A NEW PROPOSITION OF FIBONACCI NUMBER

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Abstract. **C.A.Church** and **Marjorie Bicknell** gave a version which was exponential generating function for Fibonacci number, in 1973.

In this paper, I will give some results about the Fibonacci identities.

1. introduction

Theorem 1.1. *Suppose that m,n are any natural number, $\{F_n\}_{n=0}^{\infty}$ and $\{L_n\}_{n=0}^{\infty}$ are Fibonacci Sequence and Lucas Sequence, respectively. Then we have*

$$\sum_{a+b=n} F_{ma}L_{mb} = (n+1)F_{mn} \tag{1.1}$$

and

$$\sum_{a+b=n} \binom{n}{a} F_{ma}L_{mb} = 2^n F_{mn}. \tag{1.2}$$

Proof. From Binet Formula, we know that

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \tag{1.3}$$

and

$$L_n = \alpha^n + \beta^n \tag{1.4}$$

where n be arbitrary natural number,

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad and \qquad \beta = \frac{1-\sqrt{5}}{2}. \tag{1.5}$$

Hnece,

$$\sum_{a+b=n} F_{ma}L_{mb} = \frac{1}{\sqrt{5}} \sum_{a+b=n} (\alpha^{mn} - \beta^{mn} + \alpha^{ma}\beta^{mb} - \alpha^{mb}\beta^{ma}) \tag{1.6}$$

and

$$\sum_{a+b=n} F_{mb}L_{ma} = \frac{1}{\sqrt{5}} \sum_{a+b=n} (\alpha^{mn} - \beta^{mn} + \alpha^{mb}\beta^{ma} - \alpha^{ma}\beta^{mb}) \tag{1.7}$$

Since,

$$\sum_{a+b=n} F_{ma}L_{mb} = \sum_{a+b=n} F_{mb}L_{ma}. \tag{1.8}$$

We can conclude the first identity. Likely first identity, we have

$$\sum_{a+b=n} F_{ma}L_{mb} = \frac{1}{\sqrt{5}} \sum_{a+b=n} \binom{n}{a} (\alpha^{mn} - \beta^{mn} + \alpha^{ma}\beta^{mb} - \alpha^{mb}\beta^{ma}) \tag{1.9}$$

By the proposition,

$$\sum_{a+b=n} \binom{n}{a} F_{ma}L_{mb} = \sum_{a+b=n} \binom{n}{b} F_{mb}L_{ma}. \tag{1.10}$$

We can get

$$\sum_{a+b=n} F_{ma}L_{mb} = \frac{1}{\sqrt{5}}\sum_{a+b=n}\binom{n}{b}(\alpha^{mn}-\beta^{mn}+\alpha^{mb}\beta^{ma}-\alpha^{ma}\beta^{mb}). \tag{1.11}$$

Combining the sums, we can get the second result. □

2. SOME FIBONACCI IDENTITIES

From paper whch written by **C.A.Church** and **Marjorie Bicknell**, we knew that

$$g(t) = \frac{1}{\sqrt{5}}(\exp(\alpha t) - \exp(\beta t)) = \sum_{n=0}^{\infty}\frac{F_n t^n}{n!} \tag{2.1}$$

and

$$h(t) = \exp(\alpha t) + \exp(\beta t) = \sum_{n=0}^{\infty}\frac{L_n t^n}{n!} \tag{2.2}$$

for all $|t| < 1$. Here is giving second proof for second identity

Proof. Beginning the proving, we product *g(t)* and *h(t).*

$$g(t)h(t) = \sum_{n=0}^{\infty}\sum_{a+b=n}\frac{F_a L_b}{a!b!}t^n. \tag{2.3}$$

And

$$g(t)h(t) = \sum_{n=0}^{\infty}\frac{2^n F_n t^n}{n!}. \tag{2.4}$$

Comparing the coefficient of t^n, conclude the result. □

Theorem 2.1. *Suppose that n be arbitrary natural number, then*

$$\sum_{a+b=n}\binom{2n}{2a}F_{2a}L_{2b} = 2^{2n-1}F_{2n} \tag{2.5}$$

Proof. Letting the generating functions are

$$g(t) = \sum_{n=0}^{\infty}\frac{F_{2n}t^{2n}}{(2n)!} = \frac{1}{\sqrt{5}}(\cosh(\alpha t) - \cosh(\beta t)) \tag{2.6}$$

and

$$h(t) = \sum_{n=0}^{\infty}\frac{L_{2n}t^{2n}}{(2n)!} = (\cosh(\alpha t) + \cosh(\beta t)). \tag{2.7}$$

Making a convolution, then we have

$$g(t)h(t) = \sum_{n=0}^{\infty}(\sum_{a+b=n}\frac{F_{2a}L_{2b}}{(2a)!(2b)!})t^{2n}. \tag{2.8}$$

In the other hand,

$$g(t)h(t) = \frac{1}{\sqrt{5}}(\cosh^2(\alpha t) - \cosh^2(\beta t)) = \frac{1}{2\sqrt{5}}(\cosh(2\alpha t) - \cosh(2\beta t)). \tag{2.9}$$

In fact,

$$g(t)h(t) = \frac{1}{2\sqrt{5}}(\cosh(2\alpha t) - \cosh(2\beta t)) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n}F_{2n}t^{2n}}{(2n)!}. \tag{2.10}$$

Comparing the coefficient of t^{2n}, we can conclude

$$\sum_{a+b=n} \binom{2n}{2a} F_{2a} L_{2b} = 2^{2n-1} F_{2n} \tag{2.11}$$

□

In general, the identity can be rewritten by

$$\sum_{a+b=n} \binom{2n}{2a} F_{2ma} L_{2mb} = 2^{2n-1} F_{2mn} \tag{2.12}$$

Next result was known with a period, and it is easily to show the result.

Corollary 2.2. *Suppose that n be arbitrary natural number, then*

$$\sum_{a+b=n} \binom{2n}{2a} = 2^{2n-1} \tag{2.13}$$

Proof. We are starting from

$$g(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} = \cosh(t) \tag{2.14}$$

Likely previous proving,

$$g(t)^2 = \sum_{n=1}^{\infty} \frac{t^{2n}}{(2a)!(2b)!} + 1 = \cosh^2(t) = \frac{\cosh(2t)+1}{2}. \tag{2.15}$$

And

$$g(t)^2 = \frac{\cosh(2t)+1}{2} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{2n} t^{2n}}{(2n)!} + 1. \tag{2.16}$$

Consequently, getting the result. □

Lemma 2.3. *Suppose that n be arbitrary natural number, then*

$$\sum_{a+b=n} \frac{1}{(3a)!(3b)!} = \frac{2^{3n} + 2(-1)^n}{3(3n)!} \tag{2.17}$$

Proof. It is need to prove

$$\sum_{a+b=n} \binom{3n}{3a} = \frac{2^{3n} + 2(-1)^n}{3} \tag{2.18}$$

enough. First, we consider

$$(1+1)^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n} \quad where \quad \omega = \exp(\frac{2\pi i}{3}), \tag{2.19}$$

getting the

$$\sum_{a+b=n} \binom{3n}{k} (1 + \omega^k + \omega^{2k}) = \sum_{a+b=n} \binom{3n}{k} (1 + \exp(\frac{2k\pi i}{3}) + \exp(\frac{4k\pi i}{3})). \tag{2.20}$$

Hence

$$\sum_{a+b=n} \binom{3n}{k} (1 + \exp(\frac{2k\pi i}{3}) + \exp(\frac{4k\pi i}{3})) = 3 \sum_{a+b=n} \binom{3n}{3a}. \tag{2.21}$$

And

$$(1+1)^{3n} + (1+\omega)^{3n} + (1+\omega^2)^{3n} = 2^{3n} + (\frac{1+\sqrt{3}i}{2})^{3n} + (\frac{1-\sqrt{3}i}{2})^{3n}. \tag{2.22}$$

In other words

$$2^{3n} + (\frac{1+\sqrt{3}i}{2})^{3n} + (\frac{1-\sqrt{3}i}{2})^{3n} = 2^{3n} + (\exp(\frac{\pi i}{3}))^{3n} + (\exp(\frac{5\pi i}{3}))^{3n} = 2^{3n} + 2(-1)^n. \tag{2.23}$$

So that, concluding the result. □

Lemma 2.4. *Assume the generating functions $g(t)$ and $h(t)$ are*

$$g(t) = \sum_{n=0}^{\infty} \frac{1}{(3n)!} t^{3n} \tag{2.24}$$

and

$$h(t) = \sum_{n=0}^{\infty} (\sum_{a+b=n} \frac{1}{(3a)!(3b)!}) t^{3n} = \sum_{n=0}^{\infty} (\frac{2^{3n} + 2(-1)^n}{3(3n)!}) t^{3n} \tag{2.25}$$

respectively. Then

$$g(t) = \frac{1}{3}(\exp(t) + \exp(\exp(\frac{2\pi i}{3})t) + \exp(\exp(\frac{4\pi i}{3})t)) \tag{2.26}$$

and

$$h(t) = g(t)^2. \tag{2.27}$$

Proof. Since, we knew that

$$\exp(\exp(\frac{2\pi i}{3})t) = \sum_{n=0}^{\infty} \frac{\exp(\frac{2n\pi i}{3})}{(n)!} t^n, \tag{2.28}$$

$$\exp(\exp(\frac{4\pi i}{3})t) = \sum_{n=0}^{\infty} \frac{\exp(\frac{4n\pi i}{3})}{(n)!} t^n, \tag{2.29}$$

$$\exp(t) = \sum_{n=0}^{\infty} \frac{1}{(n)!} t^n. \tag{2.30}$$

Combining all the infinite series, we have

$$g(t) = \frac{1}{3}(\exp(t) + \exp(\exp(\frac{2\pi i}{3})t) + \exp(\exp(\frac{4\pi i}{3})t)). \tag{2.31}$$

Similarly, making a convolution to $g(t)$ to conclude the second result. □

Theorem 2.5. *Suppose that n be arbitrary natural number, then*

$$\sum_{a+b=n} \binom{3n}{3a} F_{3a} L_{3b} = \frac{2^{3n} + 2(-1)^n}{3} F_{3n}. \tag{2.32}$$

Proof. By the previous lemma **2.4**, we have known that

$$\sum_{n=0}^{\infty} \frac{F_{3n}}{(3n)!} t^{3n} = \frac{1}{\sqrt{5}}(g(\alpha t) - g(\beta t)) \tag{2.33}$$

and

$$\sum_{n=0}^{\infty} \frac{L_{3n}}{(3n)!} t^{3n} = g(\alpha t) + g(\beta t). \tag{2.34}$$

Via lemma **2.4**, hence

$$\sum_{n=0}^{\infty} (\sum_{a+b=n} \frac{F_{3a} L_{3b}}{(3a)!(3b)!}) t^{3n} = \frac{1}{\sqrt{5}}(g(\alpha t)^2 - g(\beta t)^2) = \frac{1}{\sqrt{5}}(h(\alpha t) - h(\beta t)). \tag{2.35}$$

Comparing the coefficient of t^{3n}, where

$$\frac{1}{\sqrt{5}}(h(\alpha t) - h(\beta t)) = \sum_{n=0}^{\infty} (\frac{(2^{3n} + 2(-1)^n) F_{3n}}{3(3n)!}) t^{3n}. \tag{2.36}$$

As a result, we have

$$\sum_{a+b=n} \binom{3n}{3a} F_{3a} L_{3b} = \frac{2^{3n} + 2(-1)^n}{3} F_{3n}. \tag{2.37}$$

□

As before identity, we have a generalization

$$\sum_{a+b=n} \binom{3n}{3a} F_{3ma} L_{3mb} = \frac{2^{3n} + 2(-1)^n}{3} F_{3mn}. \tag{2.38}$$

3. APPLICATION

Corollary 3.1. *Assume that q be arbitrary prime number which satisfy*

$$q = 2p + 1 \tag{3.1}$$

where p be other prime number. Then q can be represented by Fibonacci Number and Lucas Number, that is,

$$2p + 1 = \frac{1}{F_{2p}} \sum_{a+b=2p} F_a L_b \tag{3.2}$$

Proof. Taking $m = 1$ and $n = 2p$ to first identity, get the result. □

Example 3.2. *Letting $q = 5$ which*

$$5 = 2 \times 2 + 1, \tag{3.3}$$

$$5 = \frac{1}{F_4} \sum_{a+b=4} F_a L_b = \frac{1}{3}(0 \times 7 + 1 \times 4 + 1 \times 3 + 2 \times 1 + 3 \times 2) \tag{3.4}$$

Example 3.3. *Letting $q = 7$ which*

$$7 = 2 \times 3 + 1, \tag{3.5}$$

$$7 = \frac{1}{F_6} \sum_{a+b=6} F_a L_b = \frac{1}{8}(0 \times 18 + 1 \times 11 + 1 \times 7 + 2 \times 4 + 3 \times 3 + 5 \times 1 + 8 \times 2) \tag{3.6}$$

References

[1] Sara Maad Sasane; Amol Sasane *A Friendly Approach To Complex Analysis*, World Scientific, Singapore, 2014.

[2] C.A.Church; Marjorie Bicknell, *EXPONENTIAL GENERATING FUNCTIONS FOR FIBONACCI NUMBER*, The Fibonacci Quarterly, **11.3** (1973), 275–281.

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