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Theoretical Study on the Kinetics of a Special Particle Swarm

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Prior studies have focused on the overall behavior of randomly moving particle swarms. However, the characteristics of ubiquitous special particle swarms that form in these swarms remain unknown. This study demonstrates a generalized diffusion equation for randomly moving particles that considers the velocity and location aggregation effects in a special circumstance (that is, in a moving reference frame $\mathcal{R}_u$ relative to a stationary reference frame $\mathcal{R}_0$). This equation can be approximated as the Schrödinger equation in the microcosmic case and describes the kinetics of the total mass distribution in the macrocosmic case. The predicted density distribution of the particle swarm in the stable aggregation state is consistent with the total mass distribution of massive, relaxed galaxy clusters (at least in the range of $r < r_\mathrm{S}$), preventing cuspy problems in the empirical Navarro-Frenk-White (NFW) profile. This article is helpful for inspiring people to think about the essence of universal gravitation.

Keywords: Randomly Moving Particles; Effects of Location Aggregation; Relaxed Galaxy Clusters; Generalized Diffusion Equation

1. Introduction

The kinetics of randomly moving particles have been extensively studied in the past. However, previous studies have been based on the case in which the means (velocity and density) of the particles in the target (sub-) domain are equal to those in the total (parent) domain (Fig. 1) or the particle swarm in the sub- and parent domains are not distinguished[1, 2, 3]. In fact, there are some special subparticle swarms with low probabilities in the particle swarm that are formed by randomly moving particles. For example, during a certain period, the subparticle swarm ($\mathcal{R}_u$) with a constant velocity relative to the parent particle swarm[4] belongs to this category (Fig. 1). These special subparticle swarms are accidental phenomena for the particles in the parent domain, but for the observers near these subparticle swarms, they are determined "gifts" from nature (survivor bias). These cases are also the more common existences we see and are meaningful to human beings (if the whole universe is regarded as composed of very small particles, the galaxy in the galaxy cluster, the solar system in the Milky Way, and the atoms on the earth are similar to this kind of phenomenon). Therefore, it is necessary to study particle swarms in common but special cases.

These special particle swarms, as a portion of the total particle swarm in a completely random state, may be in a variety of different states. In a certain period and a fixed target domain (the volume is fixed and the location can move with the average velocity of the target particle swarm, the same is done below), when a subparticle swarm is in a completely random (free) state, the location distribution of the particles in that state follows the Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location. The velocity direction distribution is also consistent with the population (the norm of the average velocity follows the same Maxwell distribution). When a subparticle swarm remains in a special accidental state for a certain period, it is equivalent to the subparticle swarm being subject to some constraints and being in a non-completely random state. According to the constraint situation of the subparticle swarm, we divide it into the following three

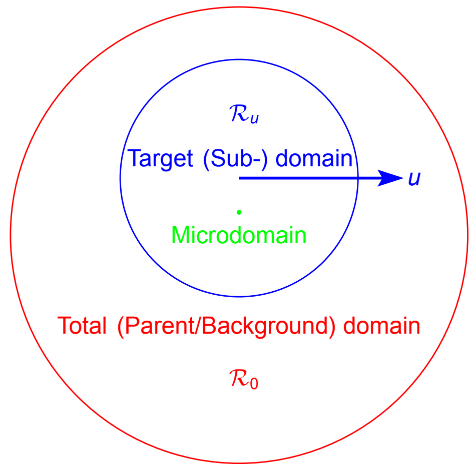

Figure 1. Relationship between the Total (Parent/Background) Domain (Red), Target (Sub-) Domain (Blue) and Microdomain (Green).

types of constrained states: For the first type of constrained state, in a certain period and a fixed target domain, the location distribution of the particles follows a Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location, but the norms of the average velocities do not follow the Maxwell distribution. The special case of this state is that the average velocity norms of all counted particles are constant at u under the unchanged location distribution condition, which is called $\mathrm{I}u$ (Fig. 2a). For the second type of constrained state, in a certain period and a fixed target domain, the norms of the average particle velocities follow the Maxwell distribution, but the location distribution of the particles in the domain does not follow the Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location. The special case of this state is that the number of particles in the fixed target domain is fixed under the condition that the velocity direction distribution remains unchanged. For the third type of constrained state, in a certain period and a fixed target domain, the norms of the average particle velocities do not follow the Maxwell distribution, and the location distribution of the particles in the domain does not follow the Poisson distribution based on time with the same strength as the Poisson distribution of the population based on location. The special case of this state is that the number of particles is fixed and the average velocity norm of all particles is fixed as u in the fixed target domain, which is called $\mathrm{III}u$ (Fig. 2b). The abovementioned subparticle swarm ($\mathcal{R}_u$) with a constant average velocity during a certain period belongs to $\mathrm{III}u$.

When a subparticle swarm in the constrained state of $\mathrm{III}u$ ($\mathcal{R}_u$ or the target domain) is observed in the total domain ($\mathcal{R}_0$), it has the characteristics of location aggregation and velocity direction aggregation, which affect the diffusion rate constant of the particles. Therefore, the kinetic phenomena of this type of particle swarm exhibit some special properties. This article focused on the particle swarm in the constrained state of $\mathrm{III}u$, deduced the diffusion equation of the particles in this case and identified the formation conditions of a non-diffusion particle swarm. The basic structure of this article is as follows. The mathematical model was deduced step-by-step based on the defined physical model. Before derivation, two verifications were performed. First, it was confirmed that the physical model contained special relativistic effects; second, the Schrödinger equation was derived from the physical model under certain conditions. The process of the two checks also clarified how to derive the mathematical model, that is, the generalized diffusion equation. The process of deriving the generalized

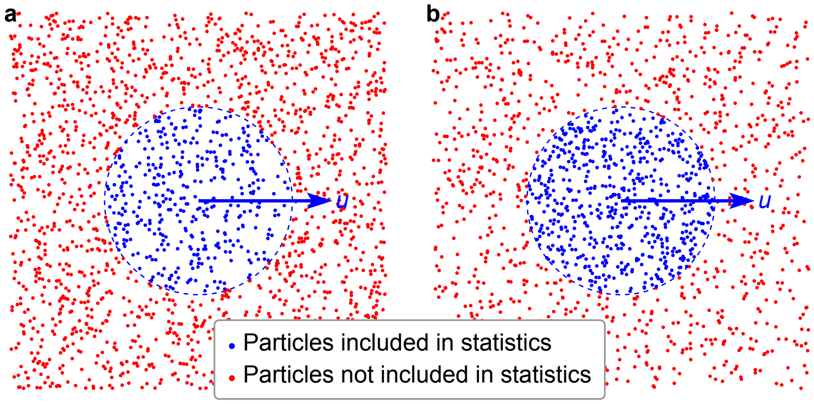

Figure 2. Relationships between the target (sub-) particles/domain and the total particles/domain. **a**, The constrained state of $\mathrm{I}u$: the number of blue particles follows the Poisson distribution based on time with the same strength as the Poisson distribution of the red particles based on location. **b**, The constrained state of $\mathrm{III}u$: the number of blue particles is fixed.

diffusion equation includes the following: (**i**) vector decomposition. The decomposition of nonmoving particles in space is extended to the decomposition of a 2-dimensional vector representing the sum of the 3-dimensional vector of moving particles at a certain point in space, which is the core of the whole derivation. (**ii**) The classic diffusion coefficient is reinterpreted and the essential key information is obtained. (**iii**) Based on (**i**) and (**ii**), the equations are assembled according to the classical diffusion principle to obtain the generalized diffusion equation. In addition, some important parts related to the equation are discussed and verified. The following is a detailed description.

2. Methods

In this study, a mathematical model was obtained by logical derivation based on a physical model. Mathematica 13.0.1.0 for Mac (*Wolfram Research Inc*.) was used for all of the mathematical calculations, and the hardware was a Mac mini (Z12P) with a macOS Monterey 12.3.1 operating system. The solutions to each specific problem can be found in Supplementary Information.

3. Results and Discussions

3.1. Physical Model

It is assumed that there are countless identical point particles with certain masses in an infinite 3-dimensional space. Their speed is c, the motion directions of each particle are evenly distributed in a 3-dimensional space, and there is no interaction between these particles. Our research object is a subset of such particles. The particles in this subset are in the special case of the third type of constrained state (i.e., $\mathrm{III}u$, the blue domain in Fig. 2b).

3.2. Special Relativistic Effects in the Constrained State of $\mathrm{I}u$

In this article, the "point particles" described above are called "particles" or "1-particles", while larger finite-mass-level particles composed of k particles are called "k-particles". The k-particles or aggregates mentioned in this section are k-generalized-particles or aggregates. The k-particle term means that only k particles are counted, but it does not matter whether they truly gather together. The 1-particles can be represented by random vectors with equal norms that are equal to the same movement speeds in Euclidean space. Thus, the "random vectors" and "randomly moving particles (or velocities)" mentioned in this article have the same meaning.

My previous study[3] has proven that the vector group in the constrained state of $\mathrm{III}u$ formed by random vectors with equivalent norms has a special relativistic effect. That is, because of the statistical effect, when the centroid of the subparticle swarm moves at a speed of u in one direction, the particles or the generalized k-particles formed by the subparticles either lose a certain degree of freedom in other directions or the movement trends in other directions decrease, resulting in the effect of special relativity. Here, the slowing ratio $\dfrac{\sqrt{c^2-u^2}}{c}$ of the particles in $\mathcal{R}_u$ or generalized aggregates they form is recorded as $\Gamma[\cdot]$ or Γ (we call it the Γ, or $\Gamma[\cdot]$, effect). Although the particles in $\mathcal{R}_u$ are in the constrained state of $\mathrm{I}u$ when observed from $\mathcal{R}_0$, they are in a completely random state when observed from $\mathcal{R}_u$. Moreover, my previous study[3] has confirmed that all the physical laws are the same as when studying a k-generalized-particle in $\mathcal{R}_0$ observed from $\mathcal{R}_0$ and in $\mathcal{R}_u$ observed from $\mathcal{R}_u$. In the constrained state of $\mathrm{I}u$, the particles themselves or the generalized particles formed by the particles show the effect of special relativity; in the constrained state of $\mathrm{III}u$, the aggregation effect also includes location aggregation (but they are not related to each other). Here, these two (aggregation) effects combined with the simultaneous effects of the velocity direction and location aggregation are collectively called the statistical effect of randomly moving particles; such particles are in the constrained state of $\mathrm{III}u$. When these statistical effects work together, the generation conditions of a non-diffusion particle swarm can be obtained. This is explained in detail below.

3.3. Establishment of the Classical Diffusion Equation in the Constrained State of $\mathrm{I}u$

Regardless of how these particles move in 3-dimensional space, their trajectories are continuous, which leads to diffusion (or agglomeration) behavior, which is the generalized diffusion of randomly moving particles in the constrained state of $\mathrm{III}u$. Considering particles of the same mass and speed, the generalized diffusivity of the corresponding random vectors is equivalent to the generalized diffusivity of random momenta (which are also vectors). It is considered that the scale of the "generalized diffusivity of vectors" is simply the scale that is most suitable for describing the invariant laws for randomly moving particles. More information will be lost if the scale is even slightly more macroscopic (e.g., the scale can be approximately described by real diffusion), and there will be no invariant statistical law to follow if the scale is even slightly more microscopic (for example, the scale described at the beginning of this paragraph). At this scale, the external behavior of the vectors in a tiny space cannot be considered isotropic. Before studying the particles in the constrained state of $\mathrm{III}u$, we first study the particles in the constrained state of $\mathrm{I}u$. For the time being, the Γ effect is not considered here; this is consistent with the scenario of a completely free state. Compared with the $\mathrm{III}u$ case, there is only diffusion without agglomeration, and the other cases are consistent. According to the Maxwell distribution, the total vector in a certain domain always points in an uncertain direction, and the norm is directly

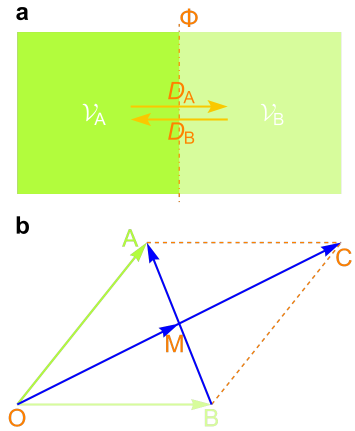

Figure 3. Illustration of the principle of the generation of a mutual diffusion potential in microdomains $\mathcal{V}_{\mathrm{A}}$ and $\mathcal{V}_{\mathrm{B}}$.

proportional to $\sqrt{k}$, where k is the number of vectors (see Part 1 of the Supplementary Information for details). Although the direction of the total vector in a tiny space cannot be determined from the Maxwell distribution, we hope to use appropriate constraints to obtain the distribution rules governing the norm and direction of the total vector at any location in space.

First, we determine the constraints acting on spatial vectors (norms and directions). Let the density of the vector sum at some point $\mathcal{P}$ in space be denoted by $\boldsymbol{\mathcal{X}}$, which is a function of location and time, that is, $\boldsymbol{\mathcal{X}}(x, y, z, t)$. It is defined as follows: at a certain time t, let $\boldsymbol{\mathcal{Y}}(\mathcal{V})$ be a function of the sum of all vectors in the closed domain $\mathcal{V}$ containing $\mathcal{P}(x, y, z)$; and $\boldsymbol{\mathcal{X}}(x, y, z, t) = \lim\limits_{\mathcal{V}\to\mathcal{P}} \dfrac{\boldsymbol{\mathcal{Y}}(\mathcal{V})}{\mathcal{V}}$ [in the following, $\boldsymbol{\mathcal{X}}$ is also a function of the spatial coordinates (x, y, z) and the time coordinate t].

$\boldsymbol{\mathcal{X}}$ is the statistical average vector. The relationship between $\boldsymbol{\mathcal{X}}$ and the number of vectors follows a Maxwell distribution. As illustrated in Fig. 3a, it is assumed that there are two microdomains $\mathcal{V}_{\mathrm{A}}$ and $\mathcal{V}_{\mathrm{B}}$ of the same size along the normal direction on both sides of the segmentation surface Φ. If the sum of all vectors in $\mathcal{V}_{\mathrm{A}}$ is $\overrightarrow{\mathrm{OA}}$ and the sum of all vectors in $\mathcal{V}_{\mathrm{B}}$ is $\overrightarrow{\mathrm{OB}}$, then their sum is $\overrightarrow{\mathrm{OC}}$, and their difference is $\overrightarrow{\mathrm{BA}}$. Let the sum and difference vectors intersect at point M (Fig. 3b). Because the velocity direction distribution is homogeneous and there is no need to consider the statistical effects due to location aggregation here, considering the previous assumption that the domains $\mathcal{V}_{\mathrm{A}}$ and $\mathcal{V}_{\mathrm{B}}$ on both sides of Φ are equal, after the particles randomly move and mix, both vectors must tend to approach their average value $\overrightarrow{\mathrm{OM}}$; that is, both $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ tend toward $\overrightarrow{\mathrm{OM}}$. The change rate of $\overrightarrow{\mathrm{OA}}$ or $\overrightarrow{\mathrm{OB}}$ to $\overrightarrow{\mathrm{OM}}$ depends on the difference between $\overrightarrow{\mathrm{OA}}$ and $\overrightarrow{\mathrm{OB}}$ and the diffusion (motion) rate of particles. Accordingly, the rate of change in $\boldsymbol{\mathcal{X}}$ along the normal direction at a particular point should be related to the time-dependent rate of change in $\boldsymbol{\mathcal{X}}$. This time-dependent rate of change is also affected by another inherent factor (i.e., the velocity of the particles forming $\boldsymbol{\mathcal{X}}$), the concrete value of which is temporally uncertain. Therefore, the above two rates of change should be directly proportional when the differences between particles caused by density (location aggregation of particles) are neglected.

In view of the similar calculus properties of vector and scalar, the derivation method for real diffusion is imitated here. If a domain $\mathcal{W}$ is enclosed by a closed surface $\varSigma$, then during the infinitesimal period $\mathrm{d}t$, the directional derivative $\dfrac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{N}}$ of $\boldsymbol{\mathcal{X}}$ along the normal direction of an infinitesimal area element $\mathrm{d}S$ on the surface $\varSigma$ is directly proportional to the vector $\mathrm{d}\boldsymbol{\mathcal{X}}$ flowing through $\mathrm{d}S$ along the normal

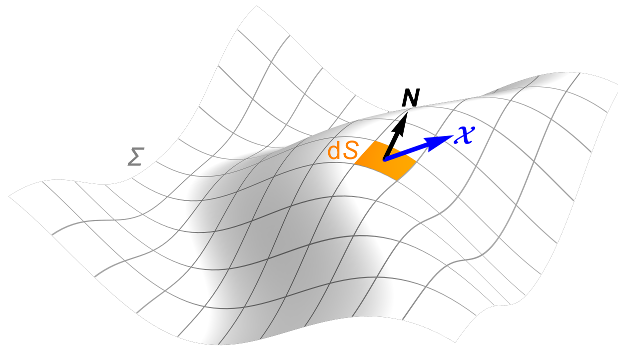

Figure 4. Illustration of the diffusion of the vector sum density $\boldsymbol{\mathcal{X}}$.

direction in the closed domain $\mathcal{W}$ enclosed by Σ (Fig. 4), under the assumption that the coefficient is a positive real number D.

From time t_a to time t_b, when the influence of the vector density on D is not considered (i.e., the diffusion coefficient is the same at every location), the variation of the vector sum $\boldsymbol{\mathcal{A}}$ inside the closed surface Σ is

$$\delta\boldsymbol{\mathcal{A}} = \int_{t_\mathrm{a}}^{t_\mathrm{b}} \left(\oiint_{\Sigma} D \frac{\partial \boldsymbol{\mathcal{X}}}{\partial \boldsymbol{N}} \mathrm{d}S \right) \mathrm{d}t. \tag{1}$$

According to the Gaussian formula, Eq. 1 can also be written in the form

$$\delta\boldsymbol{\mathcal{A}} = \int_{t_\mathrm{a}}^{t_\mathrm{b}} \left(\iiint_{\mathcal{W}} D \Delta \boldsymbol{\mathcal{X}} \mathrm{d}x \mathrm{d}y \mathrm{d}z \right) \mathrm{d}t, \tag{2}$$

where Δ is the Laplace operator, which describes the second derivative with respect to location (x, y, z). The left-hand side of Eq. 1 ($\delta\boldsymbol{\mathcal{A}}$) can also be written as

$$\delta\boldsymbol{\mathcal{A}} = \iiint_{\mathcal{W}} \left(\int_{t_\mathrm{a}}^{t_\mathrm{b}} \frac{\partial \boldsymbol{\mathcal{X}}}{\partial t} \mathrm{d}t \right) \mathrm{d}x \mathrm{d}y \mathrm{d}z. \tag{3}$$

By setting the right of Eq. 3 equal to the right of Eq. 2 and transforming the order of integration, we can obtain

$$\int_{t_\mathrm{a}}^{t_\mathrm{b}} \iiint_{\mathcal{W}} \frac{\partial \boldsymbol{\mathcal{X}}}{\partial t} \mathrm{d}x \mathrm{d}y \mathrm{d}z \mathrm{d}t = \int_{t_\mathrm{a}}^{t_\mathrm{b}} \iiint_{\mathcal{W}} D \Delta \boldsymbol{\mathcal{X}} \mathrm{d}x \mathrm{d}y \mathrm{d}z \mathrm{d}t. \tag{4}$$

Based on the observation that t_a, t_b and domain $\mathcal{W}$ are all arbitrary, the following equation can be written:

$$\frac{\partial \boldsymbol{\mathcal{X}}}{\partial t} = D \Delta \boldsymbol{\mathcal{X}}. \tag{5}$$

To facilitate the task of vector decomposition in the constrained state of $\mathrm{III}u$, a 3-dimensional vector needs to be converted into a plane vector. Next, we determine the constraints acting on plane vectors.

Although the operation in Eq. 5 is performed using 3-dimensional vectors, when differential operations are performed on a spatial vector, the (sum or) difference operations are always performed at two points on the vectors that are separated by an infinitesimal distance; thus, all 3-dimensional vectors can exhibit only relative 2-dimensional characteristics. Consequently, by solving this differential equation, only 2-dimensional constraints can be obtained. Therefore, only the derivatives of plane vectors are needed to act as the derivatives of the 3-dimensional vectors (in this case, plane vectors can retain the important information, such as the norms of the vectors and the included angle between them). Moreover, according to the Sturm-Liouville theory, the function of plane vectors obtained by solving the partial differential equation expressed in terms of plane vectors is unique and corresponds to the 3-dimensional vectors obtained from a differential equation of the same form. It is assumed that the function of plane vectors describing the density of the vectors or momenta is $\boldsymbol{\mathcal{M}}(x, y, z, t)$, which corresponds to $\boldsymbol{\mathcal{X}}$ at the point (x, y, z, t) [unless otherwise stated, in the following, $\boldsymbol{\mathcal{M}}$ is a function of the spatial coordinates (x, y, z) and the time coordinate t]. Thus, $\boldsymbol{\mathcal{X}}$ can be replaced with $\boldsymbol{\mathcal{M}}$. After this replacement, it is obvious that the norm of the plane vector does not change, but its direction will be reoriented. Finally, Eq. 5 can be written as

$$\left\| \frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} \right\| = D \left\| \Delta \boldsymbol{\mathcal{M}} \right\| . \tag{6}$$

Now, let us determine the constraints on the direction of the plane vector $\boldsymbol{\mathcal{M}}$. In view of the continuity of the trajectories of point particles, because $\boldsymbol{\mathcal{M}}$ is also characterized in terms of the statistical properties of an enormous number of particles, it should also be smooth. According to the theory of plane curves, the first and second derivatives of a plane vector in any direction in space are vertical. If an equation relating these derivatives is established according to the above derivative relationship (Eq. 6), the direction needs to be adjusted to be consistent; otherwise, the equations cannot be equal; then, the unique and definite relationship can be written in the form

$$\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = \mathbf{i}\, D \Delta \boldsymbol{\mathcal{M}}, \tag{7}$$

where $\mathbf{i}$ is an imaginary unit. By multiplying both sides of Eq. 7 by $\mathbf{i}$, the form of the Schrödinger equation (without an external field) can be obtained as

$$\mathbf{i}\, \frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = -D \Delta \boldsymbol{\mathcal{M}}. \tag{8}$$

Eq. 8 describes the distribution of a moving particle swarm (including the direction of movement) in the constrained state of $\mathrm{I}u$ (not considering the Γ effect) or in a completely free state following the same diffusion coefficient; in other words, it is the classical (vector) diffusion equation. When u is small, the constrained state of $\mathrm{I}u$ can also be approximated to a completely free state (the Γ effect can be ignored). However, when u is large or there is both a location-constrained state (i.e., the constrained state of $\mathrm{III}u$), the effect on diffusion is not clear. To more comprehensively describe this type of diffusion process (which is called generalized diffusion), further analysis is needed.

3.4. Construction of the Generalized Diffusion Equation in the Constrained State of $\mathrm{III}u$

To construct the generalized diffusion equation in the constrained state of $\mathrm{III}u$, we need to consider many aspects, including whether the generalized diffusion coefficient *Đ* should vary and how to describe it to include the characteristics of the two types of constrained states.

When particles are in the constrained state of $\mathrm{I}u$ (not considering the Γ effect) or in a completely free state, they follow a diffusion equation with the same diffusion coefficient (the Schrödinger equation). However, when such particles are in the constrained state of $\mathrm{III}u$, the effect of location aggregation on Đ should be considered, and Đ should vary with the value of the target vector. Suppose that, as illustrated in Fig. 3a, the vector sum density in the microdomain $\mathcal{V}_\mathrm{A}$ is greater than that in the $\mathcal{V}_\mathrm{B}$. If both cases are in the constrained state of $\mathrm{III}u$, there is a greater consumption of degrees of freedom for the higher density in the $\mathcal{V}_\mathrm{A}$. In terms of probability, less uncertainty is introduced into the unit volume, which inevitably affects the (average) particle movement speed. Therefore, the overall particle movement speed in the $\mathcal{V}_\mathrm{A}$ decreased. As mentioned above (or in Eq. 27 below), the particle speed is what determines D; therefore, the law governing the diffusion rate towards the right (D_A) is not the same as the law governing the diffusion rate in the $\mathcal{V}_\mathrm{B}$ towards the left (D_B) (under the assumption that Đ is a combination of D_A and D_B). Therefore, it is necessary for the generalized diffusion coefficient to vary in time with the vector sum density to reflect this inequality.

In view of the above considerations, choosing the appropriate quantitative function to describe this phenomenon (with different laws) is the main problem to be solved in this study. First, the sum of the momentum vectors in the microdomain is decomposed as follows:

3.4.1. Vector Decomposition

First, let us determine the distribution function for a certain number of nonmoving particles with equal probability (randomly) distributed in a certain domain, as follows: Suppose that the entire domain contains n particles in total. For convenience of description, the entire domain is also partitioned into n boxes of equal size. The gaps between the boxes and the wall thickness are both 0. Now, let us determine the probability of k ($k \in \mathbb{N}_+$; the same is done below) particles in a local area containing $\mathcal{M}$ boxes (suppose that the particles are small enough to fall into the box, not the wall). In view of the statement described above, the probability of particles existing in each domain is the same. Accordingly, the total number of possible cases describing how n particles can be randomly distributed among n boxes is n^n, there are $\binom{n}{k}$ total ways that k particles can be randomly chosen from among n particles, there are $\mathcal{M}^k$ total ways in which the k chosen particles can be randomly distributed among $\mathcal{M}$ boxes, and there are $(n-\mathcal{M})^{n-k}$ total ways in which the remaining $n-k$ particles can be randomly distributed among the remaining $n-\mathcal{M}$ boxes. Therefore, the probability $P(\mathcal{M},k)$ of k particles existing in $\mathcal{M}$ boxes can be expressed as

$$P(\mathcal{M},k) = \frac{\binom{n}{k}\mathcal{M}^k\,(n-\mathcal{M})^{n-k}}{n^n}. \tag{9}$$

Suppose that the number n of particles in the entire domain is infinite; then, by taking the limit of Eq. 9 as $x \to +\infty$, we find that

$$P(\mathcal{M},k) = \frac{\mathrm{e}^{-\mathcal{M}}\mathcal{M}^k}{k!}, \tag{10}$$

again, where $\mathcal{M}$ denotes the number of boxes comprising the local domain of interest (the size of the volume in 3-dimensional space), k denotes the number of particles in that domain of $\mathcal{M}$ boxes, and P denotes the probability that k particles exist in that domain. Eq. 10 is the (location-based) Poisson distribution.

It is considered that this is the most appropriate method of partitioning a whole domain (the domain can be the whole universe or simply a broad range including the objects of investigation) into uniform boxes with the same number as that of particles. In addition to reducing the parameters involved and facilitating discussion, the reasons are as follows: if the boxes are slightly larger, they will not ensure the accuracy of the following vector decomposition; if they are slightly smaller, they will not adequately reflect the grouping effect of the particles. Therefore, in this article, the whole domain is divided into a number of uniform boxes equal to the number of particles it contains, and this partitioning serves as the basis for all of the following discussions. In this article, the whole domain (environment) is called the T-domain (it is the sub-domain of sub-domain in Fig. 1), and the local domain (target) is called the S-domain; the set of all particles contained in the T-domain is called the T-particle swarm (it is the subparticle swarm of subparticles in Fig. 2), and the subset of particles contained in the S-domain is called the S-particle swarm.

Next, we will investigate the equiprobability distribution of the nonmoving particle swarm in the abovementioned S-domain $\mathcal{V}$. In Eq. 10, $\mathcal{M}$ denotes the number of boxes (volume) spanned by some S-domain (which belonged to the domain in which the target particles are distributed). Put another way, when the T-domain is partitioned into uniform boxes following the above method, $\mathcal{M}$ can also denote the average relative density of the particles in the S-domain $\mathcal{V}$, where the reference density is the average density of the T-particle swarm in the T-domain. $\mathcal{M}$ represents the corresponding multiple of the average density, k denotes the number of particles in one box, and P is the probability of k particles existing in that box. Thus, the distribution of the S-particle swarm in $\mathcal{V}$ is a Poisson distribution with density intensity $\mathcal{M}$. Next, we will analyze the Poisson distribution formula given in Eq. 10. In fact, it is the proportion of each term determined by k (when $\mathrm{e}^{\mathcal{M}}$ is expanded as a power series) to the value of $\mathrm{e}^{\mathcal{M}}$. The meaning here is that it is also the proportion of the number of boxes containing k particles each to the total number of boxes in $\mathcal{V}$ when the S-particle swarm of relative density $\mathcal{M}$ is distributed among the reference boxes determined by the above criteria and spanned by the S-domain $\mathcal{V}$ (supposing that the number of boxes spanned by $\mathcal{V}$ is sufficiently large). According to mathematical analysis, we can see that the power series expansion for this case is unique, and obviously, this ratio distribution is also unique. If the right-hand side of Eq. 10 is multiplied by k, the result, denoted by $R(\mathcal{M}, k)$, takes the following form:

$$R(\mathcal{M}, k) = \frac{\mathrm{e}^{-\mathcal{M}}\mathcal{M}^k}{(k-1)!}. \tag{11}$$

In this way, termwise addition (by k) based on this expression offers a possible form for the decomposition of $\mathcal{M}$ into infinite items. Because the power series expansion above is unique, this decomposition form of the containing power series is also unique. According to the previous statement of physical meaning, the meaning of Eq. 11 is the relative density contributed by the particles in the boxes that contain k particles each to the total relative density $\mathcal{M}$ (the average relative density in $\mathcal{V}$) after the particles of relative density $\mathcal{M}$ are dispersed among the (infinitely many) reference boxes spanned by $\mathcal{V}$ with equal probability. Multiplying Eq. 11 by the number of boxes contained in $\mathcal{V}$ yields the total number of particles in the boxes containing k particles each. Since the distribution of particles in this form is definite (following the Poisson distribution), from this point of view, the decomposition of the relative density $\mathcal{M}$ in this (containing power series) form is also unique.

If $\boldsymbol{\mathcal{M}}$ is a complex number (or plane vector), Eq. 11 can be written in vector form as follows:

$$R(\boldsymbol{\mathcal{M}}, k) = \frac{\mathrm{e}^{-\boldsymbol{\mathcal{M}}}\boldsymbol{\mathcal{M}}^k}{(k-1)!}. \tag{12}$$

The form obtained by dividing Eq. 12 by k is still the ratio of each term (complex) determined by k (when $\mathrm{e}^{\boldsymbol{\mathcal{M}}}$ is expanded as a power series) to the complex of $\mathrm{e}^{\boldsymbol{\mathcal{M}}}$. There is one more dimension here,

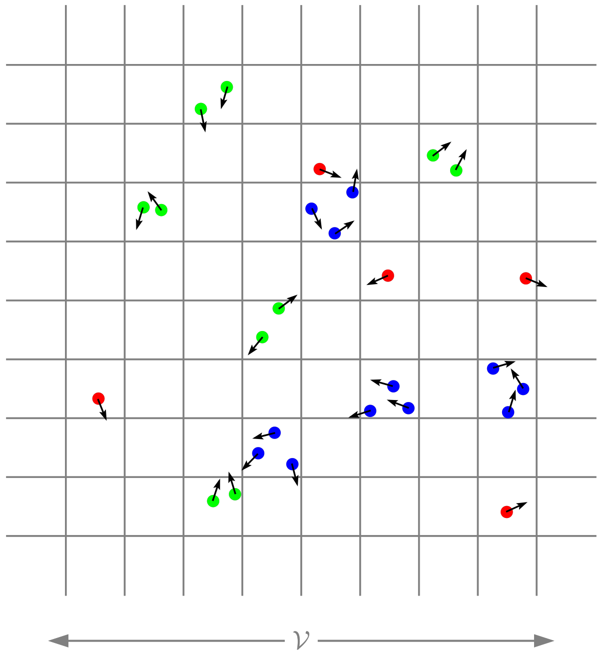

Figure 5. Illustration of the physical meaning of $\boldsymbol{\mathcal{Y}}_k$ $(k = 1, 2, 3, \cdots)$ in the S-domain $\mathcal{V}$ (a planar figure is used to represent the stereo figure). The vector sum of the red particles $(k = 1)$ is $\boldsymbol{\mathcal{Y}}_1$, the vector sum of the green particles $(k = 2)$ is $\boldsymbol{\mathcal{Y}}_2$ and the vector sum of the blue particles $(k = 3)$ is $\boldsymbol{\mathcal{Y}}_3$, $\cdots$.

and the power series expansion is still unique. Similarly, the termwise addition of Eq. 12 also provides a decomposition form for the vector $\boldsymbol{\mathcal{M}}$. This decomposition form of the containing power series is also unique.

Now, we study the distribution of the velocity of the moving S-particle swarm in the abovementioned S-domain $\mathcal{V}$. If the particles in the T-particle swarm move randomly in the T-domain, the distribution of the S-particle swarm in one time slice in a sufficiently small S-domain (when the particle speed is fast enough) can also be approximated as an equiprobable distribution. At the human scale (it will be proven with self-consistency that, in fact, at any scale range), the number of S-particles in almost every "microdomain" of the universe can be regarded as approaching infinity; therefore, the number distribution of particles in the moving S-particle swarm in a certain microdomain $\mathcal{V}$ can be described by Eq. 10. The moving particles in each type of box partitioned by k in one S-domain $\mathcal{V}$ can form a component vector (denoted by $\boldsymbol{\mathcal{Y}}_k$, as shown schematically in Fig. 5), and these components can be added together to generate the total 3-dimensional vector $\boldsymbol{\mathcal{Y}}$ in $\mathcal{V}$, that is

$$\boldsymbol{\mathcal{Y}} = \sum_{k=1}^{\infty} \boldsymbol{\mathcal{Y}}_k. \tag{13}$$

Once $\boldsymbol{\mathcal{Y}}$ formed by the moving S-particle swarm in $\mathcal{V}$, which includes the specific number of (equivalent) particles, is determined (i.e., the average speed u of the S-particles or T-particles is determined observed from $\mathcal{R}_0$), the norm (mathematical expectation) of each component vector should be (approximately) directly proportional to the number of particles forming it when the number of particles is large (see Part 2 of the Supplementary Information for details). Note that the number of samples in $\mathcal{V}$ is very large even when $k = 1$. Therefore, the ratios between the norms (mathematical expectations) of the component vectors in various boxes partitioned by k are uniquely determined by the form of (containing) the power series determined by Eq. 11. In other words, when $\mathcal{M}$ represents the relative density of the particles in $\mathcal{V}$, we have the following relationship:

$$\|\boldsymbol{\mathcal{Y}}_1\| : \|\boldsymbol{\mathcal{Y}}_2\| : \cdots = R(\mathcal{M}, 1) : R(\mathcal{M}, 2) : \cdots . \tag{14}$$

As the limiting value $\boldsymbol{\mathcal{X}}$ of the quotient of $\boldsymbol{\mathcal{Y}}$ and $\mathcal{V}$, it can still be considered as a sum of 3-dimensional vectors in the S-domain $\mathcal{V}$. Therefore, there is also a form of component vectors with the ratios of norms determined by Eq. 11 spanning various boxes partitioned by k. When the 3-dimensional component vectors (spanning various boxes partitioned by k) of the 3-dimensional vector $\boldsymbol{\mathcal{X}}$ are mapped to the 2-dimensional component vectors (spanning various boxes partitioned by k) of the plane vector $\boldsymbol{\mathcal{M}}$, it is obvious that there is also a corresponding 2-dimensional form of component vectors with the ratios of norms determined by Eq. 11 (namely, the ratios of norms follow a Poisson distribution corresponding to the number of particles), but the direction is not determined. That is, when $\boldsymbol{\mathcal{X}}_1$, $\boldsymbol{\mathcal{X}}_2$, $\cdots$ represent the component vectors of $\boldsymbol{\mathcal{X}}$ respectively and $\boldsymbol{\mathcal{M}}_1$, $\boldsymbol{\mathcal{M}}_2$, $\cdots$ represent the component vectors of $\boldsymbol{\mathcal{M}}$ respectively, we have

$$\|\boldsymbol{\mathcal{Y}}_1\| : \|\boldsymbol{\mathcal{Y}}_2\| : \cdots = \|\boldsymbol{\mathcal{X}}_1\| : \|\boldsymbol{\mathcal{X}}_2\| : \cdots = \|\boldsymbol{\mathcal{M}}_1\| : \|\boldsymbol{\mathcal{M}}_2\| : \cdots . \tag{15}$$

According to Eqs. 14 and 15, we can obtain the following relationship:

$$\|\boldsymbol{\mathcal{M}}_1\| : \|\boldsymbol{\mathcal{M}}_2\| : \cdots = R(\mathcal{M}, 1) : R(\mathcal{M}, 2) : \cdots . \tag{16}$$

According to the conclusion in Part 2 of the Supplementary Information, the norm (mathematical expectation) of each component vector is the product of the number of particles forming it and the speed of the system it located. Therefore, we can obtain

$$\|\boldsymbol{\mathcal{M}}\| = \mathcal{M} \cdot u. \tag{17}$$

Note that when $\mathcal{M}$ represents a relative scalar, $\boldsymbol{\mathcal{M}}$ represents a relative vector. Therefore, $\|\boldsymbol{\mathcal{M}}\| = \mathcal{M}$ is always true when $u = 1$, where u is the average speed of the T-particles. As a result, we have

$$\|\boldsymbol{\mathcal{M}}_1\| : \|\boldsymbol{\mathcal{M}}_2\| : \cdots = R(\|\boldsymbol{\mathcal{M}}\|, 1) : R(\|\boldsymbol{\mathcal{M}}\|, 2) : \cdots . \tag{18}$$

In other words, when $u = 1$, the ratios of norms of the component vectors of $\boldsymbol{\mathcal{M}}$ are the ratios of the power series (determined by the Poisson distribution) forms of its own norm.

When $\boldsymbol{\mathcal{M}}$ is decomposed into $\boldsymbol{\mathcal{M}}_1$, $\boldsymbol{\mathcal{M}}_2$, $\cdots$ denoted by itself (i.e., $u = 1$), the relationship between $\|\boldsymbol{\mathcal{M}}_1\|$, $\|\boldsymbol{\mathcal{M}}_2\|$, $\cdots$ must satisfy Eq. 18. In view of the uniqueness of $R(\|\boldsymbol{\mathcal{M}}\|, k)$ which is the power series form of the norms, $\boldsymbol{\mathcal{M}}_k$ must be expressed in the form of $R(\boldsymbol{\mathcal{M}}, k)$ (Eq. 12, or at least the form of $R(\boldsymbol{\mathcal{M}}, k) \cdot \mathrm{e}^{\boldsymbol{\mathcal{M}}}$) to satisfy Eq. 18. At this point, the direction of $\boldsymbol{\mathcal{M}}_k$ is uniquely determined. In view of the termwise addition (by k) of Eq. 12 is the unique decomposition of $\boldsymbol{\mathcal{M}}$, therefore, the plane mapping of the sum of all the vectors in the boxes containing the same number k of particles is the component vector determined by k in Eq. 12. When k takes all values in $\mathbb{N}_+$, the termwise sum of these terms is the unique decomposition of $\boldsymbol{\mathcal{M}}$ (spanning various boxes partitioned by k), namely,

$$\boldsymbol{\mathcal{M}} = \sum_{k=1}^{\infty} \frac{\mathrm{e}^{-\boldsymbol{\mathcal{M}}} \boldsymbol{\mathcal{M}}^k}{(k-1)!}. \tag{19}$$

The above analysis shows that two conditions must be satisfied for $\boldsymbol{\mathcal{M}}$ to be uniquely decomposed into components divided by k. On the one hand, $u = 1$ (or $\|\boldsymbol{\mathcal{M}}\| = \mathcal{M}$) must be satisfied; on the other hand, $\|\boldsymbol{\mathcal{M}}\|$ must be a relative value as $\mathcal{M}$. Therefore, it is obvious that $\boldsymbol{\mathcal{M}}$ should also be a relative vector. Furthermore, $\boldsymbol{\mathcal{M}}$ should be not only a multiple of the number of reference boxes but also a multiple of the speed of the system (that is, the norm of the average velocity of the counted particles. $u = 1$ can be satisfied only if u is regarded as a relative value u^*). Therefore, the reference value of vector $\boldsymbol{\mathcal{M}}$ is nu (where u is the absolute speed of the target domain in the background domain). Accordingly, $\boldsymbol{\mathcal{M}}$ in Section **3.3** should be exactly the relative vector sum density, which has the same direction as the absolute sum of the vectors located at that place observed from $\mathcal{R}_0$. As mentioned above, the sum and difference operations between two spatial vectors are performed in their shared plane. In this plane, they can be decomposed respectively into a sum of plane vectors, as described in Eq. 19. Therefore, the two sets of plane component vectors can also serve as their respective spatial component vectors to correspondingly perform sum, difference or derivative operations.

3.4.2. Description of Diffusion

Suppose that the standard deviation of the projection (treated as a random variable; the same is done below) of the velocities of the k equivalent particles forming a k-particle (that is the k-generalized-particle; the same is done below) onto each equivalent coordinate axis is σ. As mentioned earlier, the speeds of k-particles follow the Maxwell distribution with scale parameter $\dfrac{\sigma}{\sqrt{k}}$ (When it is in the constrained state of Iu not considering the Γ effect or in a completely free state, the speed of particle diffusion to uniform mixing in Fig. 3a is determined by the statistical average of the particle velocities, which is the inherent property of the system. Here, the particles in the target domain is regarded as a system with uniform distribution in the velocity direction, that is, the speeds of generalized particles follow the Maxwell distribution, and the average speed can be obtained according to the Maxwell distribution). Then, the average speed of k-particles is

$$\overline{v} = 2\sqrt{\frac{2}{\pi}} \cdot \frac{\sigma}{\sqrt{k}}. \tag{20}$$

For k_{a}- and k_{b}-particles, the ratio of their average speeds is

$$\frac{\overline{v_{\mathrm{a}}}}{\overline{v_{\mathrm{b}}}} = \frac{\sqrt{k_{\mathrm{b}}}}{\sqrt{k_{\mathrm{a}}}}. \tag{21}$$

Because the sizes, or masses, of all 1-particles (forming k-particles) are the same, if the masses of a k_{a}-particle and a k_{b}-particle are m_{a} and m_{b}, respectively ($m \propto k$), then according to the relationship shown in Eq. 21, the ratio of their average speeds can also be written as

$$\frac{\overline{v_{\mathrm{a}}}}{\overline{v_{\mathrm{b}}}} = \frac{\sqrt{m_{\mathrm{b}}}}{\sqrt{m_{\mathrm{a}}}}. \tag{22}$$

See Part 1 of the Supplementary Information for the detailed calculation and derivation process. According to Eq. 22, for any-particles, the product of the square root of mass and the average speed is a constant (suppose it is κ_{a}). Then, when the mass of a k-particle is m, its average speed is

$$\overline{v} = \frac{\kappa_{\mathrm{a}}}{\sqrt{m}}. \tag{23}$$

The diffusion coefficient can be defined as follows: it is the mass or mole number of a substance that diffuses vertically through a unit of area along the diffusion direction per unit time and per unit concentration gradient. Therefore, it is believed that classical real diffusion is consistent with the essence of vector diffusion described here (the two diffusions that are achieved both require the random displacement of k-particles). According to the Einstein-Brown displacement equation, the diffusion coefficient is

$$D = \frac{\overline{x}^2}{2t}, \tag{24}$$

where $\overline{x}$ is the average displacement of k-particles along the direction of the x-axis. To replace the average displacement $\overline{x}$ in Eq. 24 with the average velocity (namely, $\overline{\boldsymbol{V}}$) of k-particles along the direction of the x-axis, this diffusion coefficient can be transformed into

$$D = \frac{\left\|\overline{\boldsymbol{V}}\right\|^2}{2} t^1. \tag{25}$$

The unit of the diffusion coefficient D is $\mathrm{m}^2 \cdot \mathrm{s}^{-1}$. By combining Eq. 24 and Eq. 25 (where t^1 and the t implied in $\left\|\overline{\boldsymbol{V}}\right\|^2$ are consistent, so $t^1 = 1$ s), the abovementioned diffusion coefficient can also be regarded as follows: it is the average area over which k-particles spread out on a plane per unit time. This average area is related to the speed of a single k-particle. If the (average) speed of a single k-particle is $\overline{v}$, then the statistical average speed of these particles in one direction is

$$\left\|\overline{\boldsymbol{V}}\right\| = \frac{\overline{v}}{2}. \tag{26}$$

The k-particle swarm spreads in the plane at this rate. By substituting Eq. 26 into Eq. 25 and combining $t^1 = 1$ s into the coefficient, which we then denote by κ_{b}, we can obtain

$$D = \kappa_{\mathrm{b}} \overline{v}^2, \tag{27}$$

where κ_{b} is a constant coefficient with units of seconds (s).

By substituting Eq. 23 into Eq. 27, the diffusion coefficient of a (k-)particle swarm of (average) mass m is obtained:

$$D = \kappa_{\mathrm{b}} \left(\frac{\kappa_{\mathrm{a}}}{\sqrt{m}} \right)^2 = \frac{{\kappa_{\mathrm{a}}}^2 \kappa_{\mathrm{b}}}{m}. \tag{28}$$

In view of the diffusion coefficient D only affecting the diffusion rate, the above equation (Eq. 28) can also be thought of as the apparent diffusion coefficient of particle(s) with mass m described by the 1-particle swarm (which forms a particle of mass m after collapse) in the constrained state of $\mathrm{I}u$. Here, we suppose that

$${\kappa_{\mathrm{a}}}^2 \kappa_{\mathrm{b}} = \frac{\hbar}{2}. \tag{29}$$

As the situation in $\mathcal{R}_u$ observed from $\mathcal{R}_0$, D should also be affected by the $\Gamma[\cdot]$ effect, which is abbreviated as

$$D = \frac{\hbar \Gamma^2}{2m}. \tag{30}$$

3.4.3. Construction of the Generalized Diffusion Equation

Previously, we adopted the assumption that there is no interaction between point particles. Accordingly, in a time slice of a microdomain, the decomposition of the vector given by Eq. 19 must be exhibited, and all boxes containing the same number of particles in different microdomains containing different densities of vectors are equivalent. This is because there should be no differences between boxes of the same type (i.e., containing the same number of particles) when (the whole target domain is expressed as a system with a relative average speed of 1 and) the Poisson distribution determines the numbers of boxes of different types in different microdomains of different vector densities. Although the moving particles in the second or third constrained state can be distributed in a time slice of the microdomains with the same probability, when the overall behavior of k particles is counted, their average speed will inevitably slow down. At this time, there will be more or fewer particles in the unit volume of the domain in which they are located (or each box in the microdomain of the domain in which they are located), and the "slow down" effect will be retained according to the location characteristics; in other words, the degrees of freedom of particles will be reduced or affected by the second or third kind of constraint effect. The particles in various boxes partitioned by k move at their average relative speed, and the centroids of boxes containing k particles each are, on average, located at the center of each box. Among all boxes of the same type (i.e., containing k particles), the average relative speed of each k-particle is the same and must conform to the diffusion form of the Schrödinger equation (Eq. 8) determined by the diffusion coefficient for particles of this type. Therefore, according to the particle numbers k in the previously partitioned boxes, from 1 to ∞, we study the corresponding term $R(\boldsymbol{\mathcal{M}}, k)$, which is the component vector of $\boldsymbol{\mathcal{M}}$. First, we investigate the diffusion of individual terms, and then, we add them together to characterize the overall slowing behavior of diffusion.

Here, all the particles in each box containing k particles are regarded as forming a k-particle of a larger mass level, and together, all k-particles in all boxes containing k particles in microdomain $\mathcal{V}$ are called the k-particle swarm in that microdomain. Based on the above discussion, it can be considered that the average relative speed of each (k-)particle in the k-particle swarm is the same, and all of them have the same diffusion coefficient. According to the relationship given in Eq. 28 (the diffusion coefficient is inversely proportional to the mass of a k-particle, or the number of 1-particles forming a k-particle), if the diffusion coefficient of a 1-particle swarm is D_1, then the diffusion coefficient of a k-particle swarm is

$$D_k = D_1 \cdot \frac{1}{k}, \tag{31}$$

where $\frac{1}{k}$ is called the diffusion coefficient factor.

When the particles are in the constrained state of $\mathrm{I}u$ or in a completely random state, the diffusion behavior of interest is that of a 1-particle swarm. It is consistent with the Schrödinger equation when the target particle swarm moves along the average speed of u. Therefore, the diffusion coefficient is

$$D_1 = -\frac{\hbar \Gamma^2}{2m}. \tag{32}$$

The diffusion equation determined by this coefficient describes the kinetics of the probabilistic diffusion of a target object (or the aggregation after collapse) of mass m on the basis of the apparent diffusion rate (after deceleration) determined by the 1-particles forming it (before collapse); however, the distribution characteristics of the target object in its dispersion space is determined by the diffusion behavior of the 1-particles in the background field. When the particles are in the constrained state of

IIIu, according to the above discussion, the case of $k > 1$ must be considered. Then, the diffusion coefficient of a k-particle swarm can be obtained by substituting Eq. 32 into Eq. 31, namely,

$$D_k = -\frac{\hbar\,\Gamma^2}{2\,m} \cdot \frac{1}{k}. \tag{33}$$

This is equivalent to the proportional decline in the apparent diffusion rate of a target object (or the aggregation after collapse) of mass m due to the slowdown in the speed of the k-particles forming the target object. The meaning of the diffusion equation determined by this diffusion coefficient is similar to the case for 1-particles as considered above, that is, the kinetics of the probabilistic diffusion of a target object (or the aggregation after collapse) of mass m are described on the basis of the apparent diffusion rate (after deceleration) determined by the k-particles forming it (before collapse); however, the distribution characteristics of the target object in its dispersion space is determined by the diffusion behavior of the k-particles in the background field.

By taking the second partial derivative of $R(\boldsymbol{\mathcal{M}}, k)$ (this is the plane vector sum in the boxes containing k moving particles, namely, the k-particle swarm, which is one of the component vectors in the whole microdomain $\mathcal{V}$) with respect to location (x, y, z), $\Delta R(\boldsymbol{\mathcal{M}}, k)$ can be obtained. It should be emphasized that the absolute sizes of the two (infinitesimal) microdomains $\mathcal{V}_\mathrm{A}$ and $\mathcal{V}_\mathrm{B}$, which are selected to compare their differences, are equal when calculating the derivative of the vector $\boldsymbol{\mathcal{M}}$. After multiplying $\Delta R(\boldsymbol{\mathcal{M}}, k)$ by the diffusion coefficient for the k-particle swarm (Eq. 33) and then adding the products together from $k = 1$ to ∞, the complete generalized diffusion expression (including coefficients) can be obtained as follows:

$$-\frac{\hbar\,\Gamma^2}{2\,m} \sum_{k=1}^{\infty} \left[\frac{1}{k} \cdot \Delta R\left(\boldsymbol{\mathcal{M}}, k\right)\right]. \tag{34}$$

The diffusion calculated in this way is the generalized diffusion from the whole (infinitesimal) microdomain $\mathcal{V}_\mathrm{A}$ to $\mathcal{V}_\mathrm{B}$. Eq. 34 can be simplified as follows:

$$-\frac{\hbar\,\Gamma^2}{2\,m\,\mathrm{e}^{\boldsymbol{\mathcal{M}}}} \left[\Delta\boldsymbol{\mathcal{M}} - T^2(\boldsymbol{\mathcal{M}})\right], \tag{35}$$

where $T^2(\boldsymbol{\mathcal{M}}) = \left(\frac{\partial \boldsymbol{\mathcal{M}}}{\partial x}\right)^2 + \left(\frac{\partial \boldsymbol{\mathcal{M}}}{\partial y}\right)^2 + \left(\frac{\partial \boldsymbol{\mathcal{M}}}{\partial z}\right)^2$. By combining the left-hand side of Eq. 8 with Eq. 35, a complete expression for the generalized diffusion equation for vectors is obtained:

$$\mathbf{i}\,\frac{\partial \boldsymbol{\mathcal{M}}}{\partial t} = -\frac{\hbar\,\Gamma^2}{2\,m\,\mathrm{e}^{\boldsymbol{\mathcal{M}}}} \left[\Delta\boldsymbol{\mathcal{M}} - T^2(\boldsymbol{\mathcal{M}})\right]. \tag{36}$$

Therefore, the expression for the generalized diffusion coefficient with the two kinds of special constrained effects is

$$\boldsymbol{Đ} = -\frac{\hbar\,\Gamma^2}{2\,m\,\mathrm{e}^{\boldsymbol{\mathcal{M}}}}. \tag{37}$$

The diffusion coefficient here is not a constant but rather a natural exponential function that varies with the relative vector density of moving particles. Hence, the generalized diffusion equation and the generalized diffusion coefficient $\boldsymbol{Đ}$ for vectors in the constrained state of IIIu have been determined. In this constrained state, the ratios of norms of the spatial equivalent vectors in a microdomain can be determined in accordance with the Poisson distribution, while the norms and directions of the spatial equivalent vectors in the complex plane can be determined in accordance with Eq. 36. Thus, the basic effective information for a spatial (moving) particle swarm in the constrained state of IIIu has been derived.

The slowing down of diffusion based on spatial location is the only manifestation of the statistical effect of location aggregation (the second kind of constrained state) in diffusion. Obviously, the second kind of special constrained state effect of particles can be reflected according to the treatment method in Eq. 34. As mentioned above, the statistical effects include the location and direction aggregation. For the case of velocity direction aggregation, because the particles are in the system with a speed of u, the diffusion coefficient will be affected by the Γ effect, and the statistical effect of this case is also added to the equation. In summary, all of the statistical (constrained) effects have be incorporated into Eq. 34.

3.5. Verification of Eq. 36

The derivation process of Eq. 36 shows that $\boldsymbol{\mathcal{M}}$ is a relative vector, and the square of its first derivative is the higher-order infinitesimal of its second derivative. If the norm of the initial value (namely, the initial norm) is sufficiently small, Eq. 36 can be approximated as the Schrödinger equation without an external field when the Γ effect is not considered. For example, when solving the diffusion problem of a 3-dimensional Gaussian wave packet formed by randomly moving particles, if the initial norm is less than 10^{-2}, the solutions of the two equations are almost the same (Fig. 6a, and the relative difference is less than 1%. Note that the values of $\|\boldsymbol{\mathcal{M}}\|$ are compared here to maintain consistency with the subsequent sections). When the initial norm is sufficiently large, the particle swarm exhibits a certain degree of aggregation with time from the initial Gaussian wave packet. As shown in Fig. 6b, this aggregation is apparent at approximately $t = 0.276$. As the the initial norm increases, increasingly evident aggregation processes appear. When the initial norms was 0.250, 0.500, 0.625 and 0.750, the radial distribution profile at the time of the most visible aggregation in each process (such as the red line in Fig. 6b) was taken to obtain the profile set, as shown in Fig. 6c (each profile is normalized according to the initial norm). It is speculated that when the initial norm increases to a certain value, a completely non-diffusive particle swarm may arise. As a result, we have

$$\Delta\boldsymbol{\mathcal{M}} - T^2(\boldsymbol{\mathcal{M}}) = 0, \tag{38}$$

and $\boldsymbol{\mathcal{M}}$ does not vary with time t at this point. In the case of spherical symmetry, the boundary conditions of Eq. 38 can be given by

$$\begin{cases} \boldsymbol{\mathcal{M}}(r) = \mathcal{M}_{\mathrm{c}}, & r = r_{\mathrm{c}}, \\ \boldsymbol{\mathcal{M}}(r) = 0, & r = r_{\mathrm{e}}, \end{cases} \tag{39}$$

where r is the distance to the spherical center; r_{c}, r_{e} and $\mathcal{M}_{\mathrm{c}}$ are constants and $r_{\mathrm{c}} < r_{\mathrm{e}}$. Then, the analytical solution can be obtained by solving the simultaneous equations of Eq. 38 and Eq. 39:

$$\boldsymbol{\mathcal{M}}(r) = \ln r - \ln\left[\frac{r\left(r_{\mathrm{c}} - r_{\mathrm{e}}\,\mathrm{e}^{\mathcal{M}_{\mathrm{c}}}\right)}{r_{\mathrm{c}} r_{\mathrm{e}}\left(\mathrm{e}^{\mathcal{M}_{\mathrm{c}}} - 1\right)} + 1\right] + \ln\left[\frac{\mathrm{e}^{\mathcal{M}_{\mathrm{c}}}\left(r_{\mathrm{c}} - r_{\mathrm{e}}\right)}{r_{\mathrm{c}} r_{\mathrm{e}}\left(\mathrm{e}^{\mathcal{M}_{\mathrm{c}}} - 1\right)}\right]. \tag{40}$$

See Part 3 of the Supplementary Information for the detailed Mathematica code of the solution process. Thus, given $r_{\mathrm{c}} = \dfrac{1}{6000}$, $r_{\mathrm{e}} = 30$ and $\mathcal{M}_{\mathrm{c}} = 3 + \mathbf{i}$, the radial distribution of the mass density ($\|\boldsymbol{\mathcal{M}}\|$) projected on the plane can be obtained, as illustrated in Fig. 6d.

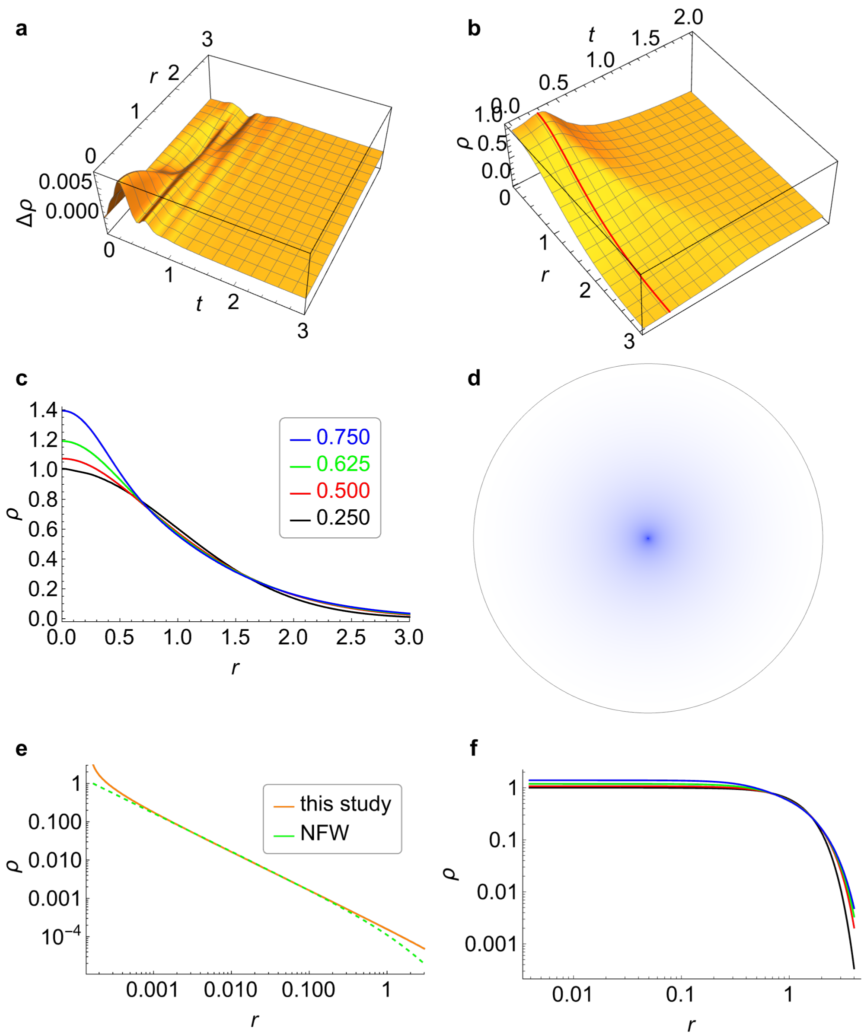

Figure 6. The prediction results ($\|\mathcal{M}\|$) of our equations in different cases. **a**, The differences in the density between the values calculated with Eq. 36 and the Schrödinger equation when the initial norm is 10^{-2}. **b**, The diffusion pattern of the Gaussian wave packet with time predicted by Eq. 36 when the initial norm is $\frac{1}{2}$. **c**, Comparison of the radial distributions for different initial norms. **d**, The radial distribution of the density (projected on the plane) integrated according to Eq. 40. **e**, Comparison of the NFW profile and Eq. 40 ($r_\mathrm{c} = \frac{1}{6000}$, $r_\mathrm{e} = 30$ and $\mathcal{M}_\mathrm{c} = 3 + \mathbf{i}$) when $r < r_\mathrm{S}$. **f**, The logarithmic profile of **c** as r varies from $0 \sim 4$.

In the universe, one of the scenarios corresponding to the particles in the constrained state of IIIu is the galaxies or galaxy clusters which are affected only by gravitation. The results predicted by Eq. 40 are consistent with the observation results of relaxed galaxies and galaxy clusters (multiple images method). The Navarro-Frenk-White (NFW) profile[5], as an empirical formula, is universally regarded as in good agreement with the observational results, which is given by

$$\rho(r) = \frac{\rho_\mathrm{c}}{r/r_\mathrm{s}(1 + r/r_\mathrm{s})^2}. \tag{41}$$

Eq. 41 shows that the shape of the profile is not affected by the parameters ρ_c and r_s. The NFW profile was obtained by adjusting the two parameters, and the result was compared to the profile obtained with Eq. 40. The two profiles are almost consistent within the scale radius of r_s (Fig. 6e). Therefore, Eq. 40 is in good agreement with the observational results of relaxed galaxy clusters within r_s, as mentioned in the literatures[6, 7]; however, Eq. 40 is not consistent with the results in the range $r > r_\mathrm{s}$. It is speculated that the inconsistency of these peripheral regions occurs because these galaxy clusters are not in completely non-diffusive states (diffusion is extremely slow when galaxy clusters are in these "relaxed" states because the principle masses are almost in non-diffusive states). The trend in Fig. 6f shows that when the initial norm increases to a certain value, the radial distribution profiles of particle swarms diffusing from Gaussian wave packets in the range of $r > r_\mathrm{s}$ are consistent with the observation results of the gravitational lens method. Furthermore, there are no cuspy problems in Eq. 40. The central part of the particle swarm described by Eq. 40 can be a structure with a specific volume and a finite concentration. The peripheral distribution forms a stable "shell" to protect the central structure from diffusion.

Traditionally, the formation of such a mass distribution of relaxed galaxies or galaxy clusters is the result of gravitations and velocities. However, there is no interaction in the randomly moving particle swarm described in Eq. 40, which generates the same effect (the particles have the same speed throughout the swarm and only when they form more massive swarms does the swarms' speed decrease to some degree). A previous study[4] proved that randomly moving particles also experience the effects of special relativity. In addition, such particles can produce non-diffusive particle swarms of different scales. Accordingly, it is speculated that galaxies or galaxy clusters (at least dark matter halos) can be formed by randomly moving particles and the essence of gravitation is the production of a third/second type of constrained state. In these constrained states, particles have less degrees of freedom in denser domains. And the apparent phenomenon of universal gravitation occurs between domains with less degrees of freedom and domains with more degrees of freedom.

4. Conclusions

Previous studies have focused on the overall behavior of randomly moving particle swarms[1, 2]. However, the characteristics of ubiquitous special particle swarms that form in these swarms remain unknown. In these special particle swarms, particular phenomena, such as the velocity or location aggregation effects, need to be considered. Based on these, this study demonstrated a generalized diffusion equation for randomly moving particles in the constrained state of $\mathrm{III}u$. When the norm of the initial value is small, the equation can be approximated as the Schrödinger equation; when the norm is large, the equation can be used to describe the aggregation process of particles. Although our model describes a noninteracting particle swarm, it includes the apparent phenomena of universal gravitation. The consistency with the observational results make us have a certain reason to believe that the essence of universal gravitation is the change of the free degree caused by the location aggregation of particles in the third/second constrained state.

In the more general case, that is, in the third kind of general constrained state, we can divide the whole system into countless fragments according to the time and domain. Each fragment can be approximated as in the constrained state of $\mathrm{III}u$. We use Eq. 36 to determine the results for each segment and splice them together. Thus, the whole problem of the third kind of general constraint can be solved.

In a future study, we will further explore the properties of Eq. 36 and deduce the corresponding relationship between $\boldsymbol{\mathcal{M}}$ and the absolute physical quantity to extend this equation to more specific fields. Furthermore, we will solve Eq. 36 on a larger scale and explore the relationship between galactic jets and electromagnetic fields.

Acknowledgements

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Supplementary Material

Supplementary Information

See the **Supplementary Information** for detailed description of the models, derivations, additional figures, and computational method.

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Appendix:

Supplementary Information
(Mathematica v13.0.1.0 code of TraditionalForm)

Theoretical Study on the Kinetics of a Special Particle Swarm

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NOTE:
1. The "Euclid Math One" regular and bold fonts are needed to display the contents correctly in this Notebook.
2. If there is no special case, the Mathematica code starts with gray "*In*[∘]:=" and is bold by default according to Mathematica's rules.

Part 1. The Square of the Norm of the Average Velocity is Proportional to the Number of Vectors

As described in the main text, the k-particle is a general particle composed of k 1-particles. Each 1-particle is moving at the same speed c and in a random direction in the 3-dimensional Cartesian coordinate system (they are in a completely free state or in the constrained state of Iu not considering the Γ effect). Suppose that the standard deviation of the projection of the velocity of any one of the k equivalent 1-particles forming a k-particle onto each equivalent coordinate axis is σ. According to the my previous study[1], the speed of k-particles (or k particles in a certain domain) follows the Maxwell distribution with scale parameter $\frac{\sigma}{\sqrt{k}}$.

Then, the average velocity of the k-particles (or k particles in a certain domain) is

In[∘]:= $\mathbf{\bar{v} = Mean\left[MaxwellDistribution\left[\frac{\sigma}{\sqrt{k}}\right]\right]}$

Out[∘]= $\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k}}$

For k_a- and k_b-particles, the ratio of their average velocity $\bar{v}_\mathrm{a}/\bar{v}_\mathrm{b}$ =

In[∘]:= $\mathbf{\frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k_a}} \Big/ \frac{2\sqrt{\frac{2}{\pi}}\,\sigma}{\sqrt{k_b}}}$

Out[∘]= $\frac{\sqrt{k_b}}{\sqrt{k_a}}$

And because: $m_\mathrm{a} = \mu\, k_\mathrm{a}$ and $m_\mathrm{b} = \mu\, k_\mathrm{b}$, where μ is the scale factor or the mass of 1-particle. $\bar{v}_\mathrm{a}/\bar{v}_\mathrm{b}$ is also equal to

In[∘]:= $\mathbf{Simplify\left[\frac{\sqrt{\frac{m_b}{\mu}}}{\sqrt{\frac{m_a}{\mu}}}, Assumptions \to \mu > 0\right]}$

Out[∘]= $\frac{\sqrt{m_b}}{\sqrt{m_a}}$

Therefore, the square of the average velocity of particles is directly proportional to the mass of particles or the number of 1-particles forming it.

References

[1] Guo, T. Study on the average speed of particles from a particle swarm derived from a stationary particle swarm. *Sci*. *Rep*. 11, 13290 (**2021**). https://doi.org/10.1038/s41598-021-92402-w

Part 2. The Norm of the Component Vector is Proportional to the Number of Vectors Forming It

When the total vector value of a specified vector swarm is determined, the mean norms between different component vectors should be proportional to the number forming them in the constrained state of IIIu. The following proves this viewpoint in detail.

According to my previous study[1], let $\mathcal{M}$k being the norm of momentum of k particles observed from $\mathcal{R}_u$, the probability density of momentum norm formed by k particles in $\mathcal{R}_u$ observed in $\mathcal{R}_0$ can be expressed as (This code takes approximately 71 seconds):

In[∘]:= **Clear["Global\`*"];**

$$\mathcal{D} = \text{TransformedDistribution}\Big[\sqrt{(k\,u)^2 + \mathcal{M}\text{k}^2 - 2\,k\,u\,\mathcal{M}\text{k}\,\text{Cos}[\text{ArcCos}[\eta]]}\,,$$
$$\Big\{\mathcal{M}\text{k} \approx \text{MaxwellDistribution}\Big[\frac{\sqrt{k}\;\sqrt{c^2-u^2}}{\sqrt{3}}\Big],\ \eta \approx \text{UniformDistribution}[\{-1, 1\}]\Big\}\Big];$$

FullSimplify[PDF[$\mathcal{D}$, x], Assumptions → $c > u > 0 \land k > 0$]

Out[∘]=
$$\begin{cases} \dfrac{\sqrt{3}\,x\left(e^{\frac{6\,u\,x}{c^2-u^2}}-1\right)e^{-\frac{3\,(k\,u+x)^2}{2\,k\,(c^2-u^2)}}}{k\,u\,\sqrt{2\,\pi\,c^2\,k-2\,\pi\,k\,u^2}} & (x>0 \land k\,u>x) \lor k\,u<x \\[2ex] -\dfrac{\sqrt{6\,\pi}\,\sqrt{c^2\,k-u\,x}\,(5\,u\,x-2\,c^2\,k)\,\text{erf}\left(\frac{\sqrt{6}\,x}{\sqrt{c^2\,k-u\,x}}\right)+4\,x\,e^{\frac{6\,x^2}{u\,x-c^2\,k}}\left(c^2\,(6\,k+2)-u\,(2\,u+3\,x)\right)-8\,x\,(c-u)\,(c+u)}{4\,\sqrt{6\,\pi}\,k^{5/2}\,u\,((c-u)\,(c+u))^{3/2}} & k\,u=x \end{cases}$$

The first branch is selected as valid.

In view of the above conclusions, we find the mean value of this distribution (This code takes approximately 50 seconds).

In[∘]:=
$$\overline{\mathcal{Y}}_k = \text{FullSimplify}\Big[$$
$$\text{Mean}\Big[\text{ProbabilityDistribution}\Big[\frac{\sqrt{3}\,x\left(e^{\frac{6\,u\,x}{c^2-u^2}}-1\right)e^{-\frac{3\,(k\,u+x)^2}{2\,k\,(c^2-u^2)}}}{k\,u\,\sqrt{2\,\pi\,c^2\,k-2\,\pi\,k\,u^2}},\ \{x, 0, +\infty\}\Big]\Big],\ \text{Assumptions} \to c>u>0 \land k>0\Big]$$

Out[∘]=
$$\frac{(c^2+(3\,k-1)\,u^2)\,\text{erf}\left(\frac{\sqrt{\frac{3}{2}}\,k\,u}{\sqrt{k\,(c-u)\,(c+u)}}\right)+\sqrt{\frac{6}{\pi}}\,u\,e^{\frac{3\,k\,u^2}{2\,(u^2-c^2)}}\,\sqrt{k\,(c-u)\,(c+u)}}{3\,u}$$

We find the limit of the ratio of this mean value $\overline{\mathcal{Y}}_k$ and k when k approaches $+\infty$.

In[∘]:=
$$\text{Simplify}\Big[\text{Limit}\Big[\frac{\overline{\mathcal{Y}}_k}{k},\ k \to +\infty\Big],\ \text{Assumptions} \to u>0\Big]$$

Out[∘]=
$$\begin{cases} -u & \arg(c^2-u^2) \geq \pi \\ u & \text{True} \end{cases}$$

The second brunch is meaningful. Therefore, when k is a large number, the norm of the mean value $\overline{\mathcal{Y}}_k$ is directly proportional to the number k forming $\overline{\mathcal{Y}}_k$, namely, $\overline{\mathcal{Y}}_k = k \cdot u$.

Eq. 11 in the main text determines the proportion of particle number distributed in various boxes partitioned by k, and these particles are distributed in each box of $\mathcal{V}$ with equal probability. That is, the particles are randomly extracted from the microdomain $\mathcal{V}$ to be distributed in each box. When the number of extractions is large enough, the norm of each component vector partitioned by k should be directly proportional to the number of particles according to the probability and the scale factor is u.

The unique expansion of scalar $\mathcal{M}$ in the form of including power series is

$$\mathcal{M} = \sum_{k=1}^{\infty} \frac{e^{-\mathcal{M}}\, \mathcal{M}^k}{(k-1)!}$$

If the corresponding terms marked by k are directly proportional between the expansion of the norm $\|\boldsymbol{\mathcal{M}}\|$ of vector $\boldsymbol{\mathcal{M}}$ and the expansion of the scalar $\mathcal{M}$ representing the number of particles, or the numbers of particles are allowed to be proportional to the norms of vectors they form, the number $\mathcal{M}$ of particles must be equal to the norm $\|\boldsymbol{\mathcal{M}}\|$ of the vector $\boldsymbol{\mathcal{M}}$ they form besides they are required to obey Poisson distribution. According to the above conclusion $\overline{\mathcal{Y}}_k = k \cdot u$, the average speed $u = 1$ is needed in the system.

References

[1] Guo, T. Study on the average speed of particles from a particle swarm derived from a stationary particle swarm. *Sci. Rep.* 11, 13290 (**2021**). https://doi.org/10.1038/s41598-021-92402-w

Part 3. Solving Process of Eq. 38 in the Main Text

To solve the partial differential equation Eq. 38 in the main text, it is assumed that the system is spherically symmetric because it is isotropic at a huge scale. Therefore, we make the conversion from rectangular to spherical coordinates (note that φ is used to denote the azimuthal angle, whereas θ is used to denote the polar angle), namely, $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$ and $z = \cos\theta$.

In the case of spherical symmetry, the change of function $\boldsymbol{\mathcal{M}}(r)$ does not depend on θ and φ, but is related to r. Therefore, after the coordinate transformation, and the first and the second derivatives are obtained, to omit the terms that depends on angles θ and φ, we can obtain (subject to the character limitation of Mathematica, $\mathcal{M}$ is used instead of $\boldsymbol{\mathcal{M}}$ in the code cell; the same is done below):

In[∘]:= $\mathbf{Simplify}\left[\frac{2}{r} D[\mathcal{M}[r], \{r, 1\}] + D[\mathcal{M}[r], \{r, 2\}] - (D[\mathcal{M}[r], \{r, 1\}])^2 \left((\mathrm{Sin}[\theta]\,\mathrm{Cos}[\varphi])^2 + (\mathrm{Sin}[\theta]\,\mathrm{Sin}[\varphi])^2 + (\mathrm{Cos}[\theta])^2\right)\right]$

Out[∘]= $\mathcal{M}''(r) - \mathcal{M}'(r)^2 + \frac{2\,\mathcal{M}'(r)}{r}$

To solve the abovementioned differential equation under the boundary condition $\boldsymbol{\mathcal{M}}(r_e) = 0$.

In[∘]:= $\mathbf{DSolve}\left[\left\{\mathcal{M}''[r] - (\mathcal{M}'[r])^2 + \frac{2}{r}\mathcal{M}'[r] == 0, \mathcal{M}[\mathrm{re}] == 0\right\}, \mathcal{M}[r], r\right]$

Out[∘]= $\{\{\mathcal{M}(r) \to \log(r) - \log(1 + c_1 r) - \log(\mathrm{re}) + \log(1 + c_1\,\mathrm{re})\}\}$

Suppose another boundary condition is $\boldsymbol{\mathcal{M}}(r_c) = \mathcal{M}_c$, then

In[∘]:= $r = \mathrm{rc};$
$\mathbf{Solve}[\mathrm{Log}[r] - \mathrm{Log}[1 + \mathrm{c1}\, r] - \mathrm{Log}[\mathrm{re}] + \mathrm{Log}[1 + \mathrm{c1}\,\mathrm{re}] == \mathcal{M}\mathrm{c}, \mathrm{c1}]$

Out[∘]= $\left\{\left\{\mathrm{c1} \to \frac{\mathrm{rc} - \mathrm{re}\, e^{\mathcal{M}\mathrm{c}}}{\mathrm{rc}\,\mathrm{re}\left(e^{\mathcal{M}\mathrm{c}} - 1\right)}\right\}\right\}$

Therefore, the solution of the above differential equation is as follows:

In[∘]:= **Clear["Global`*"];**

$$\mathbf{c1} = \frac{\mathbf{rc - re}\, e^{\mathcal{M}\mathbf{c}}}{\mathbf{rc\, re}\left(e^{\mathcal{M}\mathbf{c}} - \mathbf{1}\right)};$$

Simplify[Log[*r*] − Log[1 + c1 *r*] − Log[re] + Log[1 + c1 re]]

Out[∘]= $$-\log\left(\frac{r\left(\mathrm{rc} - \mathrm{re}\, e^{\mathcal{M}\mathrm{c}}\right)}{\mathrm{rc\, re}\left(e^{\mathcal{M}\mathrm{c}} - 1\right)} + 1\right) + \log(r) + \log\left(\frac{e^{\mathcal{M}\mathrm{c}}\,(\mathrm{rc} - \mathrm{re})}{\mathrm{rc}\left(e^{\mathcal{M}\mathrm{c}} - 1\right)}\right) - \log(\mathrm{re})$$

To restore the above solution in spherical to the solution in 3-dimensional rectangular coordinates, then

In[∘]:= $\boldsymbol{r} = \sqrt{x^2 + y^2 + z^2}$ **;**

$$\mathbf{FullSimplify}\left[-\mathbf{Log}\left[\frac{r\left(\mathbf{rc - re}\, e^{\mathcal{M}\mathbf{c}}\right)}{\mathbf{rc\, re}\left(e^{\mathcal{M}\mathbf{c}} - \mathbf{1}\right)} + \mathbf{1}\right] + \mathbf{Log}[r] + \mathbf{Log}\left[\frac{e^{\mathcal{M}\mathbf{c}}\,(\mathbf{rc - re})}{\mathbf{rc}\left(e^{\mathcal{M}\mathbf{c}} - \mathbf{1}\right)}\right] - \mathbf{Log[re]},\right.$$
$$\left.\mathbf{Assumptions} \to \mathbf{re > rc > 0}\right]$$

Out[∘]= $$-\log\left(\frac{\left(\mathrm{rc} - \mathrm{re}\, e^{\mathcal{M}\mathrm{c}}\right)\sqrt{x^2 + y^2 + z^2}}{e^{\mathcal{M}\mathrm{c}} - 1} + \mathrm{rc\, re}\right) + \log\left(\frac{e^{\mathcal{M}\mathrm{c}}\,(\mathrm{rc} - \mathrm{re})}{e^{\mathcal{M}\mathrm{c}} - 1}\right) + \frac{1}{2}\log\left(x^2 + y^2 + z^2\right)$$

To verify the above results:

In[∘]:= $$\mathcal{M}[x, y, z] := -\mathbf{Log}\left[\frac{\left(c - \mathbf{re}\, e^{\mathcal{M}\mathbf{c}}\right)\sqrt{x^2 + y^2 + z^2}}{e^{\mathcal{M}\mathbf{c}} - \mathbf{1}} + \mathbf{rc\, re}\right] + \mathbf{Log}\left[\frac{e^{\mathcal{M}\mathbf{c}}\,(\mathbf{rc - re})}{e^{\mathcal{M}\mathbf{c}} - \mathbf{1}}\right] + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{Log}\left[x^2 + y^2 + z^2\right];$$

$$\mathbf{FullSimplify}\left[\frac{\partial^2 \mathcal{M}(x, y, z)}{\partial x^2} + \frac{\partial^2 \mathcal{M}(x, y, z)}{\partial y^2} + \frac{\partial^2 \mathcal{M}(x, y, z)}{\partial z^2} - \left(\frac{\partial \mathcal{M}(x, y, z)}{\partial x}\right)^2 - \left(\frac{\partial \mathcal{M}(x, y, z)}{\partial y}\right)^2 - \left(\frac{\partial \mathcal{M}(x, y, z)}{\partial z}\right)^2\right]$$

Out[∘]= 0

Therefore, the above equation is the solution of Eq. 38 in the main text (only when $\mathrm{Im}[\mathcal{M}_0] \in [-\pi, \pi)$ and the principal values of arguments are taken in the calculation process).

Similarly, the 2-dimensional case can also be solved.

In[∘]:= **Clear["Global`*"];**

$$\mathbf{Simplify}\left[D[\mathcal{M}[r], \{r, 2\}] + \frac{\mathbf{1}}{r} D[\mathcal{M}[r], \{r, 1\}] - (D[\mathcal{M}[r], \{r, 1\}])^2\right]$$

Out[∘]= $$\mathcal{M}''(r) - \mathcal{M}'(r)^2 + \frac{\mathcal{M}'(r)}{r}$$

In[∘]:= $$\mathbf{DSolve}\left[\left\{\mathcal{M}''[r] - \mathcal{M}'[r]^2 + \frac{\mathcal{M}'[r]}{r} == \mathbf{0},\ \mathcal{M}[\mathbf{re}] == \mathbf{0}\right\},\ \mathcal{M}[r],\ r\right]$$

Out[∘]= $\{\{\mathcal{M}(r) \to \log(-\log(\mathrm{re}) + c_1) - \log(-\log(r) + c_1)\}\}$

In[∘]:= ***r* = rc;**

Solve[Log[−Log[re] + c1] − Log[−Log[*r*] + c1] == $\mathcal{M}$c, c1]

Out[∘]= $$\left\{\left\{\mathrm{c1} \to \frac{e^{\mathcal{M}\mathrm{c}}\log(\mathrm{rc}) - \log(\mathrm{re})}{e^{\mathcal{M}\mathrm{c}} - 1}\right\}\right\}$$

In[•]:=
```
Clear["Global`*"];
c1 = (ⅇ^Mc Log[rc] - Log[re])/(ⅇ^Mc - 1);
Simplify[Log[-Log[re] + c1] - Log[-Log[r] + c1]]
```

Out[•]=

$$\log\left(\frac{e^{\mathcal{M}c}(\log(rc)-\log(re))}{e^{\mathcal{M}c}-1}\right)-\log\left(\frac{e^{\mathcal{M}c}\log(rc)-\log(re)}{e^{\mathcal{M}c}-1}-\log(r)\right)$$

In[•]:=
```
r = Sqrt[x^2 + y^2];
FullSimplify[Log[(ⅇ^Mc (Log[rc] - Log[re]))/(ⅇ^Mc - 1)] - Log[(ⅇ^Mc Log[rc] - Log[re])/(ⅇ^Mc - 1) - Log[r]],
  Assumptions -> re > rc > 0]
```

Out[•]=

$$\log\left(\frac{e^{\mathcal{M}c}\log\left(\frac{rc}{re}\right)}{e^{\mathcal{M}c}-1}\right)-\log\left(\frac{\log\left(\frac{rc}{re}\right)}{e^{\mathcal{M}c}-1}+\log(rc)-\frac{1}{2}\log\left(x^2+y^2\right)\right)$$

In[•]:=
```
M[x, y] := Log[(ⅇ^Mc Log[rc/re])/(ⅇ^Mc - 1)] - Log[Log[rc/re]/(ⅇ^Mc - 1) + Log[rc] - 1/2 Log[x^2 + y^2]];
FullSimplify[∂^2 M(x, y)/∂x^2 + ∂^2 M(x, y)/∂y^2 - (∂M(x, y)/∂x)^2 - (∂M(x, y)/∂y)^2]
```

Out[•]= 0

To verify the above conclusion, the results of analytical solution and the numerical solution under the same conditions are plotted (This code takes approximately 38 seconds):

In[•]:=
```
Clear["Global`*"];
Ma[x_, y_] := Log[(ⅇ^Mc Log[rc/re])/(ⅇ^Mc - 1)] - Log[Log[rc/re]/(ⅇ^Mc - 1) + Log[rc] - 1/2 Log[x^2 + y^2]];
rc = 4/100;
re = 4;
Mc = 1 + 2 ⅈ;
Ω = ImplicitRegion[rc^2 ≤ x^2 + y^2 ≤ re^2, {x, y}];
G1 = Show[Plot3D[Norm[Ma[x, y]], {x, y} ∈ Ω, PlotRange -> {0, Sqrt[8]},
      ColorFunction -> (Hue[0.65, #3] &), MeshStyle -> None, BoundaryStyle -> None, PlotPoints -> 300,
      AxesLabel -> {Style["x  ", Italic], Style["y", Italic], Rotate[Style["Density     "], π/2]},
      AxesStyle -> Directive[Black, FontFamily -> "Arial", FontSize -> 15], TicksStyle -> Black,
      BoxStyle -> Directive[Black, Thickness -> 0.0018], BoxRatios -> Automatic, ViewPoint -> {15, -26, 16},
      Epilog -> Text[Style["a", 15, FontFamily -> "Arial", Bold, Black], {-0.07, 0.92}, {-1, 1}]],
    Table[Ω1 = ImplicitRegion[9/100 ≤ x^2 + i^2 ≤ 16, {x}]; If[i^2 ≤ 9/100, xx = Sqrt[9/100 - i^2], xx = 0];
      ParametricPlot3D[{x, i, Norm[Ma[x, i]]}, {x} ∈ Ω1, PlotStyle -> Thickness[0.0018], PlotPoints -> 300,
       ColorFunction -> (GrayLevel[0.4, 1 - #3 × Norm[Ma[xx, i]]/Norm[Ma[0, 3/10]]] &)], {i, -3.5, 3.5, 0.5}],
```

```
    Table[Ω1 = ImplicitRegion[9/100 ≤ j^2 + y^2 ≤ 16, {y}]; If[j^2 ≤ 9/100, yy = Sqrt[9/100 − j^2], yy = 0];
      ParametricPlot3D[{j, y, Norm[Ma[j, y]]}, {y} ∈ Ω1, PlotStyle → Thickness[0.0018],
        PlotPoints → 300, ColorFunction → (GrayLevel[0.4, 1 − #3× Norm[Ma[j, yy]]/Norm[Ma[0, 3/10]]] &)],
      {j, −3.5, 3.5, 0.5}], ParametricPlot3D[{4 Cos[ϕ], 4 Sin[ϕ], 0}, {ϕ, 0, 2 π},
      PlotStyle → Directive[Gray, Thickness[0.0018]], PlotPoints → 300]];
Needs["NDSolve`FEM`"];
mesh = ToElementMesh[Ω, MeshRefinementFunction →
      Function[{vertices, area}, area > 3/100000 (1/10 + 80 Norm[Mean[vertices]])]];
Mn = NDSolveValue[{∂²u(x, y)/∂x² + ∂²u(x, y)/∂y² − (∂u(x, y)/∂x)^2 − (∂u(x, y)/∂y)^2 == 0, DirichletCondition[
       u[x, y] == Mc, x^2 + y^2 == rc^2], DirichletCondition[u[x, y] == 0, x^2 + y^2 == re^2]}, u, {x, y} ∈ mesh];
G2 = Show[Plot3D[Norm[Mn[x, y]], {x, y} ∈ mesh, PlotRange → {0, Sqrt[8]},
      ColorFunction → (Hue[0.65, #3] &), MeshStyle → None, BoundaryStyle → None,
      AxesLabel → {Style["x  ", Italic], Style["y", Italic], Rotate[Style["Density      "], π/2]},
      AxesStyle → Directive[Black, FontFamily → "Arial", FontSize → 15], TicksStyle → Black,
      BoxStyle → Directive[Black, Thickness → 0.002], BoxRatios → Automatic, ViewPoint → {15, −26, 16},
      Epilog → Text[Style["b", 15, FontFamily → "Arial", Bold, Black], {−0.07, 0.92}, {−1, 1}]],
    Table[Ω2 = ImplicitRegion[9/100 ≤ x^2 + i^2 ≤ 16, {x}]; If[i^2 ≤ 9/100, xx = Sqrt[9/100 − i^2], xx = 0];
      ParametricPlot3D[{x, i, Norm[Mn[x, i]]}, {x} ∈ Ω2, PlotStyle → Thickness[0.0018], PlotPoints → 300,
        ColorFunction → (GrayLevel[0.4, 1 − #3× Norm[Mn[xx, i]]/Norm[Mn[0, 3/10]]] &)], {i, −3.5, 3.5, 0.5}],
    Table[Ω2 = ImplicitRegion[9/100 ≤ j^2 + y^2 ≤ 16, {y}]; If[j^2 ≤ 9/100, yy = Sqrt[9/100 − j^2], yy = 0];
      ParametricPlot3D[{j, y, Norm[Mn[j, y]]}, {y} ∈ Ω2, PlotStyle → Thickness[0.0018],
        PlotPoints → 300, ColorFunction → (GrayLevel[0.4, 1 − #3× Norm[Mn[j, yy]]/Norm[Mn[0, 3/10]]] &)],
      {j, −3.5, 3.5, 0.5}], ParametricPlot3D[{4 Cos[ϕ], 4 Sin[ϕ], 0}, {ϕ, 0, 2 π},
      PlotStyle → Directive[Gray, Thickness[0.0018]], PlotPoints → 300]];
s1 = GraphicsRow[{G1, G2}, ImageSize → 500, Spacings → Scaled[−0.06]];
Pane[s1, {500, 200}, ImageMargins → {{50, −30}, {−18, −25}}]
```

Out[∘]=

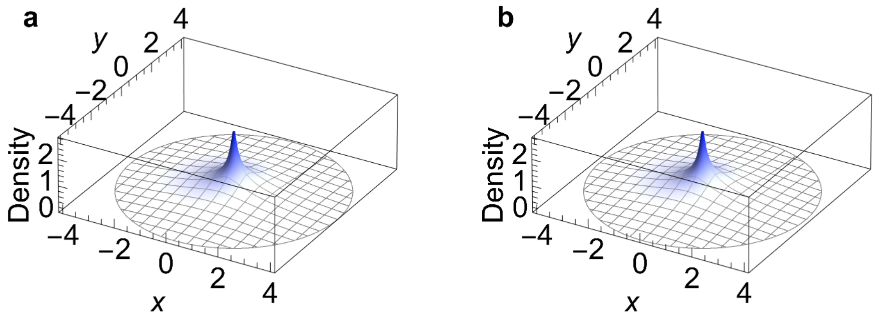

Figure S1 | Distribution of the mass density of a particle swarm meeting conditions ($\mathcal{M}(x, y) = 1 + 2i \wedge x^2 + y^2 = \frac{16}{10\,000}) \wedge (\mathcal{M}(x, y) = 0 \wedge x^2 + y^2 = 4^2)$. **a**, The analytical solution. **b**, The numerical solution.

It can be seen from Fig. S1b that the numerical solution and the analytical solution achieve a perfect agreement (only when $\mathrm{Im}[\mathcal{M}_c] \in [-\pi, \pi)$ and the principal values of arguments are taken in the calculation process).

Part 4. Figures Used in the Main Text

NOTE: To run these codes correctly, the contents in "**MyDirection = ∗∗**" in the next cell should be modified. It is similar to **MyDirection** = "**/Users/yourdirection/**". Then, run it (Shift+Enter) beforehand.

```
MyDirection = **;
Protect[MyDirection];
Off[General::wrsym];

## ## ## ## ## ## ## ## ## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##

In[∘]:= Clear["Global`*"];
head = Graphics[Polygon[0.13 * {{-1, 8.09/25}, {0, 0}, {-1, -8.09/25}, {-8.09/10, 0}, {-1, 8.09/25}}]];
aa = Graphics[{{Blue, Thickness[0.003], Circle[{0, 1/2}, 1.04]},
    {Red, Thickness[0.003], Circle[{0, 0}, 2]}, {RGBColor[0, 0, 1, 1],
      Arrowheads[{{.3, 1, {head, 0.06}}}], {Thickness[0.006], Arrow[{{0, 0.5}, {1.4, 0.5}}]}},
    {Green, PointSize[0.01], Point[{0, 1/5}]}, Text[Style["R", 18, FontFamily -> "Euclid Math One",
        Blue], {0, 1.07}], Text[Style["u", Italic, 12, FontFamily -> "Arial", Blue], {0.132, 1.03}],
    Text[Style["Target (Sub-) domain", 18, FontFamily -> "Arial", Blue], {0.01, 2/3}],
    Text[Style["Total (Parent/Background) domain", 18, FontFamily -> "Arial", Red], {0, -4/5}],
    Text[Style["R", 18, FontFamily -> "Euclid Math One", Red], {0, -1.25}],
    Text[Style["0", 12, FontFamily -> "Arial", Red], {0.132, -1.3}],
    Text[Style["Microdomain", 18, FontFamily -> "Arial", Green], {0, 0}],
    Text[Style["u", Italic, 18, FontFamily -> "Arial", Blue], {1.53, 0.51}]}];
Export[MyDirection <> "figure1.eps", aa, Background -> None];

## ## ## ## ## ## ## ## ## ## ## Figure1 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##

## ## ## ## ## ## ## ## ## ## ## Figure2 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
In[∘]:= Clear["Global`∗"];
{rr1, bb1} = Last@Reap@
    Scan[If[#[[1]]^2 + #[[2]]^2 < 1, Sow[#, "Red"], Sow[#, "Blue"]] &, RandomReal[{−2, 2}, {2000, 2}]];
𝓡1 = ImplicitRegion[x^2 + y^2 > 1, {{x, −2, 2}, {y, −2, 2}}];
𝓡2 = ImplicitRegion[x^2 + y^2 < 1, {{x, −2, 2}, {y, −2, 2}}];
{rr2, bb2} = {RandomPoint[𝓡1, 1000], RandomPoint[𝓡2, 600]};
head = Graphics[Polygon[0.1 ∗ {{−1, 8.09/25}, {0, 0}, {−1, −8.09/25}, {−8.09/10, 0}, {−1, 8.09/25}}]];
bb = Graphics[{{Blue, Dashed, Thickness[0.0016], Circle[{0, 0}, 1]}, {Red, Point[rr1]}, {Blue, Point[bb1]},
    {Blue, Dashed, Thickness[0.0016], Circle[{4.5, 0}, 1]}, {RGBColor[0, 0, 1, 1],
      Arrowheads[{{0.2, 1, {head, 0.03}}}], {Thickness[0.004], Arrow[{{0, 0}, {1.37, 0}}]}},
    Text[Style["u", 20, Italic, FontFamily → "Arial", Blue], {1.53, 0.01}],
    Text[Style["a", 20, Bold, FontFamily → "Arial", Black], {−2, 2}],
    {Red, Point[rr2 + Table[{4.5, 0}, {i, Length[rr2]}]]},
    {Blue, Point[bb2 + Table[{4.5, 0}, {i, Length[bb2]}]]},
    {Blue, Arrowheads[{{.2, 1, {head, 0.03}}}], {Thickness[0.004], Arrow[{{4.5, 0}, {5.87, 0}}]}},
    Text[Style["u", 20, Italic, FontFamily → "Arial", Blue], {6.03, 0.01}],
    Text[Style["b", 20, Bold, FontFamily → "Arial", Black], {2.5, 2}]},
   Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.004]], Directive[Red, Thickness[0.004]]},
      {Style["Particles included in statistics", FontFamily → "Arial", FontSize → 20],
        Style["Particles not included in statistics", FontFamily → "Arial", FontSize → 20]},
      Joined → {False, False}, LegendLayout → "Row", LegendFunction →
        (Framed[#, RoundingRadius → 4, Background → White, FrameStyle → GrayLevel[0.58]] &)],
     Scaled[{1/2, 0.11}]], ImageSize → 700];
Export[MyDirection <> "figure2.eps", bb, Background → None];
## ## ## ## ## ## ## ## ## ## ## Figure2 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##

## ## ## ## ## ## ## ## ## ## ## Figure3 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
In[∘]:= Clear["Global`*"];
head = Graphics[Polygon[0.3*{{-1, 8.09/25}, {0, 0}, {-1, -8.09/25}, {-8.09/10, 0}, {-1, 8.09/25}}]];
cc = Graphics[{{RGBColor[178/255, 252/255, 61/255], Rectangle[{0, 0}, {1, 1}]},
    {RGBColor[178/255, 252/255, 61/255, 0.5], Rectangle[{1, 0}, {2, 1}]}, {RGBColor[250/255, 200/255, 0],
     Arrowheads[{{0.2, 1, {head, 0.06}}}], {Thickness[0.006], Arrow[{{0.7, 0.54}, {1.3, 0.54}}]}},
    {RGBColor[250/255, 200/255, 0], Arrowheads[{{0.2, 1, {head, 0.06}}}],
     {Thickness[0.006], Arrow[{{1.3, 0.46}, {0.7, 0.46}}]}}, {RGBColor[178/255, 252/255, 61/255, 0.5],
     Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {1.2, -1.3}}]}},
    {RGBColor[178/255, 252/255, 61/255], Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {0.8, -0.3}}]}},
    {Orange, {Thickness[0.0036], DotDashed, Line[{{1, -0.05}, {1, 1.05}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{0.8, -0.3}, {2, -0.3}}]}},
    {Orange, {Thickness[0.004], Dashed, Line[{{1.2, -1.3}, {2, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{1.2, -1.3}, {0.8, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {2, -0.3}}]}},
    {Blue, Arrowheads[0.06], {Thickness[0.006], Arrow[{{0, -1.3}, {1, -0.8}}]}},
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {0.45, 0.5}],
    Text[Style["A", 17, FontFamily → "Arial", White], {0.513, 0.456}],
    Text[Style["V", 24, FontFamily → "Euclid Math One", White], {1.55, 0.5}],
    Text[Style["B", 17, FontFamily → "Arial", White], {1.616, 0.455}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.63}],
    Text[Style["A", 17, FontFamily → "Arial", Orange], {1.063, 0.59}],
    Text[Style["D", 24, FontFamily → "Arial", Orange, Italic], {0.982, 0.38}],
    Text[Style["B", 17, FontFamily → "Arial", Orange], {1.065, 0.34}],
    Text[Style["Φ", 24, FontFamily → "Arial", Orange], {1.06, 1.08}],
    Text[Style["O", 24, FontFamily → "Arial", Orange], {0, -1.39}],
    Text[Style["B", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255, 0.5]], {1.2, -1.39}],
    Text[Style["A", 24, FontFamily → "Arial", RGBColor[178/255, 252/255, 61/255]], {0.7, -0.28}],
    Text[Style["C", 24, FontFamily → "Arial", Orange], {2.02, -0.4}],
    Text[Style["M", 24, FontFamily → "Arial", Orange], {0.973, -0.932}],
    Inset[Style["a", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, 1.12}],
    Inset[Style["b", Black, Bold, FontFamily → "Arial", FontSize → 24], {0.034, -0.2}]}];
Export[MyDirection <> "figure3.png", cc, Background → None, ImageResolution → 1200];
## ## ## ## ## ## ## ## ## ## ## Figure3 ## ## ## ## ## ## ## ## ## ## ## ## ## ##
## ## ## ## ## ## ## ## ## ## ## Figure4 ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

```
In[∘]:= Clear["Global`*"];
text = Graphics[{Gray, Line[{{1, 0}, {1, 10}}], Line[{{2, 0}, {2, 10}}],
    Line[{{3, 0}, {3, 10}}], Line[{{4, 0}, {4, 10}}], Line[{{5, 0}, {5, 10}}],
    Line[{{6, 0}, {6, 10}}], Line[{{7, 0}, {7, 10}}], Line[{{8, 0}, {8, 10}}], Line[{{9, 0}, {9, 10}}],
    Line[{{0, 1}, {10, 1}}], Line[{{0, 2}, {10, 2}}], Line[{{0, 3}, {10, 3}}], Line[{{0, 4}, {10, 4}}],
    Line[{{0, 5}, {10, 5}}], Line[{{0, 6}, {10, 6}}], Line[{{0, 7}, {10, 7}}], Line[{{0, 8}, {10, 8}}],
    Line[{{0, 9}, {10, 9}}], Orange, Rectangle[{6, 4}, {7, 5}]}, PlotRangePadding → 1/1000];
dd = Show[{Plot3D[Sin[x + Cos[y]], {x, −3, 3}, {y, −3, 3}, PlotPoints → 60, MaxRecursion → 3,
     PlotStyle → Texture[text], Mesh → None, Lighting → "Neutral", PlotLabels → Placed["", {0, 0}],
     BoundaryStyle → None, Boxed → False, Axes → None, ViewPoint → {1, −1.9, 1.4}],
    Graphics3D[{{Thickness[0.007], Black,
       Arrow[{{0, 0, 0}, {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
            −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1}} +
          {{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {0.88, −0.3, Sin[0.88 + Cos[−0.3]]}}]},
      {Text[Style["N", 14, FontFamily → "Arial", Bold, Italic, Black],
        {−Evaluate[D[Sin[x + Cos[y]], x] /. {x → 0.88, y → −0.3}],
           −Evaluate[D[Sin[x + Cos[y]], y] /. {x → 0.88, y → −0.3}], 1} +
          {0.88, −0.3, Sin[0.88 + Cos[−0.3]]} + {0.02, 0.03, 0.23}]},
      {Thickness[0.007], Blue, Arrow[{{0.88, −0.3, Sin[0.88 + Cos[−0.3]]}, {1.88, −0.5, 2}}]},
      {Text[Style["X", 14, FontFamily → "Euclid Math One", Bold, Blue], {2.01, −0.5, 2.01}]},
      {Text[Style["Σ", 14, FontFamily → "Arial", Italic, Gray], {−2.14, −1.5, 0.7}]},
      {Text[Style["dS", 14, FontFamily → "Arial", Orange], {0.55, −0.8, 1.39}]}}]}];
dd = Pane[dd, {400, 300}, ImageMargins → {{−8, −52}, {−74, −39}}];
Export[MyDirection <> "figure4.png", dd, Background → None, ImageResolution → 1200];
```

Figure4 ## ## ## ## ## ## ## ## ## ## ## ## ## ##

Figure5 ## ## ## ## ## ## ## ## ## ## ## ## ## ##

```
In[∘]:= Clear["Global`*"];
head = Graphics[Polygon[0.3 * {{−1, 8.09/25}, {0, 0}, {−1, −8.09/25}, {−8.09/10, 0}, {−1, 8.09/25}}]];
headv = Graphics[Polygon[0.3 * {{−1, 8.09/25}, {0, 0}, {−1, −8.09/25}, {−1, 8.09/25}}]];
p =
  {{RandomReal[{1.1, 1.9}], RandomReal[{3.1, 3.9}]}, {RandomReal[{5.1, 5.9}], RandomReal[{7.1, 7.9}]},
   {RandomReal[{6.1, 6.9}], RandomReal[{5.1, 5.9}]}, {RandomReal[{8.1, 8.9}], RandomReal[{5.1, 5.9}]},
   {RandomReal[{8.1, 8.9}], RandomReal[{1.1, 1.9}]}, {RandomReal[{2.1, 2.5}], RandomReal[{6.1, 6.9}]},
   {RandomReal[{2.6, 2.9}], RandomReal[{6.1, 6.9}]}, {RandomReal[{3.1, 3.5}], RandomReal[{1.1, 1.9}]},
   {RandomReal[{3.6, 3.9}], RandomReal[{1.1, 1.9}]}, {RandomReal[{3.1, 3.5}], RandomReal[{8.1, 8.9}]},
   {RandomReal[{3.6, 3.9}], RandomReal[{8.1, 8.9}]}, {RandomReal[{4.1, 4.5}], RandomReal[{4.1, 4.9}]},
   {RandomReal[{4.6, 4.9}], RandomReal[{4.1, 4.9}]}, {RandomReal[{7.1, 7.5}], RandomReal[{7.1, 7.9}]},
   {RandomReal[{7.6, 7.9}], RandomReal[{7.1, 7.9}]}, {RandomReal[{4.1, 4.3}], RandomReal[{2.1, 2.9}]},
   {RandomReal[{4.4, 4.6}], RandomReal[{2.1, 2.9}]}, {RandomReal[{4.7, 4.9}], RandomReal[{2.1, 2.9}]},
   {RandomReal[{5.1, 5.3}], RandomReal[{6.1, 6.9}]}, {RandomReal[{5.4, 5.6}], RandomReal[{6.1, 6.9}]},
   {RandomReal[{5.7, 5.9}], RandomReal[{6.1, 6.9}]}, {RandomReal[{6.1, 6.3}], RandomReal[{3.1, 3.9}]},
   {RandomReal[{6.4, 6.6}], RandomReal[{3.1, 3.9}]}, {RandomReal[{6.7, 6.9}], RandomReal[{3.1, 3.9}]},
   {RandomReal[{8.1, 8.3}], RandomReal[{3.1, 3.9}]}, {RandomReal[{8.4, 8.6}], RandomReal[{3.1, 3.9}]},
   {RandomReal[{8.7, 8.9}], RandomReal[{3.1, 3.9}]}};
ee = Graphics[{Gray, Line[{{1, 0}, {1, 10}}], Line[{{2, 0}, {2, 10}}], Line[{{3, 0}, {3, 10}}],
    Line[{{4, 0}, {4, 10}}], Line[{{5, 0}, {5, 10}}], Line[{{6, 0}, {6, 10}}],
    Line[{{7, 0}, {7, 10}}], Line[{{8, 0}, {8, 10}}], Line[{{9, 0}, {9, 10}}], Line[{{0, 1}, {10, 1}}],
    Line[{{0, 2}, {10, 2}}], Line[{{0, 3}, {10, 3}}], Line[{{0, 4}, {10, 4}}], Line[{{0, 5}, {10, 5}}],
    Line[{{0, 6}, {10, 6}}], Line[{{0, 7}, {10, 7}}], Line[{{0, 8}, {10, 8}}], Line[{{0, 9}, {10, 9}}],
    {PointSize[0.02], Red, Point[p[[1]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
```

$$\text{Arrow}\left[\left\{p[[1]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[1]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Red}, \text{Point}[p[[2]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[2]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[2]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Red}, \text{Point}[p[[3]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[3]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[3]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Red}, \text{Point}[p[[4]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[4]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[4]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Red}, \text{Point}[p[[5]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[5]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[5]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[6]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[6]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[6]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[7]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[7]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[7]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[8]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[8]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[8]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[9]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[9]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[9]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[10]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[10]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[10]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[11]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[11]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[11]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[12]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[12]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[12]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[13]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[13]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[13]]\right]\right\}\right]\right\},$$

$\{\text{PointSize}[0.02], \text{Green}, \text{Point}[p[[14]]]\}, \Big\{\text{Arrowheads}[\{\{0.06, 1, \{\text{head}, 0.06\}\}\}], \text{Black},$

$$\text{Arrow}\left[\left\{p[[14]],\ \text{ReplaceAll}[\theta \to \text{RandomReal}[2\,\pi]]\left[\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\} + p[[14]]\right]\right\}\right]\right\},$$

{PointSize[0.02], Green, Point[p[[15]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[15]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[15]]]}]},
{PointSize[0.02], Blue, Point[p[[16]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[16]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[16]]]}]},
{PointSize[0.02], Blue, Point[p[[17]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[17]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[17]]]}]},
{PointSize[0.02], Blue, Point[p[[18]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[18]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[18]]]}]},
{PointSize[0.02], Blue, Point[p[[19]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[19]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[19]]]}]},
{PointSize[0.02], Blue, Point[p[[20]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[20]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[20]]]}]},
{PointSize[0.02], Blue, Point[p[[21]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[21]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[21]]]}]},
{PointSize[0.02], Blue, Point[p[[22]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[22]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[22]]]}]},
{PointSize[0.02], Blue, Point[p[[23]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[23]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[23]]]}]},
{PointSize[0.02], Blue, Point[p[[24]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[24]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[24]]]}]},
{PointSize[0.02], Blue, Point[p[[25]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[25]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[25]]]}]},
{PointSize[0.02], Blue, Point[p[[26]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[26]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[26]]]}]},
{PointSize[0.02], Blue, Point[p[[27]]]}, {Arrowheads[{{0.06, 1, {head, 0.06}}}], Black,
Arrow[{p[[27]], ReplaceAll[θ -> RandomReal[2 π]][$\begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}.\left\{\frac{2}{5}, 0\right\}$ + p[[27]]]}]},
{Arrowheads[{{0.11, 1, {headv, 0.06}}}], Gray, Thickness[0.003],
Arrow[{{5.28, −0.78}, {9.3, −0.78}}]}, {Arrowheads[{{0.11, 1, {headv, 0.06}}}],
Gray, Thickness[0.003], Arrow[{{4.72, −0.78}, {0.7, −0.78}}]},

```
Text[Style["V", 15, FontFamily → "Euclid Math One", Gray], Scaled[{0.5, 0.01634}]]}];
Export[MyDirection <> "figure5.png", ee, Background → None, ImageResolution → 1200];
## ## ## ## ## ## ## ## ## ## ## Figure5 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
## ## ## ## ## ## ## ## ## ## ## Figure6 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```

This code takes approximately 32 minutes.

```
In[∘]:= Clear["Global`*"];
usol = Block[{ϵ = $MachineEpsilon},
    NDSolveValue[{i D[M[r, t], t] == -e^(-M[r,t]) (D[M[r, t], r, r] - (D[M[r, t], r])^2 + (2 D[M[r, t], r])/r),
      M[r, 0] == 10^-2 e^(-r^2/2), M^(1,0)[ϵ, t] == 0, M[1000, t] == 0}, M, {r, ϵ, 3}, {t, 0, 3}, Method →
      {"MethodOfLines", "SpatialDiscretization" → {"TensorProductGrid", "MinPoints" → 12000}}]];
vsol = Block[{ϵ = $MachineEpsilon}, NDSolveValue[
    {i D[M[r, t], t] == 2/r D[M[r, t], {r, 1}] + D[M[r, t], {r, 2}], M[r, 0] == e^(-r^2/2),
      M^(1,0)[ϵ, t] == 0, M[1000, t] == 0}, M, {r, ϵ, 3}, {t, 0, 3}, Method →
      {"MethodOfLines", "SpatialDiscretization" → {"TensorProductGrid", "MinPoints" → 8000}}]];
G1 = Plot3D[10^2 Norm[usol[r, t]] - Norm[vsol[r, t]], {t, 0, 3}, {r, 0, 3}, PlotPoints → 60,
    MaxRecursion → 3, PlotRange → {{0, 3}, {0, 3}, {-0.003, 0.0075}},
    MeshStyle → GrayLevel[0.4], BoundaryStyle → GrayLevel[0.4],
    AxesLabel → {Style["t ", Italic], Style["r", Italic], Rotate["Δρ                    ", π/2]},
    AxesStyle → Directive[Black, Thickness → 0.002],
    BoxStyle → Directive[Black, Thickness → 0.0021], TicksStyle → Black,
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20], ViewPoint → {1, -2, 2.1}];
FindMaxValue[{(10^2 Norm[usol[r, t]] - Norm[vsol[r, t]]), r > 0, t > 0}, {r, t}] /
  (Norm[vsol[r, t]] /. Last[FindMaximum[{10^2 Norm[usol[r, t]] - Norm[vsol[r, t]], r > 0, t > 0}, {r, t}]])
vsol = Block[{ϵ = $MachineEpsilon},
    NDSolveValue[{i D[M[r, t], t] == -e^(-M[r,t]) (D[M[r, t], r, r] - (D[M[r, t], r])^2 + (2 D[M[r, t], r])/r),
      M[r, 0] == 1/2 e^(-r^2/2), M^(1,0)[ϵ, t] == 0, M[1000, t] == 0}, M, {r, ϵ, 4}, {t, 0, 2}, Method →
      {"MethodOfLines", "SpatialDiscretization" → {"TensorProductGrid", "MinPoints" → 21000}}]];
xv = NArgMax[Norm[vsol[0, t]], {t, 0.1, 0.5}];
G2 = Show[Plot3D[2 Norm[vsol[r, t]], {r, 0, 3}, {t, 0, 2},
      PlotRange -> All, MeshStyle → GrayLevel[0.4], BoundaryStyle → GrayLevel[0.4],
      AxesLabel → {Style["r ", Italic], Style["t", Italic], Rotate["ρ", π/2]},
      AxesStyle → Directive[Black, Thickness → 0.002], BoxStyle → Directive[Black, Thickness → 0.002],
      TicksStyle → Black, LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20],
      ViewPoint → {3, -2.2, 4.1}], ParametricPlot3D[{r, xv, 2 Norm[vsol[r, xv]]},
      {r, 0, 3}, PlotStyle → Directive[Red, Thickness → 0.005]]];
usol = Block[{ϵ = $MachineEpsilon}, NDSolveValue[
```

```
      {i D[M[r, t], t] == -e^(-M[r,t]) (D[M[r, t], r, r] - (D[M[r, t], r])^2 + (2 D[M[r, t], r])/r),
        M[r, 0] == 1/4 e^(-r^2/2), M^(1,0)[ϵ, t] == 0, M[1000, t] == 0}, M, {r, ϵ, 4}, {t, 0, 2}, Method →
        {"MethodOfLines", "SpatialDiscretization" → {"TensorProductGrid", "MinPoints" → 12 000}}]];
xu = NArgMax[Norm[usol[0, t]], {t, 0, 0.2}];
wsol = Block[{ϵ = $MachineEpsilon},
    NDSolveValue[{i D[M[r, t], t] == -e^(-M[r,t]) (D[M[r, t], r, r] - (D[M[r, t], r])^2 + (2 D[M[r, t], r])/r),
        M[r, 0] == 5/8 e^(-r^2/2), M^(1,0)[ϵ, t] == 0, M[1000, t] == 0}, M, {r, ϵ, 4}, {t, 0, 11/20}, Method →
        {"MethodOfLines", "SpatialDiscretization" → {"TensorProductGrid", "MinPoints" → 11 000}}]];
xw = NArgMax[Norm[wsol[0, t]], {t, 0.1, 0.5}];
xsol = Block[{ϵ = $MachineEpsilon},
    NDSolveValue[{i D[M[r, t], t] == -e^(-M[r,t]) (D[M[r, t], r, r] - (D[M[r, t], r])^2 + (2 D[M[r, t], r])/r),
        M[r, 0] == 3/4 e^(-r^2/2), M^(1,0)[ϵ, t] == 0, M[1000, t] == 0}, M, {r, ϵ, 4}, {t, 0, 11/20}, Method →
        {"MethodOfLines", "SpatialDiscretization" → {"TensorProductGrid", "MinPoints" → 12 000}}]];
xx = NArgMax[Norm[xsol[0, t]], {t, 0.1, 0.5}];
G3 = Plot[{4 Norm[usol[r, xu]], 2 Norm[vsol[r, xv]], 8/5 Norm[wsol[r, xw]], 4/3 Norm[xsol[r, xx]]},
    {r, 0, 3}, PlotRange → {{0, 3}, {-0.02, 1.42}}, PlotStyle → {{Black, Thickness → 0.005},
      {Red, Thickness → 0.005}, {Green, Thickness → 0.005}, {Blue, Thickness → 0.005}},
    Frame → {{True, False}, {True, False}}, FrameStyle → Directive[Black, Thickness → 0.002],
    FrameLabel → {Style["r", Italic], Style["ρ", Plain]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20],
    Epilog → Inset[LineLegend[{Directive[Blue, Thickness[0.005]], Directive[Green, Thickness[0.005]],
          Directive[Red, Thickness[0.005]], Directive[Black, Thickness[0.005]]}, {Style["0.750", 20,
            FontFamily → "Arial", Blue], Style["0.625", 20, FontFamily → "Arial", Green], Style[
            "0.500", 20, FontFamily → "Arial", Red], Style["0.250", 20, FontFamily → "Arial", Black]},
        LegendFunction → (Framed[#, RoundingRadius → 5, FrameStyle → GrayLevel[0.58]] &)],
      Scaled[{0.773, 0.667}]]];

M[x_, y_, z_] := -log(((rc - re e^Mc) Sqrt[x^2 + y^2 + z^2])/(e^Mc - 1) + rc re) + log((e^Mc (rc - re))/(e^Mc - 1)) + 1/2 log(x^2 + y^2 + z^2);

rc = 1/6000;
re = 30;
Mc = 3 + i;
Ω = ImplicitRegion[rc^2 ≤ x^2 + y^2 ≤ re^2, {x, y}];
G4 = DensityPlot[
    NIntegrate[Norm[M[x, y, z]], {z, -Sqrt[re^2 - x^2 - y^2], Sqrt[re^2 - x^2 - y^2]}, MaxRecursion → 15], {x, y} ∈ Ω,
    PlotRange → {{-30.07, 30.07}, {-30.07, 30.07}, {0, Sqrt[10]}}, ColorFunction → (Hue[0.65, #1] &),
    Frame → False, PlotPoints → 1000, Epilog → {Directive[Thickness[0.0014], Gray], Circle[{0, 0}, 30]}];
```

```
𝓜[r_] := −log( r (rc − re e^𝓜c) / (rc re (e^𝓜c − 1)) + 1 ) + log(r) + log( e^𝓜c (rc − re) / (rc (e^𝓜c − 1)) ) − log(re);
𝓜c = 3 + i;
rc = 1/6000;
re = 30;
A = 1/26300;
B = 22/5;
G5 = LogLogPlot[{Norm[𝓜[r]], A/((r/B) (1 + r/B)^2)}, {r, 1/6000, 3}, PlotRange → {{0, 3}, {0, 3}},
    PlotStyle → {Directive[Orange, Thickness[0.005]], Directive[Green, Dashed, Thickness[0.005]]},
    Frame → {{True, False}, {True, False}}, FrameLabel → {Style["r", Italic], " ρ"},
    FrameStyle → Directive[Black, Thickness → 0.0021],
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20],
    Epilog → Inset[LineLegend[{Directive[Orange, Thickness[0.004]], Directive[Green, Thickness[0.004]]},
        {Style["this study", FontFamily → "Arial", FontSize → 20],
         Style["NFW", FontFamily → "Arial", FontSize → 20]}, LegendFunction →
         (Framed[#, RoundingRadius → 4, FrameStyle → GrayLevel[0.58]] &)], Scaled[{0.73, 0.74}]]];
G6 = LogLogPlot[{4 Norm[usol[r, xu]], 2 Norm[vsol[r, xv]], 8/5 Norm[wsol[r, xw]], 4/3 Norm[xsol[r, xx]]},
    {r, 0, 4}, PlotRange → All, PlotStyle → {{Black, Thickness → 0.005},
      {Red, Thickness → 0.005}, {Green, Thickness → 0.005}, {Blue, Thickness → 0.005}},
    Frame → {{True, False}, {True, False}}, FrameStyle → Directive[Black, Thickness → 0.002],
    FrameLabel → {Style["r", Italic], Style["ρ", Plain]},
    LabelStyle → Directive[Black, FontFamily → "Arial", FontSize → 20],
    FrameTicks → {{{0.004, "", {0.007, 0}, Thickness → 0.0017}, {0.005, "", {0.007, 0}, Thickness → 0.0017},
        {0.006, "", {0.007, 0}, Thickness → 0.0017}, {0.007, "", {0.007, 0}, Thickness → 0.0017},
        {0.008, "", {0.007, 0}, Thickness → 0.0017}, {0.009, "", {0.007, 0}, Thickness → 0.0017},
        {0.01, "0.01", {0.01, 0}, Thickness → 0.0017}, {0.02, "", {0.007, 0}, Thickness → 0.0017},
        {0.03, "", {0.007, 0}, Thickness → 0.0017}, {0.04, "", {0.007, 0}, Thickness → 0.0017},
        {0.05, "", {0.007, 0}, Thickness → 0.0017}, {0.06, "", {0.007, 0}, Thickness → 0.0017},
        {0.07, "", {0.007, 0}, Thickness → 0.0017}, {0.08, "", {0.007, 0}, Thickness → 0.0017},
        {0.09, "", {0.007, 0}, Thickness → 0.0017}, {0.1, "0.1", {0.01, 0}, Thickness → 0.0017},
        {0.2, "", {0.007, 0}, Thickness → 0.0017}, {0.3, "", {0.007, 0}, Thickness → 0.0017},
        {0.4, "", {0.007, 0}, Thickness → 0.0017}, {0.5, "", {0.007, 0}, Thickness → 0.0017},
        {0.6, "", {0.007, 0}, Thickness → 0.0017}, {0.7, "", {0.007, 0}, Thickness → 0.0017},
        {0.8, "", {0.007, 0}, Thickness → 0.0017}, {0.9, "", {0.007, 0}, Thickness → 0.0017},
        {1, "1", {0.01, 0}, Thickness → 0.0017}, {2, "", {0.007, 0}, Thickness → 0.0017},
        {3, "", {0.007, 0}, Thickness → 0.0017}, {4, "", {0.007, 0}, Thickness → 0.0017}},
      {{0.0001, "", {0.007, 0}, Thickness → 0.0017}, {0.0002, "", {0.007, 0}, Thickness → 0.0017},
        {0.0003, "", {0.007, 0}, Thickness → 0.0017}, {0.0004, "", {0.007, 0}, Thickness → 0.0017},
        {0.0005, "", {0.007, 0}, Thickness → 0.0017}, {0.0006, "", {0.007, 0}, Thickness → 0.0017},
        {0.0007, "", {0.007, 0}, Thickness → 0.0017}, {0.0008, "", {0.007, 0}, Thickness → 0.0017},
        {0.0009, "", {0.007, 0}, Thickness → 0.0017}, {0.001, "0.001", {0.01, 0}, Thickness → 0.0017},
        {0.002, "", {0.007, 0}, Thickness → 0.0017}, {0.003, "", {0.007, 0}, Thickness → 0.0017},
        {0.004, "", {0.007, 0}, Thickness → 0.0017}, {0.005, "", {0.007, 0}, Thickness → 0.0017},
        {0.006, "", {0.007, 0}, Thickness → 0.0017}, {0.007, "", {0.007, 0}, Thickness → 0.0017},
        {0.008, "", {0.007, 0}, Thickness → 0.0017}, {0.009, "", {0.007, 0}, Thickness → 0.0017},
        {0.01, "0.01", {0.01, 0}, Thickness → 0.0017}, {0.02, "", {0.007, 0}, Thickness → 0.0017},
        {0.03, "", {0.007, 0}, Thickness → 0.0017}, {0.04, "", {0.007, 0}, Thickness → 0.0017},
        {0.05, "", {0.007, 0}, Thickness → 0.0017}, {0.06, "", {0.007, 0}, Thickness → 0.0017},
```

```
{0.07, "", {0.007, 0}, Thickness → 0.0017}, {0.08, "", {0.007, 0}, Thickness → 0.0017},
{0.09, "", {0.007, 0}, Thickness → 0.0017}, {0.1, "0.1", {0.01, 0}, Thickness → 0.0017},
{0.2, "", {0.007, 0}, Thickness → 0.0017}, {0.3, "", {0.007, 0}, Thickness → 0.0017},
{0.4, "", {0.007, 0}, Thickness → 0.0017}, {0.5, "", {0.007, 0}, Thickness → 0.0017},
{0.6, "", {0.007, 0}, Thickness → 0.0017}, {0.7, "", {0.007, 0}, Thickness → 0.0017},
{0.8, "", {0.007, 0}, Thickness → 0.0017}, {0.9, "", {0.007, 0}, Thickness → 0.0017},
{1, "1", {0.01, 0}, Thickness → 0.0017}, {2, "", {0.007, 0}, Thickness → 0.0017}}}];
ff = GraphicsGrid[{{G1, G2}, {G3, G4}, {G5, G6}}, Spacings → {10, 10}, ImageSize → 800,
   Epilog → {Text[Style["a", 21, FontFamily → "Arial", Black, Bold], Scaled[{−0.41, 1.2}]],
     Text[Style["b", 21, FontFamily → "Arial", Black, Bold], Scaled[{0.6, 1.2}]],
     Text[Style["c", 21, FontFamily → "Arial", Black, Bold], Scaled[{−0.41, 0.72}]],
     Text[Style["d", 21, FontFamily → "Arial", Black, Bold], Scaled[{0.6, 0.72}]],
     Text[Style["e", 21, FontFamily → "Arial", Black, Bold], Scaled[{−0.41, 0.212}]],
     Text[Style["f", 21, FontFamily → "Arial", Black, Bold], Scaled[{0.6, 0.212}]]}];
Export[MyDirection <> "figure6.png", ff, Background → None];
```

Out[∘]= 0.00362328

```
## ## ## ## ## ## ## ## ## ## ## Figure6 ## ## ## ## ## ## ## ## ## ## ## ## ## ## ##
```