3 + x_4 t^4 + x_5 t^5 + O(t^6) \;\; (t \to 0). \tag{7}$$

Here, $x(0) = x_0$ is the initial condition. By substituting this expansion into the differential equation, we obtain.

$$x_1 = x_0^2 + r \tag{8}$$
$$x_2 = x_0(x_0^2 + r) \tag{9}$$
$$x_3 = x_0^4 + \frac{4}{3}x_0^2 r + \frac{1}{3}r^2 \tag{10}$$
$$x_4 = x_0^5 + \frac{5}{3}x_0^3 r + \frac{2}{3}x_0 r^2 \tag{11}$$
$$x_5 = x_0^6 + 2x0^4 r + \frac{17}{15}x_0^2 r^2 + \frac{2}{15}r^3 \tag{12}$$

If r is negative, the long time behavior of the equation is given by

$$x(t) \equiv f_\infty(t) \sim -\sqrt{-r}. \quad (t \to \infty). \tag{13}$$

Since we have obtained two asymptotic series at $t = 0$ and $t = \infty$, we can apply a two points Padé approximant. We separate the asymptotic term,

$$x(t) \simeq f_\infty(t) + (f_0(t) - f_\infty(t)). \tag{14}$$

Then, we apply ordinal Padé approximant to the second term. I chose the [1/4] Padé approximant which does not have any artificial singularity point. I calculated this approximant using the computer algebra software Maple.

$$x(t) \simeq f_\infty(t) + \frac{P(t)}{Q(t)}. \tag{15}$$

Here,

$$\begin{aligned}
P(t) \equiv\ & 4r^2x_0^3 + 3(-r)^{3/2}x_0^4 - 4r^3x_0 \\
-\ & 3(-r)^{3/2}rx_0^2 + 2r\sqrt{-r}x_0^4 - 2r^2(-r)^{3/2} \\
+\ & 3r^2\sqrt{-r}x_0^2 - 3\sqrt{-r}r^3 \\
+\ & [-(2/5)r^2x_0^4 + (12/5)r^3x_0^2) + 6r(-r)^{3/2}x_0^3 \\
-\ & (2/5)r^4) + (22/5)r^2(-r)^{3/2}x_0 \\
+\ & (22/5)r^2\sqrt{-r}x_0^3 + 6r^3\sqrt{-r}x_0]t \qquad (16)\\
Q(t) \equiv\ & 3(-r)^{3/2}x_0^3 + 2r(-r)^{3/2}x_0 + 3r^2x_0^2 \\
+\ & 2r\sqrt{-r}x_0^3 - r^3 + 5r^2\sqrt{-r}x_0 \\
+\ & [-3(-r)^{3/2}x_0^4 - 3r(-r)^{3/2}x_0^2 - (12/5)r^2x_0^3 \\
-\ & 2r\sqrt{-r}x_0^4 - (2/5)r^2(-r)^{3/2} - (4/5)r^3x_0 \\
-\ & (21/5)r^2\sqrt{-r}x_0^2 - r^3\sqrt{-r}t \\
+\ & [-(3/5)r^2x_0^4 - (6/5)r^2\sqrt{-r}x_0^3 + (3/5)r^4) \\
-\ & (6/5)r^3\sqrt{-r}x_0]t^2 \\
+\ & [(8/15)r^3x_0^3 - (4/15)r^2\sqrt{-r}x_0^4 + (8/15)r^4x_0)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (4/15)\sqrt{-r}r^4]t^3 \\
&+ \quad [r^3 x_0^4/15 + (2/15)r^3\sqrt{-r}x_0^3 - r^5/15 \\
&+ \quad (2/15)r^4\sqrt{-r})x_0]t^4
\end{aligned} \tag{17}
$$

Thus, the solution of the non-linear differential equation is approximated by a combination of initial functions.

I will refer to the above approximate solution as a two points solution. First, let's compare the two points solution for $r = -1$ and several initial conditions x_0 with a numerical solution. For $r = -1$, we obtain,

$$
\begin{aligned}
P(t) \quad &\simeq \quad x_0^4 + 4x_0^3 + 6x_0^2 + 4x_0 + 1 \\
&+ \quad [(-2/5)x_0^4 - (8/5)x_0^3 - (12/5)x_0^2 - (8/5)x_0 - 2/5]t
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
Q(t) \quad &\simeq \quad x_0^3 + 3x_0^2 + 3x_0 + 1 \\
&+ \quad [-x_0^4 - (12/5)x_0^3 - (6/5)x_0^2 + (4/5)x_0 + 3/5]t \\
&+ \quad [-(3/5)x_0^4 - (6/5)x_0^3 + (6/5)x_0 + 3/5]t^2 \\
&+ \quad [-(4/15)x_0^4 - (8/15)x_0^3 + (8/15)x_0 + 4/15]t^3 \\
&+ \quad [-(1/15)x_0^4 - (2/15)x_0^3 + (2/15)x_0 + 1/15]t^4
\end{aligned} \tag{19}
$$

In Fig. 1, I compare the two points solution and the numerical solution for the case $x_0 = -2$. If the initial value is equal to attractive fixed point, we obtain the exact solution,

$$x(t) \simeq -1 \tag{20}$$

If the initial value is equal to the repulsive fixed point, we obtain the exact solution,

$$x(t) \simeq 1 \tag{21}$$

In Fig. 2, I compare the two points solution and the numerical solution for the case $x_0 = 0$. In Fig. 3, I compare the two points solution and the numerical solution for the case $x_0 = 2$. In this case, we have a divergence at finite t for both the two points solution and the numerical solution.

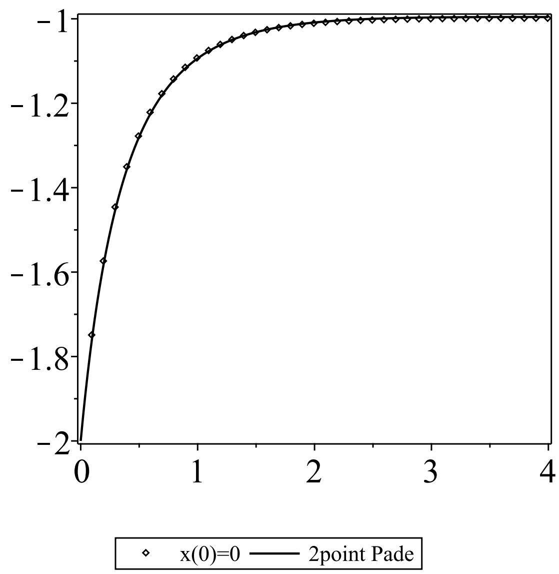

Figure 1: Comparison between the two points solution and a numerical solution for $r = -1, x_0 = -2$

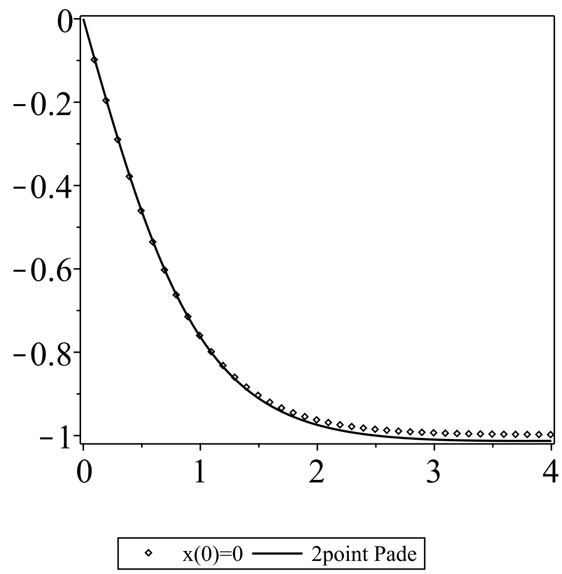

Figure 2: Comparison between the two points solution and a numerical solution for $r = -1, x_0 = 0$

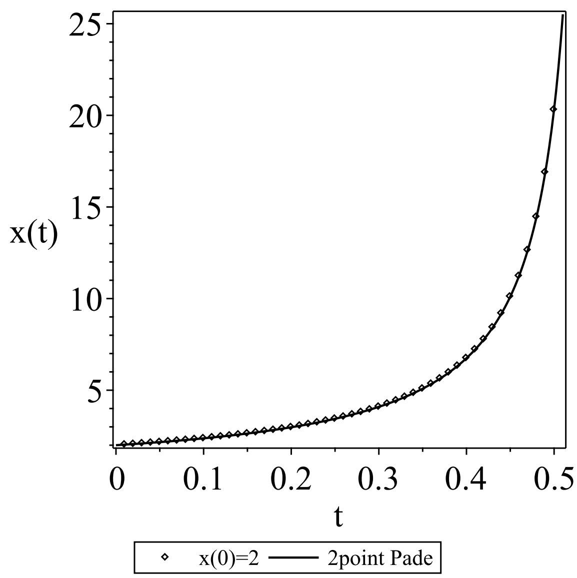

Figure 3: Comparison between the two points solution and a numerical solution for $r = -1, x_0 = 2$

Next I consider $r \to 0$. In this case, the two points solution is equal to the exact solution of the non-linear differential equation,

$$x(t) = \frac{x_0}{1 - x_0 t}. \tag{22}$$

Lastly, for $r > 0$, there is no fixed point.

4 The multipoints summation method for the equation of the pendulum

In this section, I demonstrate the multipoints summation method for the non-linear differential equation of the pendulum. The equation is

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\sin x. \tag{23}$$

For simplicity I consider the following initial condition.

$$x(0) = 0 \tag{24}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}(0) = v_0. \tag{25}$$

If v_0 is close enough to 0, the approximate solution of the equation is

$$x(t) = v_0 \sin t \tag{26}$$

If v_0 is big enough, the approximate solution is

$$x(t) = v_0 t \tag{27}$$

We have two asymptotic behavior for $v_0 = 0, \infty$ Therefore, a possible Padé approximant is given by

$$x(t) \simeq \frac{v_0 \sin(t) + P(v_0) v_0 t}{1 + Q(v_0)} \tag{28}$$

Here, $P(v_0)$ and $Q(v_0)$ are polynomials with the same order and the same coefficient of the highest order term and without a constant term. For simplicity, I take $P(v_0) = Q(v_0) = v_0$. From this rough approximation, we cannot expect a good quantitative estimate of the solution. But, we may obtain good qualitative estimate of the phase space diagram. In Fig.4, I show the phase space diagram obtained from the two points Padè approximant. In Fig.5, I show the exact phase space diagram calculated from the equation of conservation of energy., i.e.,

$$\frac{1}{2} \frac{\mathrm{d}x}{\mathrm{d}t}^2 - \cos x = E. \tag{29}$$

Here, E is a constant. These figures suggest that a qualitative description of phase space is possible using the two point Padé approximant. The quantitative estimation will be improved if we incorporate next order term of v_0 in asymptotic series at either $v_0 = 0$ or $v_0 \to \infty$

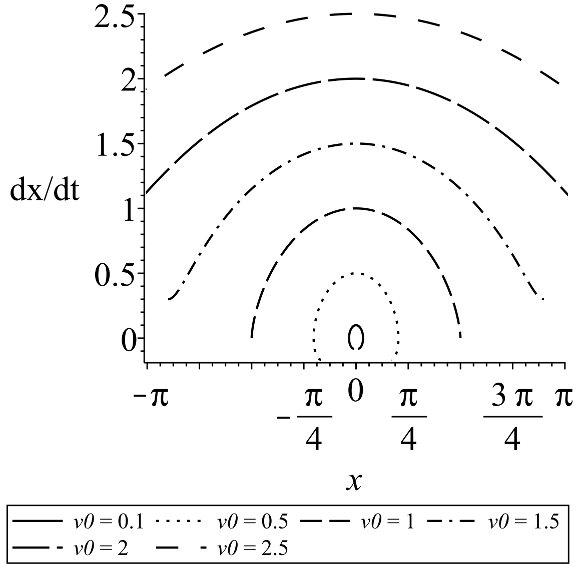

Figure 4: Phase space diagram of the pendulum obtained from a two points Padé approximant

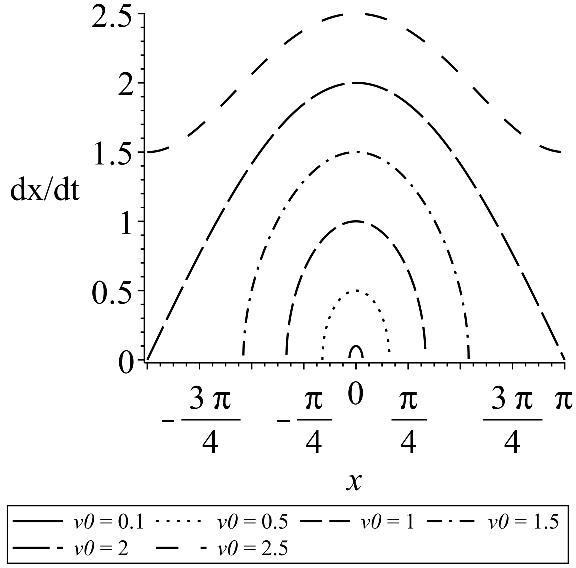

Figure 5: Exact phase space diagram of the pendulum

5 Conclusion

I described two examples of the application of multipoints summation method to non-linear differential equations. If an attractive fixed point exists, multipoints summation method can be a strong tool to solve non-linear differential equations approximately. If there is no attractive fixed point, a two points Padè approximant is available. In this case, we can estimate qualitative properties of the phase space diagram. However, for non-linear differential equation with many degrees of freedom, an approach similar to the two points Padé approximant is not yet clear. Although, multipoints summation method is still developing, it is expected to be already useful in many cases, especially for the case of one degree of freedom.

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