Rigorous proof for Riemann Hypothesis obtained by adopting Algebra-Geometry Approach in Geometric Langlands Program

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Abstract

The 1859 Riemann hypothesis conjectured all nontrivial zeros in Riemann zeta function are uniquely located on sigma = 1/2 critical line. Derived from Dirichlet eta function [proxy for Riemann zeta function] are, in chronological order, simplified Dirichlet eta function and Dirichlet Sigma-Power Law. Computed Zeroes from the former uniquely occur at sigma = 1/2 resulting in total summation of fractional exponent (–sigma) that is twice present in this function to be integer –1. Computed Pseudo-zeroes from the later uniquely occur at sigma = 1/2 resulting in total summation of fractional exponent (1 – sigma) that is twice present in this law to be integer 1. All nontrivial zeros are, respectively, obtained directly and indirectly as the one specific type of Zeroes and Pseudo-zeroes only when sigma = 1/2. Thus, it is proved (using equation-type proof) that Riemann hypothesis is true whereby this function and law rigidly comply with Principle of Maximum Density for Integer Number Solutions. The geometrical-mathematical [unified] approach used in our proof is equivalent to the algebra-geometry [unified] approach of geometric Langlands program that was formalized by Professor Peter Scholze and Professor Laurent Fargues. A succinct treatise on proofs for Polignac's and Twin prime conjectures (using algorithm-type proofs) is also outlined in this anniversary research paper.

Keywords: Algorithm-type proof, Coherent sheaf, Dirichlet Sigma-Power Law, Etale sheaf, Equation-type proof, Fargues-Fontaine curve, Geometric Langlands program, Gram's Law, Polignac's and Twin prime conjectures, Pseudo-zeroes, Riemann hypothesis, Rosser Rule, Zeroes

Mathematics Subject Classification (2010): Primary 00A05, Secondary 11M26 & 11A41

Contents

1. Introduction

Preliminary notes: *This research paper could be perceived by critics to contain largely incomprehensible text that purports to outline the rigorous proof for Riemann hypothesis with succinct treatise on proofs for Polignac's and Twin prime conjectures that seems to only occupy about half a page per each. The remaining parts of this paper seem to contain obscure materials such as semi-philosophical passages peppered with references to Langlands program and other grand schemes, and power series manipulations with weird pseudo-philosophical consideration. However, it is precisely these semi-philosophical and pseudo-philosophical aspects that are crucial to help readers fully understand the proofs.*

Riemann hypothesis is an intractable open problem in Number theory that was proposed in 1859 by famous German mathematician Bernhard Riemann (September 17, 1826 - July 20, 1866). This hypothesis conjectured all nontrivial zeros in Riemann zeta function are uniquely located on $\sigma = \frac{1}{2}$ critical line. By applying Euler formula to Dirichlet eta function [*proxy* for Riemann zeta function], we obtain simplified Dirichlet eta function whereby its computed Zeroes uniquely occur at $\sigma = \frac{1}{2}$ resulting in total summation of fractional exponent ($-\sigma$) that is twice present in this function to be integer -1. Dirichlet Sigma-Power Law is the solution from performing integration on simplified Dirichlet eta function whereby its computed Pseudo-zeroes uniquely occur at $\sigma = \frac{1}{2}$ resulting in total summation of fractional exponent ($1 - \sigma$) that is twice present in this law to be integer 1.

All nontrivial zeros are, respectively, obtained directly and indirectly as one specific type [out of three different types] of Zeroes and Pseudo-zeroes only when $\sigma = \frac{1}{2}$. Then [non-existent] virtual nontrivial zeros and [non-existent] virtual Pseudo-nontrivial zeros cannot be obtained directly and indirectly as a type of virtual Zeroes and virtual Pseudo-zeroes when $\sigma \neq \frac{1}{2}$. As per Lemma 1 on these three different types of entities, all (virtual) Pseudo-zeroes can be precisely converted to (virtual) Zeroes.

From fully solving Theorem 1, Corollary 2 and Theorem 3 (that contains Proposition 1 and Proposition 2); we confirm the following *sine qua non* statement to be true: "Valid only at unique $\sigma = \frac{1}{2}$ critical line, **geometrical** Origin intercept points in Figure 2 are precisely equivalent to **mathematical** nontrivial zeros in Eq. (1) [directly] as Zeroes and Eq. (3) [indirectly] as Pseudo-zeroes when expressed with using trigonometric identities, and in Eq. (9) [directly] as Zeroes and Eq. (10) [indirectly] as Pseudo-zeroes when expressed without using trigonometric identities". Thus, it is proved that Riemann hypothesis is true whereby this function and law rigidly obey Principle of Maximum Density for Integer Number Solutions. They additionally manifest Principle of Equidistant for Multiplicative Inverse and are [serendipitously] amendable to treatment with trigonometric identities.

Together with number theory, geometry and analysis; algebra is one of the broad areas of mathematics. In its most general form forming the unifying thread of all mathematics, algebra is the study of mathematical symbols and rules for manipulating these symbols. Geometry is concerned with properties of space that are related with distance, shape, size, and relative position of figures. We arbitrarily use the term 'mathematical' instead of 'algebra', and explain in subsection 1.2 the unified geometrical-mathematical approach used in our proof of Riemann hypothesis [that essentially unites mathematics and geometry] is essentially equivalent to algebra-geometry approach used by geometric Langlands program [that essentially unites algebra and geometry]. We provide an assortment of information on various important topics although these need not form an essential part of our proof for Riemann hypothesis: brief synopsis regarding Gram's Law and Rosser Rule for Gram points in Appendix A, and Miscellaneous Materials such as on cardinality, certain types of infinite series, Zeroes and Pseudo-zeroes in Appendix B. A succinct treatise on rigorous proofs for Polignac's and Twin prime conjectures is also outlined in the Conclusions section.

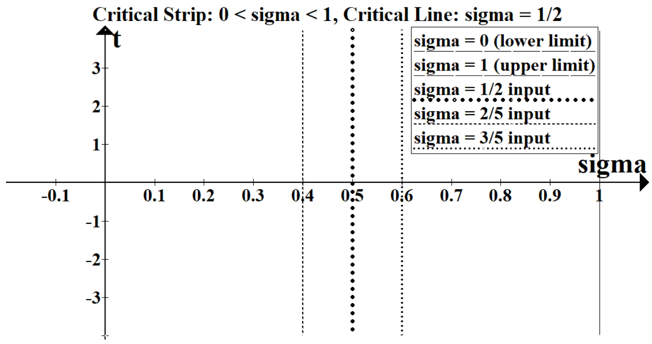

Figure 1. INPUT for $\sigma = \frac{1}{2}$, $\frac{2}{5}$, and $\frac{3}{5}$. $\zeta(s)$ has countable infinite set of Completely Predictable trivial zeros located at σ = all negative even numbers and [conjectured] countable infinite set of Incompletely Predictable nontrivial zeros located at $\sigma = \frac{1}{2}$ given by various t values.

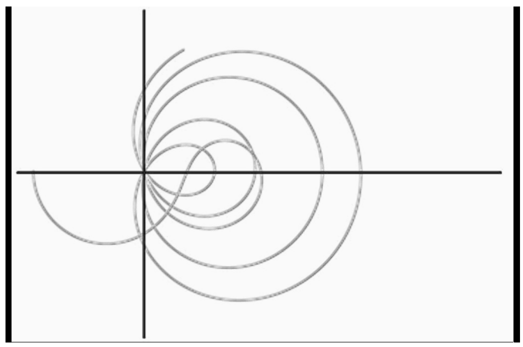

Figure 2. OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Schematically depicted polar graph of $\zeta(\frac{1}{2} + \imath t)$ plotted along critical line for real values of t running from 0 to 34, horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Total presence of all Origin intercept points.

1.1 General notations and Figures 1, 2, 3 and 4

The following is a short list of abbreviations used by this paper.
CFS: countable finite set
CIS: countable infinite set
UIS: uncountable infinite set
CP: Completely Predictable – see section 3 on CP entities
IP: Incompletely Predictable – see section 3 on IP entities
DA: Dimensional analysis – see section 4 on exact and inexact DA homogeneity
NTZ: nontrivial zeros (Gram[x=0,y=0] points) = Origin intercept points when $\sigma = \frac{1}{2}$
ζ(s): $f(n)$ Riemann zeta function containing variable n, and parameters t and σ
η(s): $f(n)$ Dirichlet eta function containing variable n, and parameters t and σ
sim-η(s): $f(n)$ simplified Dirichlet eta function containing variable n, and parameters t and σ
DSPL: $F(n)$ Dirichlet Sigma-Power Law=$\int sim - \eta(s)dn$ containing variable n, and parameters t and σ

1.2 Equivalence of our unified geometrical-mathematical approach and the approach of geometric Langlands program including p-adic Riemann zeta function $\zeta_p(s)$

An L-function consists of a Dirichlet series with a functional equation and an Euler product. Examples of L-functions come from modular forms, elliptic curves, number fields, and Dirichlet characters, as well as more generally from automorphic forms, algebraic varieties, and Artin representations. They form an integrated component of 'L-functions and Modular Forms Database' (LMFDB, located at URL https://www.lmfdb.org/) with far-reaching implications. In proper perspective, $\zeta(s)$ is then the simplest example of an L-function.

The unified geometrical-mathematical approach used in our proof on Riemann hypothesis that specifically involve only the [isolated] $\zeta(s)$ as one type of L-function must then be equivalent to the unified algebra-geometry approach of geometric Langlands program that generally involve all types of L-functions. Named after German mathematician Adolf Hurwitz

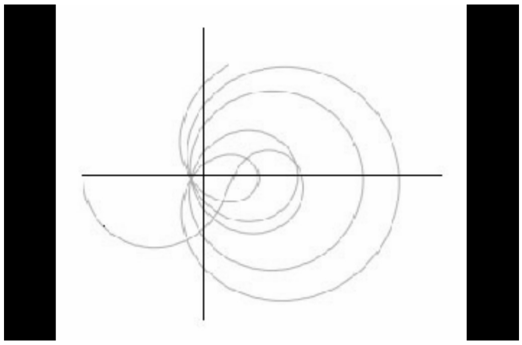

Figure 3. OUTPUT for $\sigma = \frac{2}{5}$ as virtual Gram points. Varying Loops are shifted to left of Origin with horizontal axis: $Re\{\zeta(\frac{2}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{2}{5} + \imath t)\}$. Total absence of Origin intercept points.

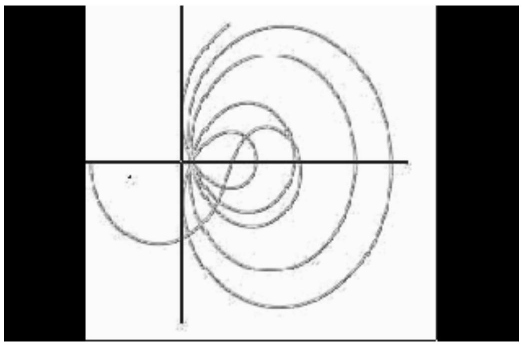

Figure 4. OUTPUT for $\sigma = \frac{3}{5}$ as virtual Gram points with horizontal axis: $Re\{\zeta(\frac{3}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{3}{5} + \imath t)\}$. Varying Loops are shifted to right of Origin. Total absence of Origin intercept points.

(March 26, 1859 - November 18, 1919), Hurwitz zeta function is one of the many zeta functions. It is formally defined for complex arguments s with Re(s) > 1 and q with Re(q) > 0 by $\zeta(s, q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}$. This series is absolutely convergent for given values of s and q, and can be extended to a meromorphic function defined for all s ≠ 1. With this scheme, our Riemann zeta function $\zeta(s)$ is equivalently given as $\zeta(s, 1)$.

Using the innovative research method of p-adic analysis popularized by renowned German mathematician Professor Peter Scholze who won the 2018 Fields Medal; a p-adic zeta function, or more generally a p-adic L-function, is a function analogous to Riemann zeta function, or more general L-functions, but whose domain and target are p-adic (where p is a prime number). In p-adic Riemann zeta function ζ_p(s), values at negative odd integers are those of Riemann zeta function ζ(s) at negative odd integers (up to an explicit correction factor). The p-adic L-functions arising in this fashion as sourced from p-adic interpolation (Koblitz, 1984) of special values of L-functions are typically referred to as analytic p-adic L-functions. The other major source of p-adic L-functions is from the arithmetic of cyclotomic fields, or more generally, certain Galois modules over towers of cyclotomic fields or even more general towers.

There is no clear delineation between algebra and analysis: The involved mathematics is considered more "algebraic" if it focuses more on structure and interaction of operations that underlie the objects of study e.g. groups, rings, fields, etc. The involved mathematics is considered more "analytic" if it focuses more on real numbers and measurable quantities, and the approximation and computation thereof e.g. calculus, Taylor series, derivatives, integrals, etc. Galois groups arise in the branch of mathematics called algebra (reflecting the way we use algebra to solve equations), and automorphic forms arise in the different branch of mathematics called analysis (which can be considered as an enhanced form of calculus).

Formulated by renowned Canadian mathematician Robert Langlands in late 1960s, Langlands correspondence classically refers to collection of results and conjectures relating number theory and representation theory. Langlands conjecture for rational numbers is further referred to as "global" Langlands correspondence [since rational number system contain all prime numbers], and for p-adics as "local" Langlands correspondence [since p-adic number systems deal with one prime number at a time]. The coined *geometric Langlands program* is a reformulation of Langlands correspondence obtained by replacing number fields appearing in original number theoretic version by function fields and applying techniques from algebraic geometry, thus relating algebraic geometry to representation theory. The aim is to find geometric objects with properties that could stand in for Galois groups and automorphic forms in Langlands' conjectures. The perfectoid spaces are adic spaces of special kind occurring in the study of problems of "mixed characteristic" such as local fields of characteristic zero which have residue fields of characteristic prime p. Based on p-adic geometry, Professor Scholze's 2012 PhD thesis on perfectoid spaces (Scholze, 2012) yields solution to a special case of weight-monodromy conjecture.

Named after renowned French mathematicians Professor Laurent Fargues and Professor Jean-Marc Fontaine (March 13, 1944 - January 29, 2019), Fargues-Fontaine curve as a geometric object is a curve whose points each represented a version of an important object called a p-adic ring. Professor Fargues and Professor Scholze subsequently came up with two different kinds of more complicated geometric objects called sheaves: coherent sheaves correspond to representations of p-adic groups, and étale sheaves to representations of Galois groups. In their paper (Fargues & Scholze, 2021) with Fargues-Fontaine curve now merging with Scholze's p-adic geometry, they develop the foundations of geometric Langlands program whereby it is proved that there is always a way to match a coherent sheaf with an étale sheaf, and as a result there is always a way to match a representation of a p-adic group with a representation of a Galois group. In this ground-breaking way of studying "local" Langlands correspondence based on these geometric objects called sheaves, they finally proved the one direction of translation for this correspondence although the other direction of translation remains an open question. This is the basic premise of Langlands program which is a broad vision for investigating Galois groups – essentially polynomials – through these types of translations.

Finally, since the infinitely many prime numbers ≥ 2 are a subset of the infinitely many integers ≥ 1; we can derive the following alternative equally valid deductions involving (positive) prime number system instead of the valid deductions as outlined in Proposition 1 involving (positive) [non-prime number] integer number system: "Only at $\sigma = \frac{1}{2}$ critical line which involves applying $f(n)$ as fractional exponent $\frac{1}{2}$ or square root on n = all perfect squares of prime numbers 4, 9, 25, 49, 121, 169, 289, 361, 529, 841... will we obtain maximum number of rational roots as consecutive prime number solutions 2, 3, 5, 7, 11, 13, 17, 19, 23, 29... (viz, all prime numbers ≥ 2). This observation uniquely comply with **Principle of Maximum Density for Prime Number Solutions** at $\sigma = \frac{1}{2}$ critical line." We immediately recognize from above commentaries that using this **Principle of Maximum Density for Prime Number Solutions** instead of **Principle of Maximum Density for Integer Number Solutions** from Proposition 1 will also crucially confer the proof for Theorem 3 to be fully complete.

2. Sketch of the Proof for Riemann hypothesis including the Modified Equations for simplified Dirichlet eta function and Dirichlet Sigma-Power Law that are expressed using trigonometric identities

Symbolically named after German mathematician Gustav Lejeune Dirichlet (February 13, 1805 - May 5, 1859), the word "Law" in DSPL represent a convenient terminology to describe this function – viz, there is resemblance to Zipf's law via power law functions in σ from s = $\sigma + it$ being exponent of a power function as similar format to n^{σ}, logarithm scale use, and ζ(s) harmonic series connection. Respectively, we use Zeroes (as three types of Gram points) and Pseudo-zeroes (as three types of Pseudo-Gram points) at $\sigma = \frac{1}{2}$ to collectively refer to corresponding $f(n)$'s and $F(n)$'s x-axis intercept points, y-axis intercept points and Origin intercept points. Respectively, we use virtual Zeroes (as two types of virtual Gram points) and virtual Pseudo-zeroes (as two types of virtual Pseudo-Gram points) at $\sigma \neq \frac{1}{2}$ to collectively refer to corresponding $f(n)$'s and $F(n)$'s x-axis intercept points and y-axis intercept points [with absent Origin intercept points].

Geometrical and mathematical definitions for Gram points and virtual Gram points. Figure 1 depicts complex variable s (= $\sigma \pm \imath t$) as INPUT with x-axis denoting real part Re{s} associated with σ, and y-axis denoting imaginary part Im{s} associated with t. The critical line: $\sigma = \frac{1}{2}$; non-critical lines: $\sigma \neq \frac{1}{2}$ viz, $0 < \sigma < \frac{1}{2}$ and $\frac{1}{2} < \sigma < 1$; and critical strip: $0 < \sigma < 1$. Both the unique $\sigma = \frac{1}{2}$ value and the non-unique $\sigma \neq \frac{1}{2}$ values ∈ Set **all σ values** whereby Set **all σ values** = $\sigma \mid \sigma$ is a real number, and $0 < \sigma < 1$. With including its complex conjugate, s = $\sigma \pm it$ is present in our chosen f(n) and F(n) whereby these are well-defined continuous [complex] functions that are always defined for any arbitrarily chosen intervals [a,b]. With f(n) = 0 and F(n) = 0 giving rise to relevant derived equations that are *dependently*-related [via Varying Loops], they generate corresponding types of IP entities. These IP entities will inherently belong to the correctly assigned *mutually exclusive CIS of Gram points and virtual Gram points* constituted by t values as transcendental numbers except for first Gram[y=0] point (and first virtual Gram[y=0] point) given by t = 0. Origin intercept points, x-axis intercept points and y-axis intercept points are geometrical definitions for IP entities of Gram[x=0,y=0] points, Gram[y=0] points and Gram[x=0] points at $\sigma = \frac{1}{2}$. These geometrical definitions are equivalent to mathematical definitions as given by the equations below in this section.

Origin intercept points at $\sigma = \frac{1}{2}$ consisting of Gram[x=0,y=0] points or NTZ are computed directly from equations η(s) = 0 and sim-η(s) = 0; and indirectly from equation DSPL = 0. x-axis intercept points at $\sigma = \frac{1}{2}$ consisting of Gram[y=0] points or (traditional) 'usual' Gram points are computed directly from equation Gram[y=0] points-sim-η(s) = 0; and indirectly from equation Gram[y=0] points-DSPL = 0. y-axis intercept points at $\sigma = \frac{1}{2}$ consisting of Gram[x=0] points are computed directly from equation Gram[x=0] points-sim-η(s) = 0; and indirectly from equation Gram[x=0] points-DSPL = 0.

Relevant functions and equations are unique mathematical objects usefully classified as three types of infinite series: *Harmonic series*, *Alternating harmonic series* or *Alternating series with trigonometric terms*. We perform crucial *de novo* analysis on these functions and equations by noting their manifested intrinsic properties. Without loss of validity in our correct and complete set of mathematical arguments, we adopt the convention of providing focused analysis predominantly on appropriately chosen Alternating series with trigonometric terms throughout our presentation. The complex f(n) ζ(s) is a Harmonic series that does not converge in critical strip. The complex f(n) η(s) is an Alternating harmonic series that converge in critical strip. Through analytic continuation, η(s) must act as *proxy* function for ζ(s) in this strip. [Caveat: the limit of an analytic continuation is not the analytic continuation of the limit.] Derived as Euler formula application to η(s) is the complex f(n) sim-η(s), and derived as $\int sim - \eta(s) dn$ is the complex F(n) DSPL. Both sim-η(s) and DSPL are Alternating series with trigonometric terms that converge in critical strip.

The f(n) η(s) will converge infinitely often to a zero value as η(s) = 0 equation giving rise to all NTZ or Gram[x=0,y=0] points. This event will only happen when η(s) is substituted with one unique σ value which is conjectured to be $\sigma = \frac{1}{2}$ by Riemann hypothesis. Being an Alternating harmonic series [without trigonometric terms that graphically cater for all possible types of x-axis and y-axis intercept points], we inherently cannot derive valid functions to obtain corresponding equations Gram[y=0] points-η(s) = 0 and Gram[x=0] points-η(s) = 0 that will enable mathematical computations of Gram[y=0] points as x-axis intercept points and Gram[x=0] points as y-axis intercept points. Then, computed Zeroes are mathematically defined as η(s) = 0 and sim-η(s) = 0 when parameter $\sigma = \frac{1}{2}$; computed virtual Zeroes are mathematically defined as $\eta(s) \neq 0$ and sim-η(s) = 0 when parameter $\sigma \neq \frac{1}{2}$; computed Pseudo-Zeroes are mathematically defined as DSPL = 0 when parameter $\sigma = \frac{1}{2}$; and computed virtual Pseudo-zeroes are mathematically defined as DSPL = 0 when parameter $\sigma \neq \frac{1}{2}$.

For $0 \leq \delta \leq 1$, let $f(n) = sin(n) \pm \delta$ and $f(n) = cos(n) \pm \delta$ represent two [simple] trigonometric functions which are periodic transcendental-type functions. Both $sin(n) \pm \delta = 0$ and $cos(n) \pm \delta = 0$ as equations will generate infinitely many CP x-axis intercept points (Zeroes) for any given values of δ. This will additionally include the solitary Origin intercept point (Zero) obtained from $sin(n) \pm \delta = 0$ when $\delta = 0$. For both $sin(n) \pm \delta$ and $cos(n) \pm \delta$, only when $\delta = 0$ will their progressive / cummulative *Areas Above the horizontal axis* be overall identical to *Areas Below the horizontal axis*. Otherwise, these mentioned Areas will not be overall identical to each other when $\delta \neq 0$. We now provide analogical reasoning for existence of infinitely many substituted σ values (including $\sigma = \frac{1}{2}$) that will all contribute to two conditions sim-η(s) = 0 and DSPL = 0 being satisfied while simultaneously giving rise to (i) IP Zeroes and IP Pseudo-zeroes [when $\sigma = \frac{1}{2}$], and (ii) IP virtual Zeroes and IP virtual Pseudo-zeroes [when $\sigma \neq \frac{1}{2}$]. With (complex) sine and/or cosine terms present in f(n) sim-η(s) and F(n) DSPL also being periodic transcendental-type functions, we intuitively deduce $\sigma = \frac{1}{2}$ and $\sigma \neq \frac{1}{2}$ must respectively act as the analogical equivalence of $\delta = 0$ and $\delta \neq 0$. This deduction allows intuitive and valid explanations for our two conditions to be satisfied by the infinitely many substituted σ values. Consequently, we must rigorously prove additional property of sim-η(s) and DSPL that they will characteristically, inevitably and uniquely comply with Principle of Maximum Density for Integer Number Solutions only when $\sigma = \frac{1}{2}$ with this Principle signifying complete presence of NTZ in sim-η(s) or Pseudo-NTZ in DSPL as one unique type of Gram points or Pseudo-Gram points [which are otherwise totally absent when $\sigma \neq \frac{1}{2}$].

Figures 2, 3 and 4 are $\zeta(\sigma + \imath t)$ Polar Graphs [see Remark 10 on intimate relationship between Cartesian Coordinates and Polar Coordinates] with x-axis denoting real part $\mathrm{Re}\{\zeta(s)\}$ and y-axis denoting imaginary part $\mathrm{Im}\{\zeta(s)\}$ generated by $\zeta(s)$'s output as real values of t running from 0 to 34. There are infinite types-of-spirals (Varying Loops) possibilities associated with each σ value arising from all infinite σ values in $0 < \sigma < 1$ critical strip whereby the unique and solitary $\sigma = \frac{1}{2}$ value that denote critical line is located in this strip. We observe that Figure 3 [with $\sigma = \frac{2}{5}$] and Figure 4 [with $\sigma = \frac{3}{5}$] show associated shifts of Varying Loops that manifest Principle of Equidistant for Multiplicative Inverse – see Proposition 2 from section 7. From observing Figure 2, we can geometrically define NTZ (or Gram[x=0,y=0] points) as Origin intercept points occurring when $\sigma = \frac{1}{2}$. Then, two remaining types of Gram points as part of continuous Varying Loops are consequently defined as x-axis intercept points and y-axis intercept points occurring when $\sigma = \frac{1}{2}$.

Lemma 2 confirms the paired IP two types of Gram points [as Zeroes] situation, paired IP two types of virtual Gram points [as virtual Zeroes] situation, paired IP two types of Pseudo-Gram points [as Pseudo-zeroes] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] situation are always $\frac{1}{2}\pi$ out-of-phase with each other in every one of these situations. Lemma 1 confirms IP Zeroes, IP virtual Zeroes, IP Pseudo-zeroes and IP virtual Pseudo-zeroes are precisely related as $\frac{1}{2}\pi$ (for NTZ case) or $\frac{3}{4}\pi$ (for Gram[y=0] points and Gram[x=0] points cases) out-of-phase with each other. Thus from Lemma 1, corresponding three types of F(n)'s Pseudo-zeroes or Pseudo-Gram points and two types of F(n)'s virtual Pseudo-zeroes or virtual Pseudo-Gram points can be precisely converted to three types of f(n)'s Zeroes or Gram points and two types of f(n)'s virtual Zeroes or virtual Gram points. Then, Statement (I) – (IV) are valid whereby $\sigma = \frac{1}{2}$'s derived entities from Statement (III) can be precisely converted to those from Statement (I), and $\sigma \neq \frac{1}{2}$'s derived virtual entities from Statement (IV) can be precisely converted to those from Statement (II):

Statement (I) The f(n)'s Zeroes at $\sigma = \frac{1}{2}$ [directly] equates to three types of Gram points.

Statement (II) The f(n)'s virtual Zeroes at $\sigma \neq \frac{1}{2}$ [directly] equates to two types of virtual Gram points.

Statement (III) The F(n)'s Pseudo-zeroes at $\sigma = \frac{1}{2}$ [indirectly] equates to three types of Gram points.

Statement (IV) The F(n)'s virtual Pseudo-zeroes at $\sigma \neq \frac{1}{2}$ [indirectly] equates to two types of virtual Gram points.

Remark 1. Of particular relevance to Riemann hypothesis, we mathematically deduce from above materials that f(n)'s NTZ or Gram[x=0,y=0] points as one type of Gram points will conjecturally only exist at unique $\sigma = \frac{1}{2}$ critical line [but not at non-unique $\sigma \neq \frac{1}{2}$ non-critical lines]. This can be equivalently stated as: F(n)'s Pseudo-NTZ or Pseudo-Gram[x=0,y=0] points as one type of Pseudo-Gram points will conjecturally only exist at unique $\sigma = \frac{1}{2}$ critical line [but not at non-unique $\sigma \neq \frac{1}{2}$ non-critical lines].

Useful analogy for Remark 2: A line consists of infinitely many points. Graphically, the Origin is a zero-dimensional [single] point; x-axis or horizontal axis and y-axis or vertical axis are one-dimensional lines [containing infinitely many points].

Remark 2. In Figure 3 and Figure 4, we note Origin intercept points as Gram[x=0,y=0] points or NTZ cannot exist when $\sigma \neq \frac{1}{2}$. In Figure 2, we note Origin intercept points as Gram[x=0,y=0] points or NTZ only exist when $\sigma = \frac{1}{2}$. Of particular relevance to Riemann hypothesis, we deduce sim-η(s) as periodic transcendental-type function only contain one solitary σ-valued type of Origin intercept points (when $\sigma = \frac{1}{2}$ for Gram[x=0,y=0] points or NTZ as conjectured by Riemann hypothesis) but infinitely many different σ-valued types of x-axis intercept points and y-axis intercept points (constituted by solitary $\sigma = \frac{1}{2}$ value for Gram[y=0] points and Gram[x=0] points as well as infinitely many $\sigma \neq \frac{1}{2}$ values for virtual Gram[y=0] points and virtual Gram[x=0] points). We can conjure up an equivalent statement for DSPL as periodic transcendental-type function whereby we replace NTZ and (virtual) Gram points with their counterparts Pseudo-NTZ and (virtual) Pseudo-Gram points.

We can now propose Theorem 1 (with $\sigma = \frac{1}{2}$ connoting exact DA homogeneity) and Corollary 2 (with $\sigma \neq \frac{1}{2}$ connoting inexact DA homogeneity) to fully represent Remark 1 and Remark 2. Their successful proofs will firstly, denote rigorous proof for Riemann hypothesis that involves conjecture on location of NTZ as one type of Gram points [viz, Origin intercept points] and secondly, provide precise explanations for remaining two types of Gram points [viz, x-axis intercept points and y-axis intercept points]. In addition, we incorporate Theorem 3 on rigid compliance by sim-η(s) and DSPL with Principle of Maximum Density for Integer Number Solutions whereby its successful proof will only eventuate when $\sigma = \frac{1}{2}$.

Theorem 1. Rigidly complying with exact DA homogeneity, f(n) sim-η(s) and F(n) DSPL as relevant equations can incorporate three types of Gram points and Pseudo-Gram points onto solitary $\sigma = \frac{1}{2}$ critical line thus fully supporting Riemann hypothesis to be true.

Proof. Using f(n) sim-η(s) and F(n) DSPL, Riemann hypothesis propose all NTZ are located on $\sigma = \frac{1}{2}$ critical line in these functions. The three types of Gram points and Pseudo-Gram points are each infinite in magnitude consisting of mutually exclusive entities. Amounting to direct Proof by Positive, we show CIS of Gram[x=0,y=0] points or NTZ constitutes one type of Gram points only when $\sigma = \frac{1}{2}$ thus fully supporting Riemann hypothesis to be true. The preceding sentence is equally valid when we replace Gram[x=0,y=0] points, NTZ and Gram points with corresponding Pseudo-Gram[x=0,y=0] points, Pseudo-NTZ and Pseudo-Gram points. Respectively, the conveniently defined term of exact DA homogeneity denote [exact] integer -1 and 1 derived from $\sum$(all fractional exponents) = $2(-\sigma)$ and $2(1-\sigma)$. These act as surrogate markers in sim-η(s) and DSPL on [solitary] $\sigma = \frac{1}{2}$ situation. Generated by relevant functions and laws when $\sigma = \frac{1}{2}$, the three types of Gram points are mathematically defined as equations sim-η(s) = 0, Gram[y=0] points-sim-η(s) = 0 and Gram[x=0] points-sim-η(s) = 0; and the three types of Pseudo-Gram points are mathematically defined as equations DSPL = 0, Gram[y=0] points-DSPL = 0 and Gram[x=0] points-DSPL = 0. They all correspond to relevant geometrically defined Origin intercept points, x-axis intercept points and y-axis intercept points. Thus, three types of IP Gram points

[IP Zeroes] and IP Pseudo-Gram points [IP Pseudo-Zeroes] are mathematically and geometrically defined to be located on $\sigma = \frac{1}{2}$ critical line. Based solely on these definitive definitions, we can uniquely incorporate three types of IP Gram points [IP Zeroes] and IP Pseudo-Gram points [IP Pseudo-zeroes] onto $\sigma = \frac{1}{2}$ critical line. *The proof is now complete for Theorem 1*□.

Corollary 2. Rigidly complying with inexact DA homogeneity, f(n) sim-η(s) and F(n) DSPL as relevant equations can incorporate two types of virtual Gram points and virtual Pseudo-Gram points onto infinitely many $\sigma \neq \frac{1}{2}$ non-critical lines thus also fully supporting Riemann hypothesis to be true.

Proof. Using f(n) sim-η(s) and F(n) DSPL, Riemann hypothesis equivalently propose all NTZ are not located on $\sigma \neq \frac{1}{2}$ non-critical lines in these functions. The two types of virtual Gram points and virtual Pseudo-Gram points are each infinite in magnitude consisting of mutually exclusive entities. Amounting to indirect Proof by Contrapositive, we show [non-existent] virtual Gram[x=0,y=0] points or virtual NTZ will not constitute one type of [non-existent] virtual Gram points when $\sigma \neq \frac{1}{2}$ thus also fully supporting Riemann hypothesis to be true. The preceding sentence is equally valid when we replace virtual Gram[x=0,y=0] points, virtual NTZ and virtual Gram points with corresponding virtual Pseudo-Gram[x=0,y=0] points, virtual Pseudo-NTZ and virtual Pseudo-Gram points. Respectively, conveniently defined term of inexact DA homogeneity denote [inexact] fractional (non-integer) number $\neq -1$ and $\neq 1$ derived from $\sum$(all fractional exponents) = $2(-\sigma)$ and $2(1-\sigma)$. These act as surrogate markers in sim-η(s) and DSPL on [infinitely many] $\sigma \neq \frac{1}{2}$ situations. Generated by relevant functions and laws when $\sigma \neq \frac{1}{2}$, the two types of virtual Gram points are mathematically defined as equations virtual Gram[y=0] points-sim-η(s) = 0 and virtual Gram[x=0] points-sim-η(s) = 0; and the two types of virtual Pseudo-Gram points are mathematically defined as equations virtual Gram[y=0] points-DSPL = 0 and virtual Gram[x=0] points-DSPL = 0. They all correspond to relevant geometrically defined x-axis intercept points and y-axis intercept points. Thus, two types of IP virtual Gram points [IP virtual Zeroes] and IP virtual Pseudo-Gram points [IP virtual Pseudo-Zeroes] are mathematically and geometrically defined to be located on $\sigma \neq \frac{1}{2}$ non-critical lines. Based solely on these definitive definitions, we can uniquely incorporate two types of IP virtual Gram points [IP virtual Zeroes] and IP virtual Pseudo-Gram points [IP virtual Pseudo-zeroes] onto $\sigma \neq \frac{1}{2}$ non-critical lines. *The proof is now complete for Corollary 2*□.

Theorem 3. Conforming to the solitary $\sigma = \frac{1}{2}$ critical line [and not the infinitely many $\sigma \neq \frac{1}{2}$ non-critical lines e.g. $\sigma = \frac{1}{3}$ or $\frac{2}{3}$] whereby σ forms part of relevant fractional exponents from base quantities (2n) and (2n-1) in sim-η(s) [as Riemann sum $\Delta n \longrightarrow 1$ with variable n involving all integers ≥ 1] or DSPL [as definite integral $\Delta n \longrightarrow 0$ with variable n involving all real numbers ≥ 1]; square roots of perfect squares [and not e.g. cube roots of perfect cubes or squared cube roots of perfect cubes] when applied to combined base quantities (2n) and (2n-1) in sim-η(s) or DSPL will generate the maximum number of integer solutions (constituted by all integers ≥ 1) that uniquely comply with Principle of Maximum Density for Integer Number Solutions while also manifesting Principle of Equidistant for Multiplicative Inverse.

Proof. $\int$sim-η(s)dn = DSPL. Whereas the two subsets of rational roots as integers and irrational roots as irrational numbers can be generated by combined base quantities (2n) and (2n-1) from sim-η(s) [as Riemann sum $\Delta n \longrightarrow 1$ with variable n involving all integers ≥ 1], so must these two exact same subsets be generated by combined base quantities (2n) and (2n-1) from DSPL [as definite integral $\Delta n \longrightarrow 0$ with variable n involving all real numbers ≥ 1]. Thus in sim-η(s) or DSPL, its computed CIS rational roots (subset) as integers [rational numbers] + computed CIS irrational roots (subset) as irrational numbers = computed CIS total roots. These two mutually exclusive subsets belong to UIS real numbers. Using subset rational roots as integers at $\sigma = \frac{1}{2}$ critical line, and by comparing and contrasting this subset with [different] subset rational roots as integers at $\sigma = \frac{1}{3}$ or $\frac{2}{3}$ non-critical lines corollary situation; we will show that square roots of perfect squares [and not e.g. cube roots of perfect cubes or squared cube roots of perfect cubes] when applied to combined base quantities (2n) and (2n-1) from sim-η(s) or DSPL giving rise to maximum number of integer solutions (constituted by all integers ≥ 1) must uniquely comply with Principle of Maximum Density for Integer Number Solutions (see Proposition 1 in section 6) while also manifesting Principle of Equidistant for Multiplicative Inverse (see Proposition 2 in section 7). We apply concepts from elegant Gauss Circle Problem and Primitive Circle Problem in section 5 onto materials on aptly-named Gauss Areas of Varying Loops to justifiably obtain correct and complete set of mathematical arguments that fully support Theorem 3. *The proof is now complete for Theorem 3*□.

By conveniently employing only sim-η(s) for analysis here [with analysis using DSPL being equally valid], Theorem 1 and Corollary 2 above can also be insightfully combined as follows. Let Set **G** = all Gram points = Gram[x=0,y=0] points + Gram[y=0] points + Gram[x=0] points and Set **vG** = all virtual Gram points = virtual Gram[y=0] points + virtual Gram[x=0] points with virtual Gram[x=0,y=0] points = null set $\emptyset$. We can apply **inclusion-exclusion principle**

$|\mathbf{G} \cup \mathbf{vG}| = |\mathbf{G}| + |\mathbf{vG}| - |\mathbf{G} \cap \mathbf{vG}| = |\mathbf{G}| + |\mathbf{vG}|$ because $|\mathbf{G} \cap \mathbf{vG}| = 0$. Since exclusive presence of Gram points and absence of virtual Gram points on critical line denotes exclusive absence of Gram points and exclusive presence of virtual Gram points on non-critical lines; then Gram points and virtual Gram points as mutually exclusive entities must mathematically and geometrically be incorporated, respectively, onto unique (solitary) critical line and non-unique (infinitely many) non-critical lines of sim-η(s).

Derived f(n) = 0 and F(n) = 0 equations *– see $\sigma = \frac{1}{2}$ (via Proposition 4.3 and Proposition 5.3) and $\frac{2}{5}$ (via Corollary 4.4 and Corollary 5.4) representative examples given in (Ting, 2020, p. 27-28 & p. 29-30) and section 4 below – comply with exact DA homogeneity at $\sigma = \frac{1}{2}$ critical line and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ non-critical lines.* NTZ are synonymous with Gram[x=0,y=0] points which is one type of Gram points. Whenever applicable, all modified equations below are expressed using trigonometric identities. Together with Gram[y=0] points and Gram[x=0] points as remaining two types of Gram points, these three types of Gram points are fully **located** in their complex equations (akin to *Complex Containers*) as IP entities whereby their overall location [but not actual positions] are **intrinsically incorporated** in these complex equations – see section 3 for additional clarification. Eqs. (1), (3), (5), (6), (7) and (8) that comply with exact DA homogeneity at $\sigma = \frac{1}{2}$ all have fractional exponents $\frac{1}{2}$. Eqs. (2) and (4) that comply with inexact DA homogeneity at $\sigma = \frac{2}{5}$ have fractional exponents $\frac{2}{5}$ in the former and $\frac{3}{5}$ in the later that are mixed with fractional exponents $\frac{1}{2}$.

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}2^{\frac{1}{2}}\cos(t\ln(2n)+\frac{1}{4}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}2^{\frac{1}{2}}\cos(t\ln(2n-1)+\frac{1}{4}\pi)=0 \tag{1}$$

With exact DA homogeneity, Eq. (1) is f(n) sim-η(s) at $\sigma = \frac{1}{2}$ that will incorporate all NTZ [as Zeroes]. There is total absence of (non-existent) virtual NTZ [as virtual Zeroes].

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}2^{\frac{1}{2}}\cos(t\ln(2n)+\frac{1}{4}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}2^{\frac{1}{2}}\cos(t\ln(2n-1)+\frac{1}{4}\pi)=0 \tag{2}$$

With inexact DA homogeneity, Eq. (2) is f(n) sim-η(s) at $\sigma = \frac{2}{5}$ that will incorporate all (non-existent) virtual NTZ [as virtual Zeroes]. There is total absence of NTZ [as Zeroes].

$$\frac{1}{2^{\frac{1}{2}}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}\cdot\left[(2n)^{\frac{1}{2}}\cos(t\ln(2n)-\frac{1}{4}\pi)-(2n-1)^{\frac{1}{2}}\cos(t\ln(2n-1)-\frac{1}{4}\pi)+C\right]_1^{\infty}=0 \tag{3}$$

With exact DA homogeneity, Eq. (3) is F(n) DSPL at $\sigma = \frac{1}{2}$ that will incorporate all NTZ [as Pseudo-zeroes to Zeroes conversion]. There is total absence of (non-existent) virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion].

$$\frac{1}{2^{\frac{1}{2}}}\left(t^2+\frac{9}{25}\right)^{\frac{1}{2}}\cdot\left[(2n)^{\frac{3}{5}}\cos(t\ln(2n)-\frac{1}{4}\pi)-(2n-1)^{\frac{3}{5}}\cos(t\ln(2n-1)-\frac{1}{4}\pi)+C\right]_1^{\infty}=0 \tag{4}$$

With inexact DA homogeneity, Eq. (4) is F(n) DSPL at $\sigma = \frac{2}{5}$ that will incorporate all (non-existent) virtual NTZ [as virtual Pseudo-zeroes to virtual Zeroes conversion]. There is total absence of NTZ [as Pseudo-zeroes to Zeroes conversion].

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}\sin(t\ln(2n))-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}\sin(t\ln(2n-1))=0 \tag{5}$$

Eq. (5) can also be equivalently written as
$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}\cos(t\ln(2n)-\frac{1}{2}\pi)-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}\cos(t\ln(2n-1)-\frac{1}{2}\pi)=0.$

With exact DA homogeneity, Eq. (5) is f(n) Gram[y=0] points-sim-η(s) at $\sigma = \frac{1}{2}$ that will incorporate all Gram[y=0] points [as Zeroes]. There is total absence of virtual Gram[y=0] points [as virtual Zeroes].

$$-\frac{1}{2(t^2+\frac{1}{4})^{\frac{1}{2}}}\cdot\left[(2n)^{\frac{1}{2}}(\cos(t\ln(2n)-\frac{1}{4}\pi)-\cos(t\ln(2n-1)-\frac{1}{4}\pi))+C\right]_1^{\infty}=0 \tag{6}$$

Eq. (6) can also be equivalently written as

$$\frac{1}{2(t^2+\frac{1}{4})^{\frac{1}{2}}}\cdot\left[(2n)^{\frac{1}{2}}(\cos(t\ln(2n)+\frac{3}{4}\pi)-\cos(t\ln(2n-1)+\frac{3}{4}\pi))+C\right]_1^{\infty}=0.$$

With exact DA homogeneity, Eq. (6) is F(n) Gram[y=0] points-DSPL at $\sigma=\frac{1}{2}$ that will incorporate all Gram[y=0] points [as Pseudo-zeroes to Zeroes conversion]. There is total absence of virtual Gram[y=0] points [as virtual Pseudo-zeroes to virtual Zeroes conversion].

$$\sum_{n=1}^{\infty}(2n)^{-\frac{1}{2}}\cos(t\ln(2n))-\sum_{n=1}^{\infty}(2n-1)^{-\frac{1}{2}}\cos(t\ln(2n-1))=0 \tag{7}$$

With exact DA homogeneity, Eq. (7) is f(n) Gram[x=0] points-sim-η(s) at $\sigma=\frac{1}{2}$ that will incorporate all Gram[x=0] points [as Zeroes]. There is total absence of virtual Gram[x=0] points [as virtual Zeroes].

$$\frac{1}{2(t^2+\frac{1}{4})^{\frac{1}{2}}}\cdot\left[(2n)^{\frac{1}{2}}(\cos(t\ln(2n)-\frac{3}{4}\pi)-\cos(t\ln(2n-1)-\frac{3}{4}\pi))+C\right]_1^{\infty}=0 \tag{8}$$

With exact DA homogeneity, Eq. (8) is F(n) Gram[x=0] points-DSPL at $\sigma=\frac{1}{2}$ that will incorporate all Gram[x=0] points [as Pseudo-zeroes to Zeroes conversion]. There is total absence of virtual Gram[x=0] points [as virtual Pseudo-zeroes to virtual Zeroes conversion].

We outline sim-η(s) as Eq. (2) and DSPL as Eq. (4) that comply with inexact DA homogeneity at $\sigma=\frac{2}{5}$ non-critical line (depicted by Figure 3) whereby $\sigma=\frac{2}{5}$ [instead of $\sigma=\frac{1}{2}$] is substituted into these two equations. Using [selective] trigonometric identity for linear combination of sine and cosine function whenever applicable to relevant f(n) = 0 and F(n) = 0 equations, we outline exact DA homogeneity at $\sigma=\frac{1}{2}$ critical line (depicted by Figure 2) for Gram[x=0,y=0] points (NTZ) as Eq. (1), Gram[y=0] points as Eq. (5) and Gram[x=0] points as Eq. (7). However, f(n) = 0 equations for Gram[y=0] points as Eq. (5) and Gram[x=0] points as Eq. (7) with exact DA homogeneity at $\sigma=\frac{1}{2}$ critical line are not amendable to treatments using trigonometric identity with implication that their corollary situation endowed with inexact DA homogeneity at $\sigma\neq\frac{1}{2}$ non-critical lines (depicted by Figures 3 and 4) will only manifest solitary [unmixed] $\neq\frac{1}{2}$ fractional exponents. We provide [self-explanatory] corresponding f(n) = 0 equations below for Gram[y=0] points and Gram[x=0] points corollary situation when $\sigma=\frac{2}{5}$.

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}\sin(t\ln(2n))-\sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}\sin(t\ln(2n-1))=0$$

$$\sum_{n=1}^{\infty}(2n)^{-\frac{2}{5}}\cos(t\ln(2n))-\sum_{n=1}^{\infty}(2n-1)^{-\frac{2}{5}}\cos(t\ln(2n-1))=0$$

We arbitrarily chose single cosine wave with format R$cos(n\pm\alpha)$ to use above where R is scaled amplitude and α is phase shift. For equations regarding NTZ, Gram[y=0] points and Gram[x=0] points; all their approximate Areas of Varying Loops $\propto$ precise Areas of Varying Loops with R validly treated as a proportionality factor. We analyze f(n) = 0 and F(n) = 0 equations at $\sigma=\frac{1}{2}$ critical line for NTZ situation where R = $2^{\frac{1}{2}}(2n)^{-\frac{1}{2}}$ or $2^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}$ in f(n)'s Eq. (1) and R = $\frac{1}{2^{\frac{1}{2}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $\frac{1}{2^{\frac{1}{2}}\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$ in F(n)'s Eq. (3).

Remark 3. Whereas for NTZ F(n) Eq. (3) that exactly represent precise Areas of Varying Loops and f(n) Eq. (1) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R from Eq. (3) **which is dependent on parameter t** and Eq. (1) **which is independent of parameter t** represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

We analyze f(n) = 0 equations [relevant to approximate Areas of Varying Loops] at $\sigma=\frac{1}{2}$ critical line for Gram[y=0] points as Eq. (5) and Gram[x=0] points as Eq. (7) whereby we validly designate R = $(2n)^{-\frac{1}{2}}$ or $(2n-1)^{-\frac{1}{2}}$ as the assigned scaled amplitude and [unwritten] $\alpha=0$ as the assigned phase shift.

Relevant to precise Areas of Varying Loops at $\sigma=\frac{1}{2}$ critical line for Gram[y=0] points F(n) Eq. (6) with R =

$-\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $-\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$ and Gram[x=0] points F(n) Eq. (8) with R = $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$, we observe the former R to be the negative of the later R. However, this observation is context-sensitive because when Eq. (6) is written in its equivalent format above, the former R is identical to the later R. Both R are now just given by $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n)^{\frac{1}{2}}$ or $\frac{1}{2\left(t^2+\frac{1}{4}\right)^{\frac{1}{2}}}(2n-1)^{\frac{1}{2}}$.

Remark 4. Whereas for Gram[y=0] points F(n) Eq. (6) that exactly represent precise Areas of Varying Loops and f(n) Eq. (5) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R in Eq. (6) **which is dependent on parameter t** and Eq. (5) **which is independent of parameter t** represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Remark 5. Whereas for Gram[x=0] points F(n) Eq. (8) that exactly represent precise Areas of Varying Loops and f(n) Eq. (7) [when interpreted as Riemann sum] that exactly represent approximate Areas of Varying Loops in a proportionate manner; so must the associated scaled amplitude R in Eq. (8) **which is dependent on parameter t** and Eq. (7) **which is independent of parameter t** represent [in a surrogate manner] corresponding precise and approximate Areas of Varying Loops in a proportionate manner.

Finally, we analyze f(n) = 0 and F(n) = 0 equations at $\sigma = \frac{1}{2}$ critical line for NTZ situation where phase shift $\alpha = \frac{1}{4}\pi$ in NTZ f(n) Eq. (1) and $-\frac{1}{4}\pi$ in NTZ F(n) Eq. (3); and F(n) = 0 equations at $\sigma = \frac{1}{2}$ critical line for Gram[y=0] points and Gram[x=0] points situations where phase shift $\alpha = -\frac{1}{4}\pi$ (or $\frac{3}{4}\pi$ when written in its equivalent format above) in Gram[y=0] points F(n) Eq. (6) and $-\frac{3}{4}\pi$ in Gram[x=0] points F(n) Eq. (8). Always being $\frac{1}{2}\pi$ out-of-phase with each other, trigonometric functions sine and cosine are cofunctions with sin n = cos ($\frac{\pi}{2}$ - n) or cos (n - $\frac{\pi}{2}$), cos n = sin ($\frac{\pi}{2}$ - n) or sin (n + $\frac{\pi}{2}$), $\frac{d(\sin n)}{dn} = \cos n$, $\frac{d(\cos n)}{dn} = -\sin n$, $\int \sin n \cdot dn = -\cos n + C$ [= sin (n - $\frac{\pi}{2}$) + C] and $\int \cos n \cdot dn = \sin n + C$ [= cos (n - $\frac{\pi}{2}$) + C]. Last two integrals explain relation between f(n)'s Zeroes and F(n)'s Pseudo-zeroes when they involve simple sine and/or cosine terms viz, f(n)'s CP Zeroes = F(n)'s CP Pseudo-zeroes – $\frac{1}{2}\pi$ with CP Zeroes and CP Pseudo-zeroes being $\frac{1}{2}\pi$ out-of-phase with each other.

Lemma 1. NTZ obtained directly from IP Zeroes and indirectly from IP Pseudo-zeroes behave in accordance with complex sine and/or cosine terms present in their equations that are $\frac{1}{2}\pi$ out-of-phase with each other.

Proof. Involving trigonometric functions as complex sine and/or cosine terms: f(n)'s IP NTZ or [non-existent] f(n)'s IP virtual NTZ (in t values) = F(n)'s IP Pseudo-NTZ or [non-existent] F(n)'s IP virtual Pseudo-NTZ (in t values) – $\frac{1}{2}\pi$; f(n)'s IP Gram[y=0] points or f(n)'s IP virtual Gram[y=0] points (in t values) = F(n)'s IP Pseudo-Gram[y=0] points or F(n)'s IP virtual Pseudo-Gram[y=0] points (in t values) – $\frac{3}{4}\pi$; and f(n)'s IP Gram[x=0] points or f(n)'s IP virtual Gram[x=0] points (in t values) = F(n)'s IP Pseudo-Gram[x=0] points or F(n)'s IP virtual Pseudo-Gram[x=0] points (in t values) – $\frac{3}{4}\pi$.

$\int f(n)dn = F(n) + C$ where $F'(n) = f(n)$. f(n) and F(n) are literally [connected] **bijective (both injective and surjective or a one-to-one correspondence) functions**. Underlying f(n) as equation and F(n) as law (equation) that generate their CIS of IP Zeroes, IP virtual Zeroes, IP Pseudo-zeroes and IP virtual Pseudo-zeroes are precisely related as $\frac{1}{2}\pi$ (for NTZ case) or $\frac{3}{4}\pi$ (for Gram[y=0] points and Gram[x=0] points cases) out-of-phase with each other. Peculiar to IP NTZ as Origin intercept points, we crucially note only they will uniquely behave in accordance with complex sine and/or cosine terms present in their equations that generate corresponding IP Zeroes and IP Pseudo-zeroes which are $\frac{1}{2}\pi$ [but not $\frac{3}{4}\pi$] out-of-phase with each other. *The proof is now complete for Lemma 1*□.

Lemma 2. Corresponding paired IP two types of Gram points [as Zeroes] situation, paired IP two types of virtual Gram points [as virtual Zeroes] situation, paired IP two types of Pseudo-Gram points [as Pseudo-zeroes] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] situation are always $\frac{1}{2}\pi$ out-of-phase with each other in every one of these situations.

Proof. The x-axis and y-axis are orthogonal to each other with angle between them = $\frac{1}{2}\pi$ radian. Involving trigonometric functions as complex sine and/or cosine terms: f(n)'s IP Gram[y=0] points or f(n)'s IP virtual Gram[y=0] points (in t values) = f(n)'s IP Gram[x=0] points or f(n)'s IP virtual Gram[x=0] points (in t values) + $\frac{1}{2}\pi$; and F(n)'s IP Pseudo-Gram[y=0] points or F(n)'s IP virtual Pseudo-Gram[y=0] points (in t values) = F(n)'s IP Pseudo-Gram[x=0] points or F(n)'s IP virtual Pseudo-Gram[x=0] points (in t values) + $\frac{1}{2}\pi$.

These observations imply underlying f(n) as equation and F(n) as law (equation) that generate corresponding paired IP two types of Gram points [as Zeroes] situation, paired IP two types of virtual Gram points [as virtual Zeroes] situation, paired IP two types of Pseudo-Gram points [as Pseudo-zeroes] situation, and paired IP two types of virtual Pseudo-Gram points [as virtual Pseudo-zeroes] situation are always $\frac{1}{2}\pi$ out-of-phase with each other in every one of these mentioned situations. *The proof is now complete for Lemma 2*□.

3. The Completely Predictable and Incompletely Predictable entities

The word "number" [singular noun] or "numbers" [plural noun] used in reference to CP even and odd numbers, IP prime and composite numbers, IP NTZ and two other types of Gram points can interchangeably be replaced with the word "entity" [singular noun] or "entities" [plural noun]. For i = all integers ≥ 0 or i = all integers ≥ 1; the ith position of ith CP numbers and ith IP numbers is simply given by *i*. Apart from the very first Gram[y=0] point and the very first virtual Gram[y=0] point being both 0, we note all Gram points and virtual Gram points will consist of t-valued transcendental numbers whose positions are IP with the infinitely many digits after the decimal point in each transcendental number again being IP.

We outline an innovative method to classify appropriately chosen equation or algorithm in two ways by using relevant locational properties of its output. This output consist of generated entities either from function-based equations or from algorithms. Our novel [albeit loose] classification systems named *"Mathematics for Completely Predictable problems"* that is associated with conveniently-coined *simple calculations*, and *"Mathematics for Incompletely Predictable problems"* that is associated with conveniently-coined *complex calculations*, are respectively formalized by providing definitions for CP entities obtained from CP equations or algorithms, and IP entities obtained from IP equations or algorithms.

CP simple equation or algorithm generates CP numbers. A generated CP number is **locationally defined** as a number whose ith position is *independently* determined by simple calculations without needing to know related positions of all preceding numbers. IP complex equation or algorithm generates IP numbers. A generated IP number is **locationally defined** as a number whose ith position is *dependently* determined by complex calculations with needing to know related positions of all preceding numbers. Container is a useful analogical term that metaphorically group CP entities (e.g. even and odd numbers) and IP entities (e.g. nontrivial zeros, prime and composite numbers) to be exclusively located in, respectively, Simple Container and Complex Container.

Simple properties are inferred from a sentence such as "This simple equation or algorithm by itself will intrinsically incorporate *overall location [and actual positions]* of all CP numbers". Examples: simple equations E = (2 X i) for i = all integers ≥ 0 [or i = all real numbers ≥ 0] and O = (2 X i) - 1 for i = all integers ≥ 1 [or i = all real numbers ≥ 1] will respectively and intrinsically incorporate or generate CIS of all [non-negative] CP even number E_i = 0, 2, 4, 6,... and CIS of all [non-negative] CP odd numbers O_i = 1, 3, 5, 7,... whereby even number (**n**) is defined as "Any integer that can be divided exactly by 2 with last digit always being 0, 2, 4, 6 or 8" and odd number (**n**) is defined as "Any integer that cannot be divided exactly by 2 with last digit always being 1, 3, 5, 7 or 9". Congruence **n** ≡ 0 (mod 2) holds for even **n** and congruence **n** ≡ 1 (mod 2) holds for odd **n**. Note the zeroth even number is given by E_0 = 0.

Complex properties, or meta-properties, are inferred from a sentence such as "This complex equation or algorithm by itself will intrinsically incorporate *overall location [but not actual positions]* of all IP numbers". Examples: complex algorithms $P_{i+1} = P_i + pGap_i$ and $C_{i+1} = C_i + cGap_i$ for i = 1, 2, 3,..., ∞ with $P_1 = 2$ and $C_1 = 4$ will respectively and intrinsically incorporate CIS of all IP prime number 2, 3, 5, 7,... and CIS of all IP composite numbers 4, 6, 8, 9,... whereby prime numbers are defined as "All Natural numbers apart from 1 that are evenly divisible by itself and by 1" and composite numbers are defined as "All Natural numbers apart from 1 that are evenly divisible by numbers other than itself and 1". E.g. via computed Pseudo-zeroes that can be converted to Zeroes at $\sigma = \frac{1}{2}$ critical line, complex equation DSPL will intrinsically incorporate the CIS of all IP NTZ [given as t values rounded off to six decimal places]: 14.134725, 21.022040, 25.010858, 30.424876, 32.935062, 37.586178,... and complex equation Gram[y=0] points-DSPL will intrinsically incorporate the CIS of all IP Gram[y=0] points [given as t values rounded off to six decimal places]: 0, 3.436218, 9.666908, 17.845599; 23.170282, 27.670182,.... Choice of index n for Gram[y=0] points is crudely chosen in the past to be -3, -2, -1, 0, 1, 2, 3,... [≡ i = 1, 2, 3, 4, 5, 6, 7,...] whereby the first Gram[y=0] point is historically denoted by $n = 1$ [≡ i = 5] with t value 17.845599 (on the critical line) being larger than first NTZ's t value of 14.134725 (on the critical line).

The Even-Odd Pairing. For i = 1, 2, 3,..., ∞; let mutually exclusive i^{th} Even numbers = E_i and i^{th} Odd numbers = O_i, and i^{th} even number gaps = $eGap_i$ and i^{th} odd number gaps = $oGap_i$. The i^{th} positions of E_i and O_i are CP, and are independent from each other.

E_i	2		4		6		8		10		12	.....
$eGap_i$		2		2		2		2		2		2

We employ simple equations E = (2 X i) and O = (2 X i) - 1. E.g., we can precisely, easily and independently calculate E_5 = (2 X 5) = 10 and O_5 = (2 X 5) - 1 = 9.

O_i	1		3		5		7		9		11	.....
$oGap_i$		2		2		2		2		2		2

The Prime-Composite Pairing. For i = 1, 2, 3,..., ∞; let mutually exclusive i^{th} Prime numbers = P_i and i^{th} Composite numbers = C_i, and i^{th} prime number gaps = $pGap_i$ and i^{th} composite number gaps = $cGap_i$. The i^{th} positions of P_i and C_i are IP, and are dependent on each other.

P_i	2		3		5		7		11		13	.....
$pGap_i$		1		2		2		4		2		4

We employ complex algorithms $P_{i+1} = P_i + pGap_i$ and $C_{i+1} = C_i + cGap_i$. E.g., we precisely, tediously and dependently calculate $P_6 = 13$ as 2 is 1^{st} prime number, 3 is 2^{nd} prime number, 4 is 1^{st} composite number, 5 is 3^{rd} prime number, 6 is 2^{nd} composite number, 7 is 4^{th} prime number, 8 is 3^{rd} composite number, 9 is 4^{th} composite number, 10 is 5^{th} composite number, 11 is 5^{th} prime number, 12 is 6^{th} composite number, and our desired 13 is 6^{th} prime number.

C_i	4		6		8		9		10		12	.....
$cGap_i$		2		2		1		1		2		2

The $\sigma = \frac{1}{2}$ NTZ computed from Eq. (1) – $\sigma \neq \frac{1}{2}$ (non-existent) virtual NTZ computed from Eq. (2) Pairing. For i = 1, 2, 3,..., ∞; let mutually exclusive i^{th} NTZ = NTZ_i and i^{th} virtual NTZ = $vNTZ_i$, and i^{th} NTZ gaps = NTZ-Gap_i and i^{th} virtual NTZ gaps = vNTZ-Gap_i. Eq. (1) and Eq. (2) are dependently identical except for associated σ values. They are used to precisely, tediously and dependently calculate all NTZ_i and $vNTZ_i$ with their i^{th} positions being IP.

4. The exact and inexact Dimensional analysis homogeneity for Equations

For 'base quantities' *length*, *mass* and *time*; their fundamental SI 'units of measurement' meter (m) is defined as distance travelled by light in vacuum for time interval 1/299 792 458 s with speed of light c = 299,792,458 ms^{-1}, kilogram (kg) is defined by taking fixed numerical value Planck constant h to be 6.626 070 15 X 10^{-34} Joules·second (Js) [whereby Js is equal to kgm^2s^{-1}] and second (s) is defined in terms of ΔvCs = $\Delta(^{133}Cs)_{hfs}$ = 9,192,631,770 s^{-1}. Derived SI units such as J and ms^{-1} respectively represent 'base quantities' *energy* and *velocity*. 'Dimension' is commonly used to indicate 'units of measurement' in well-defined equations. DA is a traditional analytic tool with DA homogeneity and DA non-homogeneity (respectively) denoting valid and invalid equation occurring when 'units of measurements' for 'base quantities' are "balanced" and "unbalanced" across both sides of equation. E.g. equation 2 m + 3 m = 5 m is valid but equation 2 m + 3 kg = 5 'm·kg' is invalid (respectively) manifesting DA homogeneity and non-homogeneity.

We conveniently adopt concepts from DA which are mathematically correct and valid. Let (2n) and (2n-1) be 'base quantities' in equation DSPL. Fractional exponents as 'units of measurement' given by $(1-\sigma)$ in equation DSPL when $\sigma = \frac{1}{2}$ coincide with exact DA homogeneity; and $(1-\sigma)$ in equation DSPL when $\sigma \neq \frac{1}{2}$ coincide with inexact DA homogeneity. Respectively, exact DA homogeneity at $\sigma = \frac{1}{2}$ denotes ∑(all fractional exponents) as $2(1-\sigma)$ equates to [exact] integer 1; and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ denotes ∑(all fractional exponents) as $2(1-\sigma)$ equates to [inexact] fractional number ≠1 [Range: $0 < 2(1-\sigma) < 1$ and $1 < 2(1-\sigma) < 2$]. Computations based on exact and inexact DA homogeneity in equation DSPL explicitly give rise to $\sigma = \frac{1}{2}$ critical line Gram points (given indirectly as

Pseudo-zeroes t-values which can be converted to Zeroes t-values) and $\sigma \neq \frac{1}{2}$ non-critical lines virtual Gram points (given indirectly as virtual Pseudo-zeroes t-values which can be converted to virtual Zeroes t-values).

Performing exact and inexact DA homogeneity on equation sim-η(s) is equally valid. With same 'base quantities', fractional exponents as 'units of measurement' are now given by $(-\sigma)$. Respectively, exact DA homogeneity at $\sigma = \frac{1}{2}$ denotes $\sum$(all fractional exponents) as $2(-\sigma)$ equates to [exact] integer -1; and inexact DA homogeneity at $\sigma \neq \frac{1}{2}$ denotes $\sum$(all fractional exponents) as $2(-\sigma)$ equates to [inexact] fractional number ≠–1 [Range: $-2 < 2(-\sigma) < -1$ and $-1 < 2(-\sigma) < 0$]. Computations using equation sim-η(s) [when interpreted as Riemann sum] explicitly give rise to $\sigma = \frac{1}{2}$ critical line Gram points (given directly as Zeroes t-values) while representing exact DA homogeneity and $\sigma \neq \frac{1}{2}$ non-critical lines virtual Gram points (given directly as virtual Zeroes t-values) while representing inexact DA homogeneity.

For calculations involving $2(1-\sigma)$ or $2(-\sigma)$, we note it is inconsequential whether σ values from the fractional exponents of 'base quantities' (2n) or (2n-1) are formatted in simplest form or not. For example, since $\frac{1}{2} \equiv \frac{2}{4}$; performing the $\sigma = \frac{1}{2}$ exact DA homogeneity on exponent $\frac{1}{2}$ in $(2n)^{\frac{1}{2}}$ when depicted in simplest form will be equivalent to performing the [same] $\sigma = \frac{1}{2}$ exact DA homogeneity on exponent $\frac{1}{4}$ in $(2^2n^2)^{\frac{1}{4}}$ when *not* depicted in simplest form.

5. Gauss Circle Problem and Primitive Circle Problem

Equation of a circle centered at Origin with radius r and precise Area = πr^2 is given in Cartesian coordinates as $x^2 + y^2 = r^2$. The number of integer lattice points $N(r)$ on and inside a circle [viz, pairs of integers (m,n) such that $m^2 + n^2 \leq r^2$] can be exactly determined by following two equations whereby $N(r)$ is considered the most accurate surrogate marker of approximate Area for a given circle. Named after German mathematician Carl Friedrich Gauss (April 30, 1777 - February 23, 1855), Gauss Circle Problem is the problem of determining how many integer lattice points as approximate Area for a given circle. For i and r = 0, 1, 2, 3,...,∞ and through which it can be given by several series such as in terms of a sum involving the floor function; $N(r)$ is expressed as equation $N(r) = 1 + 4\sum_{i=0}^{\infty}\left(\left\lfloor \frac{r^2}{4i+1}\right\rfloor - \left\lfloor \frac{r^2}{4i+3}\right\rfloor\right)$ whereby this equation is a consequence of Jacobi's two-square theorem which follows almost immediately from Jacobi triple product. A much simpler sum appears if sum of squares function $r_2(n)$ that is defined as number of ways of writing number n as sum of two squares is used. Then, we have alternative equation $N(r) = \sum_{n=0}^{r^2} r_2(n)$. The first few $N(r)$ values for r = 0, 1, 2, 3, 4, 5, 6, 7, 8,... are 1, 5, 13, 29, 49, 81, 113, 149,... whereby these are Incompletely Predictable entities complying with relationship: [simple] equation for precise Area of circle = πr^2 is proportional to above two most accurate and equivalent [complex] equations for approximate Area of circle = $N(r)$.

We expect $N(r) = \pi r^2 + E(r)$ for some error term $E(r)$ of relatively small absolute value. Gauss managed to prove $|E(r)| \leq 2\sqrt{2}\pi r$. Modern proofs on upper bound value [in 2000] and lower bound value [in 1915] for $E(r)$ have since been derived. We recognize r does not have to be an integer. After $N(4) = 49$, we obtain $N(\sqrt{17}) = 57, N(\sqrt{18}) = 61, N(\sqrt{20}) = 69, N(5) = 81$. At these places, $E(r)$ increases by $8, 4, 8, 12$ after which it decreases at a rate of $2\pi r$ until the next time it increases.

Finally, the identity $N(x) - \frac{r_2(x^2)}{2} = \pi x^2 + x\sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{n}}J_1(2\pi x\sqrt{n})$ has implicitly been observed to be related to number of integer lattice points, $N(r)$, where J_1 denotes Bessel function of first kind with order 1. It was discovered by English mathematician Godfrey H. Hardy (February 7, 1877 - December 1, 1947) (Landau, 1969).

Primitive Circle Problem as least accurate surrogate marker of approximate Area for a given circle involves calculating the number of coprime integer solutions (m,n) to the inequality $m^2 + n^2 \leq r^2$. If the number of such solutions is denoted $V(r)$ then the values of $V(r)$ for r taking small integer values are 0, 4, 8, 16, 32, 48, 72, 88, 120, 152, 192,.... Using the same ideas as usual Gauss Circle Problem and the fact that probability two integers are coprime is $\frac{6}{\pi^2}$, it is relatively straightforward to show $V(r) = \frac{6}{\pi}r^2 + O(r^{1+\varepsilon})$. We solve problematic part of Primitive Circle Problem by reducing the exponent in the error term. This exponent is presently best known to be $221/304 + \varepsilon$ since we can now validly assume Riemann hypothesis to be true in this paper.

Remark 6. Let A denote Area of a given circle with radius r. The computed precise A using A = πr^2 method, computed approximate A using [most accurate] approximate $N(r)$ method of Gauss Circle Problem and computed approximate A using [least accurate] approximate $A(r)$ method of Primitive Circle Problem will explicitly confirm A $\propto r^2$ for all three methods.

6. Gauss Areas of Varying Loops and Principle of Maximum Density for Integer Number Solutions

We translate concepts from Gauss Circle Problem and Primitive Circle Problem in section 5 onto Gauss Areas of Varying Loops to fully support all materials below.

Proposition 1. We can validly and fully demonstrate that only when $\sigma = \frac{1}{2}$ [and not when $\sigma \neq \frac{1}{2}$] in sim-η(s) or DSPL will the maximum number of integer solutions (constituted by all integers $\geq$ 1) arise that must uniquely comply with Principle of Maximum Density for Integer Number Solutions.

Proof. For n classically involving all integers ≥1 in sim-η(s) as $\Delta n \longrightarrow 1$ or n classically involving all real numbers ≥1 in DSPL as $\Delta n \longrightarrow 0$; their base quantities (2n) and (2n-1), respectively, generate CIS even numbers commencing from 2 and CIS odd numbers commencing from 1. These base quantities are subjected to algebraic function square roots at $\sigma = \frac{1}{2}$ critical line [viz, when $\sigma = \frac{1}{2}$] and cube roots at $\sigma = \frac{1}{3}$ non-critical line or twice cube roots at $\sigma = \frac{2}{3}$ non-critical line [viz, when $\sigma \neq \frac{1}{2}$] thus giving rise to corresponding subset of rational roots and subset of irrational roots. We now concentrate on combined (2n)'s and (2n-1)'s obtained integer lattice points [≥ 1] to derive solitary subset of rational roots for n = 1 to 100 range in sim-η(s) or DSPL when:

(I) $\sigma = \frac{1}{2}$ involving a **neither even nor odd function** with no symmetry viz, $f(-n) \neq f(n)$ and $f(-n) \neq -f(n)$ by applying $f(n)$ as fractional exponent $\frac{1}{2}$ or square root on n = ten perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 giving rise to the (maximum) ten rational roots as consecutive integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

(II) $\sigma = \frac{1}{3}$ involving a **odd function** with Origin symmetry viz, $f(-n) = -f(n)$ by applying $f(n)$ as fractional exponent $\frac{1}{3}$ or cube root on n = four perfect cubes 1, 8, 27, 64 giving rise to the (non-maximum) four rational roots as consecutive integer solutions 1, 2, 3, 4.

(III) $\sigma = \frac{2}{3}$ involving an **even function** with y-axis symmetry viz, $f(-n) = f(n)$ by applying $f(n)$ as fractional exponent $\frac{2}{3}$ or squared cube root on n = four perfect cubes 1, 8, 27, 64 giving rise to the (non-maximum) four rational roots as non-consecutive integer solutions 1, 4, 9, 16.

Only at $\sigma = \frac{1}{2}$ critical line which involves applying $f(n)$ as fractional exponent $\frac{1}{2}$ or square root on n = all perfect squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100... will we obtain maximum number of rational roots as consecutive integer solutions 1, 2, 3, 4, 5, 6, 7, 8, 9, 10... (viz, all integers ≥1). This observation uniquely comply with **Principle of Maximum Density for Integer Number Solutions** at $\sigma = \frac{1}{2}$ critical line. *The proof is now complete for Proposition 1*□.

Notation: **Term-(2n)** denote **(2n)-complex term with algebraic functions** X **(2n)-complex term with transcendental functions**; and **Term-(2n-1)** denote **(2n-1)-complex term with algebraic functions** X **(2n-1)-complex term with transcendental functions**. sim-η(s) or DSPL is complex function or law with single variable n and parameters σ, t. Their derived equations [Eqs. (1) to (8)] have **(2n)- or (2n-1)-complex term with algebraic functions** consisting of powers, fractional powers, root extraction and scaled amplitude R that are **dependent on parameter** σ, and **(2n)- or (2n-1)-complex term with transcendental functions** consisting of sine, cosine, single cosine wave, single sine wave, natural logarithm that are **independent of parameter** σ.

Remark 7. Corresponding to Areas of Varying Loops = 0 in f(n) sim-η(s) or F(n) DSPL, Term-(2n) must precisely cancel Term-(2n-1) in order to obtain $\sigma = \frac{1}{2}$ f(n)'s Zeroes and F(n)'s Pseudo-zeroes or to obtain $\sigma \neq \frac{1}{2}$ f(n)'s virtual Zeroes and F(n)'s virtual Pseudo-zeroes.

Applicable to sim-η(s) and DSPL, we note the computed CIS rational roots (subset) as integers [rational numbers] + CIS irrational roots (subset) as irrational numbers = CIS total roots.

Remark 8. Complex function F(n) = DSPL [representive of precise Area under the Curve] generates the most accurate precise Areas of Varying Loops [when all rational and irrational roots from combined base quantities (2n) and (2n-1) are utilized] and the least accurate precise Areas of Varying Loops [when only rational roots from combined base quantities (2n) and (2n-1) are utilized]; and complex function f(n) = sim-η(s) when interpreted as Riemann sum [representive of approximate Area under the Curve] generates the most accurate approximate Areas of Varying Loops [when all rational and irrational roots from combined base quantities (2n) and (2n-1) are utilized] and the least accurate approximate Areas of Varying Loops [when only rational roots from combined base quantities (2n) and (2n-1) are utilized].

Our [metaphoric] varying radius r in sim-η(s) or DSPL is defined as r = Term-(2n) – Term-(2n-1) whereby perpetually recurring $r = 0$ will correspond to Areas of Varying Loops = 0 in order to obtain $\sigma = \frac{1}{2}$ f(n)'s Zeroes and F(n)'s Pseudo-zeroes or to obtain $\sigma \neq \frac{1}{2}$ f(n)'s virtual Zeroes and F(n)'s virtual Pseudo-zeroes. In effect, Areas of Varying Loops is conceptionally synonymous with varying radius r whereby varying radius r could also be visualized as [metaphoric] varying distance d between Term-(2n) and Term-(2n-1).

Remark 9. Whether involving the most accurate method using total roots or the least accurate method using rational roots to determine DSPL's precise or sim-η(s)'s approximate Areas of Varying Loops, we can explicitly conclude all the infinitely-many obtained Areas of Varying Loops are proportional and equal to varying radius r with these Varying Loops being synthesized in a perpetually dynamic, cyclical and Incompletely Predictable manner.

7. Shift of Varying Loops in $\zeta(\sigma + \imath t)$ Polar Graph and Principle of Equidistant for Multiplicative Inverse with General Equations for simplified Dirichlet eta function and Dirichlet Sigma-Power Law

We reiterate that Gram[x=0,y=0] points, Gram[y=0] points and Gram[x=0] points are three types of IP Gram points [Zeroes] occurring at $\sigma = \frac{1}{2}$ critical line (Figure 2) based on, respectively, Origin intercept points, x-axis intercept points and y-axis intercept points. They can be dependently computed from relevant types of sim-η(s) = 0 equations whereby sim-η(s) is obtained by applying Euler formula to η(s). Gram[x=0,y=0] points are synonymous with NTZ and Gram[y=0] points are synonymous with 'usual' Gram points. Virtual Gram[y=0] points and virtual Gram[x=0] points are two types of IP virtual Gram points [virtual Zeroes] occurring at $\sigma \neq \frac{1}{2}$ non-critical lines based on, respectively, x-axis intercept points and y-axis intercept points – see Figure 3 for $\sigma = \frac{2}{5}$ and Figure 4 for $\sigma = \frac{3}{5}$. They are also dependently computed from these same equations.

Proposition 2. Both $f(n)$ sim-η(s) and $F(n)$ DSPL will manifest Principle of Equidistant for Multiplicative Inverse.

Proof. Let $\delta = \frac{1}{10}$. This will generate in Figure 3 and Figure 4 the δ induced shift of [infinitely many] Varying Loops in reference to Origin; viz, the simple relationship of [more negative] left-shift given by $\zeta(\frac{1}{2} - \delta + \imath t)$ [Figure 3] < [neutral] nil-shift given by $\zeta(\frac{1}{2} + \imath t)$ [Figure 2] < [more positive] right-shifted given by $\zeta(\frac{1}{2} + \delta + \imath t)$ [Figure 4] will always be consistently true.

Given $\delta = \frac{1}{10}$, the $\sigma = \frac{1}{2} - \delta$ non-critical line (represented by Figure 3) and $\sigma = \frac{1}{2} + \delta$ non-critical line (represented by Figure 4) are equidistant from $\sigma = \frac{1}{2}$ critical line (represented by Figure 2). The additive inverse operation of sin(δ) + sin(-δ) = 0 indicating symmetry with respect to Origin [or cos(δ) - cos(-δ) = 0 indicating symmetry with respect to y-axis] is not applicable to our complex single sine wave [or single cosine wave] since **(2n)- or (2n-1)-complex term with transcendental functions** consisting of sine, cosine, single sine wave, single cosine wave, natural logarithm are **independent of parameter σ**. However, **(2n)- or (2n-1)-complex term with algebraic functions** consisting of powers, fractional powers, root extraction [and scaled amplitude R as alluded to by Remarks 3, 4 and 5 on its (in)dependency on parameter t] are **dependent on parameter σ**.

Let x = $(2n)$ or $\frac{1}{(2n)}$ or $(2n-1)$ or $\frac{1}{(2n-1)}$. With multiplicative inverse operation of $x^{\delta} \cdot x^{-\delta} = 1$ or $\frac{1}{x^{\delta}} \cdot \frac{1}{x^{-\delta}} = 1$ that is applicable, this imply intrinsic presence of Multiplicative Inverse in sim-$\eta(s)$ or DSPL for all σ values with this function or law rigidly obeying relevant trigonometric identity. This phenomenon is **Principle of Equidistant for Multiplicative Inverse**. Finally, we note by letting $\delta = 0$, we will always generate Figure 2 representing $\sigma = \frac{1}{2}$ critical line. *The proof is now complete for Proposition 2*□.

For complex functions and complex equations in this paper, s = $\sigma \pm it$ whereby we commonly invoke s = $\sigma + it$ for discussion. For all f(n) and F(n) general equations depicted below without trigonometric identity application, we note presence of mixed sine and cosine terms in these general equations except for f(n)'s Gram[y=0] points-sim-η(s) and f(n)'s Gram[x=0] points-sim-η(s).

I. NTZ or Gram [x=0,y=0] points as geometrical Origin intercept points are mathematically defined by $\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0$. General equation for f(n)'s sim-η(s) as Zeroes is given by

$$\sum_{n=1}^{\infty} -(2n)^{-\sigma}(\sin(t\ln(2n)) - \cos(t\ln(2n)))$$

$$-\sum_{n=1}^{\infty} -(2n-1)^{-\sigma}(\sin(t\ln(2n-1)) - \cos(t\ln(2n-1))) = 0 \tag{9}$$

General equation for F(n)'s DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

$$\begin{aligned}\frac{1}{2\left(t^2+(\sigma-1)^2\right)} &\cdot \Big[(2n)^{1-\sigma}((t+\sigma-1)\sin(t\ln(2n)) + (t-\sigma+1) \\ &\cdot \cos(t\ln(2n))) - (2n-1)^{1-\sigma}((t+\sigma-1) \\ &\cdot \sin(t\ln(2n-1)) + (t-\sigma+1)\cos(t\ln(2n-1))) + C\Big]_1^{\infty} = 0\end{aligned} \tag{10}$$

II. Gram[y=0] points as geometrical x-axis intercept points are mathematically defined by $\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + 0,$ or simply $\text{Im}\{\eta(s)\} = 0$. General equation for f(n)'s Gram[y=0] points-sim-η(s) as Zeroes is given by

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\sin(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\sin(t\ln(2n-1)) = 0 \tag{11}$$

General equation for F(n)'s Gram[y=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

$$-\frac{1}{2\left(t^2+(\sigma-1)^2\right)} \cdot \Big[(2n)^{1-\sigma}((\sigma-1)\sin(t\ln(2n)) + t\cos(t\ln(2n))) -$$

$$(2n-1)^{1-\sigma}((\sigma-1)\sin(t\ln(2n-1)) + t\cos(t\ln(2n-1))) + C\Big]_1^{\infty} = 0 \tag{12}$$

III. Gram[x=0] points as geometrical y-axis intercept points are mathematically defined by $\sum ReIm\{\eta(s)\} = 0 + Im\{\eta(s)\},$ or simply $\text{Re}\{\eta(s)\} = 0$. General equation for f(n)'s Gram[x=0] points-sim-η(s) as Zeroes is given by

$$\sum_{n=1}^{\infty}(2n)^{-\sigma}\cos(t\ln(2n)) - \sum_{n=1}^{\infty}(2n-1)^{-\sigma}\cos(t\ln(2n-1)) = 0 \tag{13}$$

General equation for F(n)'s Gram[x=0] points-DSPL with ability for Pseudo-zeroes to Zeroes conversion is given by

$$\frac{1}{2\left(t^2+(\sigma-1)^2\right)} \cdot \Big[(2n)^{1-\sigma}(t\sin(t\ln(2n)) - (\sigma-1)\cos(t\ln(2n))) -$$

$$(2n-1)^{1-\sigma}(t\sin(t\ln(2n-1)) - (\sigma-1)\cos(t\ln(2n-1))) + C\Big]_1^{\infty} = 0 \tag{14}$$

Remark 10. The Cartesian Coordinates (x,y) is intimately related to Polar Coordinates (r,θ) with $r = \sqrt{x^2+y^2}$ and $\theta = tan^{-1}(\frac{y}{x})$. In anti-clockwise direction, it has four quadrants defined by the + or - of (x,y); viz, Quadrant I as (+,+), Quadrant II as (-,+), Quadrant III as (-,-), and Quadrant IV as (+,-).

NTZ are Origin intercept points or Gram [x=0,y=0] points. With 'gap' being synonymous with 'interval', NTZ gap is given by initial NTZ t-value minus next NTZ t-value. Running a Full cycle from 0π to 2π, size of each IP Varying Loop in Figure 2 is proportional to magnitude of its corresponding IP NTZ varying gap. We note the 2π here as observed in Figure 2 [on Gram points at $\sigma = \frac{1}{2}$], Figure 3 [on virtual Gram points at $\sigma = \frac{2}{5}$] and Figure 4 [on virtual Gram points at $\sigma = \frac{3}{5}$] refers to IP Varying Loops transversed by parameter t with NTZ (Gram [x=0,y=0] points) corresponding to t values as Origin intercept on Origin's solitary (0,0) part (point); Gram [y=0] points and virtual Gram [y=0] points corresponding to t values as x-axis intercept on x-axis' (+ve) 0π part and (-ve) 1π part; and Gram [x=0] points and virtual Gram [x=0] points corresponding to t values as y-axis intercept on y-axis' (+ve) $\frac{\pi}{2}$ part and (-ve) $\frac{3\pi}{2}$ part. Virtual NTZ entities do not exist; viz, Origin intercept points do not occur in Figure 3 and Figure 4.

With η(s) being *proxy* function for ζ(s), NTZ are defined by η(s) = 0 or sim-η(s) = 0. This mathematically-defined NTZ (or Gram[x=0,y=0] points) are precisely equivalent to the geometrically-defined Origin intercept points. Then, NTZ given by relevant computed IP t values are validly deduced to be infinite in magnitude since the sim-η(s) = 0 equation contains [complex] sine and/or cosine functions which are well-defined continuous functions having infinitely many computed Origin intercept points located on infinitely many Varying Loops generated by $0 < t < +\infty$ or [its complex conjugate] $-\infty < t < 0$ domain with unlimited range.

Riemann hypothesis is the original 1859-dated conjecture that all NTZ are located on $\sigma = \frac{1}{2}$ critical line of $\zeta(s)$. Mathematically proving all NTZ location on critical line as denoted by solitary $\sigma = \frac{1}{2}$ value equates to geometrically proving all Origin intercept points occurrence at solitary $\sigma = \frac{1}{2}$ value. Both result in rigorous proof for Riemann hypothesis. Locations of first 10,000,000,000,000 NTZ on critical line have previously been computed to be correct. Hardy (Hardy, 1914), and with Littlewood (Hardy & Littlewood, 1921), showed infinitely many NTZ on $\sigma = \frac{1}{2}$ critical line by considering moments of certain functions related to ζ(s).

Remark 11. The discovery by Hardy and Littlewood showing infinitely many NTZ on $\sigma = \frac{1}{2}$ critical line cannot constitute rigorous proof for Riemann hypothesis because they have not exclude theoretical existence of NTZ in the region located away from the critical line [whereby this region is denoted by the infinitely many $\sigma \neq \frac{1}{2}$ non-critical lines]. Furthermore, it is literally a mathematical impossibility ("mathematical impasse") to be able to computationally check [in a successful manner] locations of all the infinitely many NTZ are on the critical line.

The monumental task of solving Riemann hypothesis is completed by deriving $F(n)$ DSPL from $f(n)$ sim-η(s) with its computed Pseudo-zeroes and virtual Pseudo-zeroes which can all be converted to corresponding Zeroes and virtual Zeroes since F(n)'s IP Pseudo-zeroes and IP virtual Pseudo-zeroes (t values) = f(n)'s IP Zeroes and IP virtual Zeroes (t values) + $\frac{\pi}{2}$ [for NTZ situation] whereby both f(n) and F(n) have parameters σ and t. Correctly deducing exact DA homogeneity in DSPL symbolizes rigorous proof for Riemann hypothesis which is depicted as Pseudo-zeroes to Zeroes conversion that obeys relevant trigonometric identities.

Three types of [traditionally] finite-interval Riemann Sums: Left / Right / Midpoint Riemann Sum uses left endpoints / right endpoints / midpoints of the subintervals. With n = 1, 2, 3,..., ∞ and therefore $\Delta n = 1$, we note f(n) can analogically be interpreted as approximate Area under the Curve (AUC) [right infinite-interval] Riemann sum $\sum_{n=1}^{\infty} f(n)\Delta n = \sum_{n=1}^{\infty} f(n)$ $= \sum_{n=1}^{2} f(n) + \sum_{n=3}^{4} f(n) + \sum_{n=5}^{6} f(n) + ... + \sum_{n=\infty-1}^{\infty} f(n)$. Corresponding solution to exact AUC improper integral $\int_{n=1}^{n=\infty} f(n)dn$ can be validly expanded as $\int_{n=1}^{n=2} f(n)dn + \int_{n=2}^{n=3} f(n)dn + \int_{n=3}^{n=4} f(n)dn + ... + \int_{n=\infty-1}^{n=\infty} f(n)dn = [F(n) + C]_1^2 + [F(n) + C]_2^3 + [F(n)+C]_3^4 + ... + [F(n)+C]_{\infty-1}^{\infty}$ which, for all sufficiently large n as n⟶ ∞, will manifest *divergence by oscillation* (viz. for all sufficiently large n as n⟶ ∞, this cummulative total will not diverge in a particular direction to a solitary well-defined limit value since the [complex] sine and/or cosine terms present in sim-η(s) and DSPL are periodic transcendental-type functions). Evaluation of definite integrals Eq. (3) or Eq. (10), Eq. (6) or Eq. (12) and Eq. (8) or Eq. (14) using limit as $n \rightarrow +\infty$ for $0 < t < +\infty$ enable countless computations resulting in t values for (respectively) CIS NTZ, CIS Gram[y=0] points and CIS Gram[x=0] points [all as Pseudo-zeroes to Zeroes conversion]. Larger n values used for computations will correspond to increasing accuracy of these entities.

Remark 12. Whereas exact AUC from F(n) given by DSPL = $\int_{n=1}^{n=\infty} sim-\eta(s)dn$ and approximate AUC from f(n) given by sim-η(s) = $\sum_{n=1}^{\infty}$sim-η(s) [when interpreted as Riemann sum] are proportional; the Zeroes when indirectly derived from DSPL [as Pseudo-zeroes converted to Zeroes] and the Zeroes when directly derived from sim-η(s) must agree with each other at $\sigma = \frac{1}{2}$ critical line.

8. Riemann zeta function, Dirichlet eta function, simplified Dirichlet eta function and Dirichlet Sigma-Power Law

$\zeta(s)$ is a function of complex variable s (= $\sigma \pm \imath t$) that analytically continues sum of infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$. The common convention is to write s as $\sigma + \imath t$ with $\imath = \sqrt{-1}$, and with σ and t real. Valid for $\sigma > 0$, we write $\zeta(s)$ as $Re\{\zeta(s)\}+\imath Im\{\zeta(s)\}$ and note that $\zeta(\sigma + \imath t)$ when $0 < t < +\infty$ is the complex conjugate of $\zeta(\sigma - \imath t)$ when $-\infty < t < 0$.

Also known as alternating zeta function, $\eta(s)$ must act as *proxy* for $\zeta(s)$ in critical strip (viz. $0 < \sigma < 1$) containing critical line (viz. $\sigma = \frac{1}{2}$) because $\zeta(s)$ only converges when $\sigma > 1$. This implies $\zeta(s)$ is undefined to left of this $\sigma > 1$ region [in the critical strip] which then requires $\eta(s)$ representation instead. They are related to each other as $\zeta(s) = \gamma \cdot \eta(s)$ with proportionality factor $\gamma = \dfrac{1}{(1-2^{1-s})}$ and $\eta(s) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^s} = \dfrac{1}{1^s} - \dfrac{1}{2^s} + \dfrac{1}{3^s} - \cdots$.

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\
&= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\
&= \Pi_{p\ prime} \frac{1}{(1-p^{-s})} \\
&= \frac{1}{(1-2^{-s})} . \frac{1}{(1-3^{-s})} . \frac{1}{(1-5^{-s})} . \frac{1}{(1-7^{-s})} . \frac{1}{(1-11^{-s})} \cdots \frac{1}{(1-p^{-s})} \cdots
\end{aligned} \tag{15}$$

Eq. (15) is defined for only $1 < \sigma < \infty$ region where $\zeta(s)$ is absolutely convergent with no zeros located here. In Eq. (15), equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] also represents $\zeta(s) \implies$ all prime and, by default, composite numbers are (intrinsically) encoded in $\zeta(s)$. Brief diversion: On April 17, 2013, Zhang announced a ground-breaking proof (Zhang, 2014) stating there are infinitely many pairs of prime numbers that differ by 70 million or less. This result implies the existence of an infinitely repeatable prime 2-tuple, thus establishing a theorem akin to the twin prime conjecture.

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \cdot \Gamma(1-s) \cdot \zeta(1-s) \tag{16}$$

With $\sigma = \frac{1}{2}$ as symmetry line of reflection, Eq. (16) is Riemann's functional equation valid for $-\infty < \sigma < \infty$. It can be used to find all trivial zeros on horizontal line at $\imath$t = 0 occurring when σ = -2, -4, -6, -8, -10,…, ∞ whereby $\zeta(s) = 0$ because factor $\sin(\dfrac{\pi s}{2})$ vanishes. Γ is gamma function, an extension of factorial function [a product function denoted by ! notation whereby n! $= n(n-1)(n-2) \ldots (n-(n-1))$] with its argument shifted down by 1, to real and complex numbers. That is, if n is a positive integer, $\Gamma(n) = (n-1)!$

$$\begin{aligned}
\zeta(s) &= \frac{1}{(1-2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\
&= \frac{1}{(1-2^{1-s})} \left(\frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \right)
\end{aligned} \tag{17}$$

Eq. (17) is defined for all $\sigma > 0$ values except for simple pole at $\sigma = 1$. As alluded to above, ζ(s) without $\dfrac{1}{(1-2^{1-s})}$ viz. $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^s}$ is η(s). It is a holomorphic function of s defined by analytic continuation and is mathematically defined at σ = 1 whereby analogous trivial zeros with presence for η(s) [but not for ζ(s)] on vertical straight line σ = 1 are found at $s = 1 \pm \imath \dfrac{2\pi k}{\ln(2)}$ where k = 1, 2, 3, 4,…, ∞.

Euler formula can be stated as $e^{\imath n} = \cos n + \imath \cdot \sin n$. Euler identity (where $n = \pi$) is $e^{\imath\pi} = \cos\pi + \imath \cdot \sin\pi = -1 + 0$ [or stated as $e^{\imath\pi} + 1 = 0$]. The n^s of $\zeta(s)$ is expanded to $n^s = n^{(\sigma+\imath t)} = n^\sigma \mathrm{e}^{t\ln(n)\cdot\imath}$ since $n^t = e^{t\ln(n)}$. Apply Euler formula to n^s result in $n^s = n^\sigma(\cos(t\ln(n)) + \imath \cdot \sin(t\ln(n))$. This is written in trigonometric form [designated by short-hand notation $n^s(Euler)$] whereby n^σ is modulus and $t\ln(n)$ is polar angle (argument).

We apply $n^s(Euler)$ to Eq. (17) to obtain f(n) general sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ NTZ versus (non-existent) $\sigma \neq \frac{1}{2}$ virtual NTZ (Ting, 2020, section 4, p. 24 - 28). At $\sigma = \frac{1}{2}$, this is given as Eq. (9) and with the trigonometric identity application as Eq. (1). Integrate f(n) general sim-$\eta(s)$ to obtain F(n) general DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeroes versus (non-existent) $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeroes. Pseudo-zeroes and (non-existent) virtual Pseudo-zeroes can be converted to Zeroes (NTZ) and (non-existent) virtual Zeroes (virtual NTZ). At $\sigma = \frac{1}{2}$, this is given as Eq. (10) and with the trigonometric identity application as Eq. (3).

We provide f(n) general Gram[y=0] points-sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ Gram[y=0] points versus $\sigma \neq \frac{1}{2}$ virtual Gram[y=0] points (Ting, 2020, section 5, p. 28 - 30). At $\sigma = \frac{1}{2}$, this is given as Eq. (11) but we are unable to apply trigonometric identity. Integrate f(n) general Gram[y=0] points-sim-$\eta(s)$ to obtain F(n) general Gram[y=0] points-DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeroes versus $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeroes. Pseudo-zeroes and virtual Pseudo-zeroes can be converted to Zeroes (Gram[y=0] points) and virtual Zeroes (virtual Gram[y=0] points). At $\sigma = \frac{1}{2}$, this is given as Eq. (12) and with the trigonometric identity application as Eq. (6).

We provide f(n) general Gram[x=0] points-sim-$\eta(s)$ for determining $\sigma = \frac{1}{2}$ Gram[x=0] points versus $\sigma \neq \frac{1}{2}$ virtual Gram[x=0] points (Ting, 2020, section 5, p. 28 - 30). At $\sigma = \frac{1}{2}$, this is given as Eq. (13) but we are unable to apply trigonometric identity. Integrate f(n) general Gram[x=0] points-sim-$\eta(s)$ to obtain F(n) general Gram[x=0] points-DSPL for determining $\sigma = \frac{1}{2}$ Pseudo-zeroes versus $\sigma \neq \frac{1}{2}$ virtual Pseudo-zeroes. Pseudo-zeroes and virtual Pseudo-zeroes can be converted to Zeroes (Gram[x=0] points) and virtual Zeroes (virtual Gram[x=0] points). At $\sigma = \frac{1}{2}$, this is given as Eq. (14) and with the trigonometric identity application as Eq. (8).

9. Compare and Contrast Riemann hypothesis versus Polignac's and Twin prime conjectures

We provide the following succinct exposition on Riemann hypothesis, Polignac's and Twin prime conjectures.

The rigorous proof for the intractable open problem in Number theory of Riemann hypothesis concerns the unique location of nontrivial zeros (as one type of Gram points) given as equation-type proof with associated explanations for the closely-related two types of Gram points. The rigorous proofs for the intractable open problems in Number theory of Polignac's and Twin prime conjectures concern the infinite in magnitude properties of odd prime numbers and even number prime gaps given as algorithm-type proofs.

Derived from one common [equation-type] Riemann zeta function when its parameter $\sigma = \frac{1}{2}$; the three mathematically-depicted sets of nontrivial zeros (or Gram[x=0,y=0] points), Gram[y=0] points and Gram[x=0] points will precisely correspond to (respectively) the three geometrically-depicted sets of Origin intercept points, x-axis intercept points and y-axis intercept points. These three dependent sets are infinite in magnitude with all their associated entities constituted by t parameter as transcendental numbers except for the very first Gram[y=0] point which is constituted by t parameter as integer 0. The critical strip is defined as parameter σ having values between 0 and 1. The 1859 Riemann hypothesis proposed all nontrivial zeros in this function are uniquely located on its critical line which is simply defined as parameter σ having the value $\frac{1}{2}$. Alternatively, Riemann hypothesis proposed all nontrivial zeros in this function are not located on its infinitely many non-critical lines which are defined as parameter σ not having the value $\frac{1}{2}$ (viz, parameter σ endowed with values in the critical strip between 0 and $\frac{1}{2}$, and between $\frac{1}{2}$ and 1).

The set of natural numbers 1, 2, 3, 4, 5,... is constituted by the set containing solitary integer 1, the set containing all prime numbers 2, 3, 5, 7, 11,... and the set containing all composite numbers 4, 6, 8, 9, 10,... whereby the first, third and fourth sets are all infinite in magnitude with all four sets dependently related to each other. We define Complement-Sieve-of-Eratosthenes as all the remaining natural numbers [apart from integer 1] obtained after Sieve-of-Eratosthenes faithfully generates all the prime numbers – these numbers will now constitute all the composite numbers. Respectively, prime numbers and composite numbers are generated dependently by [algorithm-type] Sieve-of-Eratosthenes and [algorithm-type] Complement-Sieve-of-Eratosthenes. The first prime number 2 is the only even prime number having the unique solitary odd number prime gap 1. The remaining prime numbers 3, 5, 7, 11, 13,... are all odd prime numbers endowed with even number prime gaps. The 1849 Polignac's conjecture proposed even number prime gaps 2, 4, 6, 8, 10,... are infinite in magnitude with each even number prime gap generating unique odd prime numbers which are infinite in magnitude. The 1846 Twin prime conjecture, which is a subset of Polignac's conjecture, proposed even number prime gap 2 will generate unique odd prime numbers which are infinite in magnitude. All composite numbers with associated composite gaps of 1 or 2 are infinite in magnitude constituted by unique even numbers and odd numbers which are both infinite in magnitude.

In summary, we provide rigorous proofs for open problems in Number theory of Riemann hypothesis [concerning the unique location of nontrivial zeros given as equation-type proof with associated explanations for the closely-related two types of Gram points], Polignac's and Twin prime conjectures [concerning the infinite in magnitude properties of odd prime numbers and even number prime gaps given as algorithm-type proofs]. We additionally note that proofs for Polignac's and Twin prime conjectures that involves prime numbers must inevitably and complementarily also involve composite numbers.

10. Conclusions

Previously regarded as **primary spin-offs** in his anniversary research paper (Ting, 2020), correct and complete mathematical arguments for solving the 1859 Riemann hypothesis, and explaining the closely related Gram[y=0] points and

Gram[x=0] points, can inherently be classified as belonging to Mathematics for Incompletely Predictable problems.

"*With this one solution* [for Riemann hypothesis]*, we have proven five hundred theorems or more at once*". Previously regarded as **secondary spin-offs** (Ting, 2020) arising out of solving Riemann hypothesis, this profound statement apply to many important theorems in Number theory (mostly on prime numbers) that rely on properties of Riemann zeta functions such as where trivial zeros and nontrivial zeros are / are not located.

Derived innovative *Fic-Fac Ratio* was previously regarded as **tertiary spin-offs** (Ting, 2020) serving as medical or epidemiological tool to assist understanding of SARS-CoV-2 causing COVID-19 and 2020 Coronavirus Pandemic. Unprecedented negative global health and economic impacts have arised from this event. Fic-Fac Ratio connects seemingly unrelated subject of Medicine with frontier Mathematics from Number theory.

There are concrete analogies between the Completely Predictable entities even and odd numbers [that are all located on unique 'linear' lines] versus the Incompletely Predictable entities prime and odd numbers, NTZ, Gram[y=0] points and Gram[x=0] points [that are all located on unique 'non-linear' lines]. We can either conceptionally or mathematically derive valid intrinsic properties such as the actual gaps/intervals between any two adjacent entities, and the various slope/gradient (involving the calculus of differentiation) and Area-under-the-Curve (involving the calculus of integration) of these lines which will all be given by continuous functions that are always defined for any arbitrarily chosen intervals [a,b] except the following. The corresponding lines computed from complex algorithms that generate all prime and composite numbers are only defined at two end-points a,b but not for interval [a,b] as these algorithms are simply not well-defined functions. We immediately recognize these complex algorithms [which are not functions] are not amendable to differentiation or integration. Then as succinctly outlined below, the previously published quantitative and qualitative rigorous algorithm-type proofs (Ting, 2020) for Polignac's and Twin prime conjectures cannot be stated using functions.

Quantitative proof: We validly exclude first and only even prime number (**P**) '2', and show from following mathematical arguments that Polignac's and Twin prime conjectures are true with appearance of $\aleph_0$ cardinality 'uniformity' conforming to Dimensional analysis homogeneity. Let (i) cardinality T = $\aleph_0$ for Set **all odd P** derived from even number (**E**) prime gaps 2, 4, 6,..., ∞, (ii) cardinality $T_2 = \aleph_0$ for Subset **odd P** derived from **E** prime gap 2, cardinality $T_4 = \aleph_0$ for Subset **odd P** derived from **E** prime gap 4, cardinality $T_6 = \aleph_0$ for Subset **odd P** derived from **E** prime gap 6, etc. Paradoxically, (as sets) $T = T_2 + T_4 + T_6 + ... + T_\infty$ equation is valid despite (their cardinality) $T = T_2 = T_4 = T_6 = ... = T_\infty$; and **E** prime gaps are 'infinite in magnitude' can justifiably be perceived instead as 'arbitrarily large in magnitude' since cumulative sum total of **E** prime gaps is relatively much slower to attain the 'infinite in magnitude' status when compared to cumulative sum total of **P** which rapidly attain this status.

Qualitative proof: Plus-Minus Gap 2 Composite Number Alternating Law refers to some sort of "rhythmic patterns of alternating presence and absence" for relevant Gap 2 Composite Numbers. It has built-in intrinsic mechanism to automatically generate all the prime numbers from prime gaps ≥ 4 in a mathematically consistent *ad infinitum* manner. Plus Gap 2 Composite Number Continuous Law refers to the special situation of "(non-)rhythmic patterns with continual presence" for relevant Gap 2 Composite Numbers. It has built-in intrinsic mechanism to automatically generate all the prime numbers from prime gap = 2 appearances in a mathematically consistent *ad infinitum* manner. These two deduced Laws ***"that must crucially involve both prime and composite numbers being dependently and algorithmically tabulated together with subsequent analysis on their [consequently combined] corresponding gaps"*** will qualitatively confirm Polignac's and Twin prime conjectures to be true.

In this paper, we have intrinsically treated and analyzed in a *de novo* fashion simple and complex single-variable function f(n) or F(n) and their simple and complex single-variable equation f(n) = 0 or F(n) = 0 as Completely Predictable or Incompletely Predictable mathematical objects. Being mutually exclusive and Incompletely Predictable entities with $\sigma = \frac{1}{2}$ critical line depicted in Figure 2, and $\sigma \neq \frac{1}{2}$ non-critical lines as exemplified by $\sigma = \frac{2}{5}$ depicted in Figure 3 and $\sigma = \frac{3}{5}$ depicted in Figure 4; **the $\sigma = \frac{1}{2}$ NTZ computed from Eq. (1) – $\sigma \neq \frac{1}{2}$ (non-existent) virtual NTZ computed from Eq. (2) Pairing** outlined at the end of section 3 mathematically serve to distinguish and separate the unique complete set of nontrivial zeros from the infinitely many non-unique complete sets of (non-existent) virtual nontrivial zeros. The critical line of Riemann zeta function is denoted by $\sigma = \frac{1}{2}$ whereby all nontrivial zeros are proposed to be located in the 1859 Riemann hypothesis. Our Dirichlet Sigma-Power Law symbolizes the end-product proof on Riemann hypothesis.

We reiterate the following important criteria: The three types (three separate "containers") of Gram points at $\sigma = \frac{1}{2}$ and two types (two separate "containers") of virtual Gram points at $\sigma \neq \frac{1}{2}$ are labelled together as *Zeroes*. After performing integration on relevant $f(n)$ resulting in $F(n)$, we obtain corresponding three types (three separate "containers") of Pseudo-Gram points at $\sigma = \frac{1}{2}$ and two types (two separate "containers") of virtual Pseudo-Gram points at $\sigma \neq \frac{1}{2}$ which are labelled together as *Pseudo-zeroes*.

With groundings in *Mathematics for Incompletely Predictable problems*, we advocate that we have now provided a comparatively elementary and rigorous equation-type proof on Riemann hypothesis while explaining existence of mutually exclusive three types of [Incompletely Predictable] Gram points and two types of [Incompletely Predictable] virtual Gram points. These achievements are completed with appropriate analysis on complex (meta-) properties present in Dirichlet Sigma-Power Law, Gram[y=0] points-Dirichlet Sigma-Power Law and Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant Pseudo-Gram points; and in virtual Gram[y=0] points-Dirichlet Sigma-Power Law and virtual Gram[x=0] points-Dirichlet Sigma-Power Law that give rise to relevant virtual Pseudo-Gram points. Exact Dimensional analysis homogeneity [occurring only once at $\sigma = \frac{1}{2}$ critical line] in these Laws is endowed with ability to convert their computed Pseudo-zeroes to Zeroes resulting in nontrivial zeros (Origin intercept points or Gram[x=0,y=0] points) as one type of Gram points plus Gram[y=0] points and Gram[x=0] points as two remaining types of Gram points. Inexact Dimensional analysis homogeneity [occurring infinitely often at $\sigma \neq \frac{1}{2}$ non-critical lines] in these Laws is endowed with ability to convert their computed virtual Pseudo-zeroes to virtual Zeroes resulting in virtual Gram[y=0] points and virtual Gram[x=0] points as two types of virtual Gram points.

Conflict of Interest Statement
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Appendix

A. Gram's Law and Rosser Rule for Gram points

Named after Danish mathematician Jørgen Pedersen Gram (June 27, 1850 – April 29, 1916), ['traditional'/'usual'] Gram points or (mathematical) Gram[y=0] points or (geometrical) x-axis intercept points are other conjugate pairs values in Riemann zeta function $\zeta(s)$ on $\sigma = \frac{1}{2}$ critical line. Then s = $\frac{1}{2} + \imath t$ gives rise to $\zeta(\frac{1}{2} + \imath t)$ on critical line; and Gram points when defined in terms of $\zeta(s)$ is given by $\sum ReIm\{\zeta(s)\} = Re\{\zeta(s)\} + 0$, or simply $\text{Im}\{\zeta(s)\} = 0$. Alternatively defined using expression denoting $\zeta(s)$ on critical line $\zeta(\frac{1}{2} + \imath t) = Z(t)e^{-\imath\theta(t)}$ whereby Hardy's function, Z, is real for real t, and θ is Riemann–Siegel theta function given in terms of gamma function as $\theta(t) = \arg\left(\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\right) - \frac{\log \pi}{2} t$ for real values of t; we note that $\zeta(s)$ is real when sin(θ(t)) = 0. This implies that θ(t) is an integer multiple of π which allows for location of Gram points to be calculated easily by inverting the formula for θ. Gram points are historically [crudely] numbered as g_n for n = 0, 1, 2, 3,..., whereby g_n is the unique solution of θ(t) = nπ. Here, n = 0 is the [first] g_0 value of 17.8455995405... which is larger than the smallest [first] positive nontrivial zeros (NTZ) value of 14.13472515.... Thus, n = -3 correspond to $g_{-3} = 0$, n = -2 correspond to $g_{-2} = 3.4362182261...$, and n = -1 correspond to $g_{-1} = 9.6669080561....$

Paired [infinite-length] integer sequences with prestigious connections:
A100967+0, which is A100967 (Noe, 2004), is precisely defined as "Least k such that binomial(2k+1, k-n-1) ≥ binomial(2k, k) viz. (2k+1)!k!k! ≥ (2k)!(k-n-1)!(k+n+2)!". The terms commencing from Position 0, 1, 2, 3,... of A100967+0 are listed below: 3, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 225, 263, 303, 347, 393, 442, 494, 549, 606, 667, 730, 797, 866, 938, 1013, 1091, 1172, 1255, 1342, 1431, 1524, 1619, 1717, 1818, 1922, 2029, 2138, 2251, 2366, 2485, 2606, 2730, 2857, 2987, 3119, 3255, 3394, 3535,....

A100967+1 is precisely defined as "Add 1 to each and every terms from A100967+0". The terms commencing from Position 0, 1, 2, 3,... of A100967+1 are listed below: 4, 10, 19, 30, 45, 62, 82, 105, 131, 160, 192, 226, 264, 304, 348, 394, 443, 495, 550, 607, 668, 731, 798, 867, 939, 1014, 1092, 1173, 1256, 1343, 1432, 1525, 1620, 1718, 1819, 1923, 2030, 2139, 2252, 2367, 2486, 2607, 2731, 2858, 2988, 3120, 3256, 3395, 3536,....

A228186 (Ting, 2013) is precisely defined as "Smallest natural number k > n such that (k+n+1)!(k-n-2)! < 2k!(k-1)!" or alternatively defined as "Greatest natural number k > n such that calculated peak values for ratio R = $\frac{CombinationsWithRepetition}{CombinationsWithoutRepetition} = \frac{(k+n-1)!(n-k)!}{n!(n-1)!}$ belong to maximal rational numbers < 2". The terms commencing from Position 0, 1, 2, 3,... of A228186 are listed below: 4, 9, 18, 29, 44, 61, 81, 104, 130, 159, 191, 226, 263, 304, 347, 393, 442, 494, 549, 607, 667, 731, 797, 866, 938, 1013, 1091, 1172, 1256, 1342, 1432, 1524, 1619, 1717, 1818, 1922, 2029, 2139, 2251, 2367, 2485, 2606, 2730, 2857, 2987, 3120, 3255, 3394, 3535,....

Unexpected connection [and unrelated to NTZ and Gram points]: A228186 can be considered an innovative [infinite-length] *"Hybrid integer sequence"* identical to *"non-Hybrid integer sequence"* A100967+0 except for the interspersed [finite] 21 'exceptional' terms located at Position 0, 11, 13, 19, 21, 28, 30, 37, 39, 45, 50, 51, 52, 55, 57, 62, 66, 70, 73, 77, and 81 with their corresponding 21 values exactly specified by [infinite-length] *"non-Hybrid integer sequence"* A100967+1.

A114856-"bad"-Gram-points, which is A114856 (Weisstein, 2006), is precisely defined as "Indices n of Gram points g_n for which $(-1)^n Z(g_n) < 0$ with Z(t) being Riemann-Siegel Z-function and full given range of values n = 0, 1, 2, 3,...". The terms of A114856-"bad"-Gram-points are listed below: 126, 134, 195, 211, 232, 254, 288, 367, 377, 379, 397, 400, 461, 507, 518, 529, 567, 578, 595, 618, 626, 637, 654, 668, 692, 694, 703, 715, 728, 766, 777, 793, 795, 807, 819, 848, 857, 869, 887, 964, 992, 995, 1016, 1028, 1034, 1043, 1046, 1071, 1086,....

A114856-"good"-Gram-points, given by "total"-Gram points minus A114856-"bad"-Gram-points, is precisely defined as "Indices n of Gram points g_n for which $(-1)^n Z(g_n) > 0$ with Z(t) being Riemann-Siegel Z-function and full given range of values n = 0, 1, 2, 3,...". The derived terms of A114856-"good"-Gram-points: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50,....

A216700 (Greathouse IV, 2012) is precisely defined as "Violations of Rosser Rule: numbers n such that the Gram block [g_n, g_{n+k}] contains fewer than k points t such that Z(t) = 0 with Z(t) being Riemann-Siegel Z-function and full given range of values n = 0, 1, 2, 3,...". The terms of A216700 are listed below: 13999525, 30783329, 30930927, 37592215, 40870156, 43628107, 46082042, 46875667, 49624541, 50799238, 55221454, 56948780, 60515663, 61331766, 69784844, 75052114, 79545241, 79652248, 83088043, 83689523, 85348958, 86513820, 87947597,....

Expected connection: All NTZ (as conjectured by Riemann hypothesis) and Gram points (by definition) are located on the same critical line of Riemann zeta function. Counting NTZ can be validly reduced to counting all Gram points where Gram's Law is satisfied and adding count of NTZ inside each Gram block. With this process, we need not locate NTZ but just have to accurately compute Z(t) to show that it changes sign.

Gram's Law is the observation that there is [usually] exactly one NTZ (Gram[x=0,y=0] points or Origin intercept points) between any two "good" Gram points. Examples of closely related statements equivalent to Gram's law are: $(-1)^n Z(g_n)$ is [usually] positive or Z(t) [usually] has opposite sign at consecutive Gram points. Thus, a t-valued Gram point is called a "good" Gram point if $\zeta(s)$ is positive at $\frac{1}{2} + \imath t$ with $(-1)^n Z(g_n) > 0$ and a "bad" Gram point if $\zeta(s)$ is negative at $\frac{1}{2} + \imath t$ with $(-1)^n Z(g_n) < 0$. The indices of "bad" Gram points where Z has the 'wrong' sign are given by A114856 in OEIS. A Gram block $[g_n, g_{n+k}]$ is a half-open interval bounded by two "good" Gram points g_n and g_{n+k} such that all Gram points g_{n+1},..., g_{n+k-1} between them are "bad" Gram points. A refinement of Gram's Law is known as Rosser Rule (Rosser, Yohe & Schoenfeld, 1969) which stated that Gram blocks [usually] have the expected number of NTZ in them (identical to number of Gram intervals), even though some of the individual Gram intervals in the block may not have exactly one NTZ in them. Example, the interval bounded by g_{125} and g_{127} is a Gram block containing a unique "bad" Gram point g_{126} and the expected number 2 of NTZ although neither of its two Gram intervals contains a unique NTZ.

Gram's Law and Rosser Rule both imply that in some sense NTZ do not stray too far from their expected positions, and that they hold most of the time but are violated infinitely often (in an Incompletely Predictable manner) (Trudgian, 2011 & Trudgian, 2014). Professor Timothy Trudgian in 2011 explicitly showed that both Gram's Law and Rosser Rule fail in a positive proportion of cases. In particular, it is expected that in about 73% [$\approx\frac{3}{4}$] one NTZ is enclosed by two successive Gram points [and thus Gram's Law fails for about 27% [$\approx\frac{1}{4}$] of all Gram intervals to contain exactly one NTZ], but in about 14% no NTZ and in about 13% two NTZ are in such a Gram interval on the long run.

B. Miscellaneous Materials

Cardinality: With increasing size, arbitrary Set **X** can be CFS, CIS or UIS. Cardinality of Set **X**, |**X**|, measures *number of elements* in Set **X**. E.g. Set **negative Gram[y=0] point** has CFS of negative Gram[y=0] point with |**negative Gram[y=0] point**| = 1, Set **even Prime number** has CFS of even **Prime number** with |**even Prime number**| = 1, Set **Natural numbers** has CIS of **Natural numbers** with |**Natural numbers**| = $\aleph_0$, and Set **Real numbers** has UIS of **Real numbers** with |**Real numbers**| = $\mathfrak{c}$ (cardinality of the continuum). Let $\mathbb{C}$ = UIS complex numbers, **R** = UIS real numbers, **Q** = CIS rational numbers that include fractional numbers and rational roots, **R-Q** = UIS total irrational numbers, **A** = CIS algebraic numbers, **R-A** = UIS transcendental irrational numbers, **Z** = CIS integers which are literally fractional numbers with denominator 1, **W** = CIS whole numbers, **N** = CIS natural numbers, **E** = CIS even numbers, **O** = CIS odd numbers, **P** = CIS prime numbers, and **C** = CIS composite numbers. CIS **N** = Set **E** [whereby we did not include the zeroth even number $E_0 = 0$] + Set **O**; CIS **N** = CIS **P** + CIS **C** + CFS Number 1; and CIS **N** $\subset$ CIS **W** $\subset$ CIS **Z** $\subset$ CIS **Q** $\subset$ UIS **R** $\subset$ UIS $\mathbb{C}$. CIS **A** as $\mathbb{C}$ (including **R**) = CIS **Q** that include fractional numbers and rational roots + CIS irrational roots whereby both rational and irrational roots are derived from non-zero polynomials.

The following refined definitions are useful: UIS total irrational numbers = CIS irrational roots (numbers) + UIS transcendental irrational numbers whereby transcendental irrational numbers $\gg$ [algebraic] irrational numbers. Whereas CIS rational roots (numbers), CIS irrational roots (numbers) and UIS transcendental numbers are treated separately as mutually exclusive numbers; so must the existing algebraic functions that generate CIS rational roots (numbers) and CIS irrational roots (numbers), and the existing transcendental functions that generate UIS transcendental numbers be treated separately as mutually exclusive functions.

Certain types of infinite series: An algebraic function [such as rational functions, square root, cube root function, etc] satisfies a polynomial equation. A transcendental function [such as exponential function, natural logarithm, trigonometric functions, hyperbolic functions, gamma, elliptic, zeta functions, etc] is an analytic function that does not satisfy a polynomial equation. Thus a transcendental function "transcends" algebra since it cannot be expressed in terms of a finite sequence of algebraic operations consisting of addition, subtraction, multiplication, division, powers, and fractional powers or root extraction. All integers, rational numbers, rational or irrational roots of real and complex numbers are algebraic numbers e.g. a root of polynomial $x^2 - x - 1$ = golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2} = 1.618033\ldots$, square root of 2 viz, $\sqrt[2]{2}$ or $\sqrt{2} = 2^{\frac{1}{2}} = 1.414213...$, or cube root of 2 viz, $\sqrt[3]{2} = 2^{\frac{1}{3}} \approx 1.259921$. Real and complex numbers that are not algebraic numbers e.g. π and e are transcendental numbers. However, we note sine and cosine as transcendental functions generally give rise to mutually exclusive sets of transcendental numbers except at discrete points such as $\sin\frac{\pi}{6} = \sin 30^\circ = \cos\frac{2\pi}{6} = \cos 60^\circ = \frac{\sqrt{1}}{2} = \frac{1}{2}$ [viz, transcendental functions generating an algebraic number as rational root (number) at certain discrete points].

Following [side-note] treatise of interest involve infinite series. A property of irrational number $\sqrt{2}$ is $\frac{1}{\sqrt{2}-1} = \sqrt{2}+1$ since $\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right) = 2-1 = 1$. This is related to the property of silver ratios. $\sqrt{2}$ can also be expressed in terms of copies of imaginary unit i using only square root and arithmetic operations, if the square root symbol is interpreted suitably for complex numbers i and -i: $\frac{\sqrt{i}+i\sqrt{i}}{i}$ and $\frac{\sqrt{-i}-i\sqrt{-i}}{-i}$. Multiplicative inverse (reciprocal) of $(2)^{\frac{1}{2}}$ or $\sqrt{2}$ is $(2)^{-\frac{1}{2}}$ or $\sqrt{\frac{1}{2}}$ which is a unique [irrational number] constant since $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2} = cos\frac{\pi}{4} = sin\frac{\pi}{4}$. Transcendental numbers such as $\frac{\pi}{4}$ (given by Leibniz series $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \approx 0.78539816$); and $\frac{\pi^2}{6}$ (given by $\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \approx 1.6449340668482$), respectively, encode complete set of alternating odd and, by default, alternating even numbers; and natural numbers. Also known as alternating zeta function, Dirichlet eta function $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$ when expanded, will intrinsically encode complete set of alternating natural numbers e.g. $\eta(1) = ln(2)$ (given by $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \sum_{n=2}^{\infty} \frac{1}{2^n}[\zeta(n)-1] + \frac{1}{2} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \approx 0.69314718056$). Equivalent Euler product formula for $\zeta(s)$ with product over prime numbers [instead of summation over natural numbers] will intrinsically encode complete set of prime and, by default, composite numbers. As an extra point, complete set of alternating prime and, by default, alternating composite numbers is encoded in converging alternating series $\sum_{k=1}^{\infty} \frac{(-1)^k}{p_k} \approx -0.2696063519$ (transcendental number) when fully expanded whereby p_k is kth prime number.

Zeroes and Pseudo-zeroes: There are three types of stationary points in a given [simple] periodic f(n) involving sine and/or cosine functions that act as x-axis intercept points via three types of f(n)'s Zeroes with corresponding three types of F(n)'s Pseudo-zeroes: maximum points e.g. with f(n) or F(n) = sin n - 1; minimum points e.g. with f(n) or F(n) = sin n + 1; and points of inflection e.g. with f(n) or F(n) = sin n [*which also has Origin intercept point as a Zero or Pseudo-zero*]. A fourth type of f(n)'s Zeroes and F(n)'s Pseudo-zeroes consist of non-stationary points occurring e.g. with f(n) or F(n) = sin n + 0.5. One can analogically assimilate these concepts to aesthetically explain the more "exotic" characteristics manifested by [complex] periodic f(n) or F(n) involving sine and/or cosine functions that are present in f(n) sim-η(s) or F(n) DSPL at (solitary) $\sigma = \frac{1}{2}$ critical line and (infinitely many) $\sigma \neq \frac{1}{2}$ non-critical lines.

With (j - i) = (l - k) = 2π [viz, one Full cycle], let a given Zero be located in f(n)'s interval [i,j] viz, i < Zero < j; and its corresponding Pseudo-zero be located in F(n)'s Pseudo-interval [k,l] viz, k < Pseudo-zero < l. For this Zero and Pseudo-zero characterized by either point of inflection or non-stationary point; both will comply with preserving positivity [going from (-ve) below x-axis to (+ve) above x-axis] as explained using the Zero case [with the Pseudo-zero case following similar lines of explanations]. This can be stated as follow for interval [i,j]: If j > i, then computed f(j) > computed f(i). In particular, the condition "If i ≥ 0, then computed f(i) ≥ 0" must not be present for these two particular types of Zero to validly exist in interval [i,j]. With reversal of inequality signs, converse situation for j < Zero < i and corresponding l < Pseudo-zero < k is equally true in preserving negativity [going from (+ve) above x-axis to (-ve) below x-axis]. These are useful properties on Zeroes and Pseudo-zeroes.

Preservation or conservation of Net Area Value and Total Area Value with definitions (Ting, 2020, p. 10 - 13): $\int f(n)dn = F(n) + C$ with $F'(n) = f(n)$. Consider a nominated function $f(n)$ for interval [a,b]. We define Net Area Value (NAV) calculated using its antiderivative $F(n)$ as the net difference between positive area value(s) [above horizontal x-axis] and negative area value(s) [below horizontal x-axis] in interval [a,b]; viz, NAV = all +ve value(s) + all -ve value(s). Again calculated using $F(n)$, we define Total Area Value (TAV) as the total sum of (absolute value) positive area value(s) [above horizontal x-axis] and (absolute value) negative area value(s) [below horizontal x-axis] in interval [a,b]; viz, TAV = all |+ve value(s)| + all |-ve value(s)|. Calculated NAV and TAV are precise using antiderivative $F(n)$ obtained from integration of $f(n)$ but are only approximate when using Riemann sum on $f(n)$. For f(n)'s interval [a,b] whereby a = initial Zero and b = next Zero, and F(n)'s Pseudo interval [c,d] whereby c = initial Pseudo-zero and d = next Pseudo-zero;

then compliance with preservation or conservation of NAV and TAV will simultaneously occur in both f(n)'s Zeroes and F(n)'s Pseudo-zeroes given by their sine and/or cosine functions only when Zero gap = (b - a) = Pseudo-zero gap = (d - c) = 2π [viz, involving one Full cycle]. For our purpose, NAV = 0 condition is validly preserved or conserved for f(n) sim-η(s)'s IP Zeroes and F(n) DSPL's IP Pseudo-zeroes at parameter $\sigma = \frac{1}{2}$. *Ditto* for f(n) sim-η(s)'s IP virtual Zeroes and F(n) DSPL's IP virtual Pseudo-zeroes at parameter $\sigma \neq \frac{1}{2}$; viz, NAV = 0 condition is validly preserved or conserved for f(n) sim-η(s)'s IP virtual Zeroes and F(n) DSPL's IP virtual Pseudo-zeroes.

For single-term trigonometric function f(n) = sin(n), it is an odd function with Origin symmetry since -f(n) = f(-n) for all n. The f(n) = sin(n) has an infinite number of CP x-axis intercept points (Zeroes) and a solitary unique Origin intercept point (Zero) since it belong to a class of odd functions that is defined at n = 0 and must pass through the Origin. Otherwise, the other class of odd functions such as f(n) = sin($\frac{1}{n}$) with infinite number of CP x-axis intercept points (Zeroes) but without Origin intercept point [since sin($\frac{1}{n}$) is undefined at n = 0] can remain symmetrical about the Origin without actually passing through it. For single term trigonometric function f(n) = cos(n) with symmetry about the y-axis, it is an even function since f(n) = f(-n) for all n. It has an infinite number of CP x-axis intercept points (Zeroes). Being undefined at n = 0, it will never have Origin intercept point.

For dual terms trigonometric functions f(n) = cos(n) - sin(n) and f(n) = cos(n) + sin(n), they are neither even nor odd without any symmetry. They both have an infinite number of CP x-axis intercept points (Zeroes). Being undefined at n = 0, they will never have Origin intercept point. Special properties for Addition and Multiplication: The sum or difference of two even functions is even. The sum or difference of two odd functions is odd. The sum or difference of an even and odd function is neither even nor odd unless one function is zero; viz, there is (exactly) one function that is both even and odd, and it is the zero function f(n) = 0. The product of two even functions is an even function. The product of two odd functions is an even function. The product of an even function and an odd function is an odd function.

Trigonometric identity for the linear combination of sine and cosine functions: Here, we again use simple single-variable function $f(n)$ or $F(n)$. The trigonometric identity for linear combination of sine and cosine a$cos(n)$ + b$sin(n)$ can be freely, arbitrarily and interchangeably written as either [simple] single cosine wave R$cos(n-\alpha)$ or [simple] single sine wave R$sin(n+\alpha)$ whereby R is the scaled amplitude and α is the phase shift. R = $\sqrt{a^2+b^2} = (a^2+b^2)^{\frac{1}{2}}$. Since sin($\alpha$) = $\frac{b}{\sqrt{a^2+b^2}}$ and cos(α) = $\frac{a}{\sqrt{a^2+b^2}}$, then $\alpha = \tan^{-1}\frac{b}{a}$. Below, we assign $\sqrt{2}$ to equivalently denote $2^{\frac{1}{2}}$.

With a = 1, b = -1, R = $\sqrt{2}$; $\cos(n) - \sin(n) = \sqrt{2}\cos\left(n + \frac{1}{4}\pi\right) = \sqrt{2}\sin\left(n + \frac{3}{4}\pi\right)$.

With a = -1, b = 1, R = $\sqrt{2}$; $-\cos(n) + \sin(n) = \sqrt{2}\sin\left(n - \frac{1}{4}\pi\right) = \sqrt{2}\cos\left(n - \frac{3}{4}\pi\right)$.

With a = 1, b = 1, R = $\sqrt{2}$; $\cos(n) + \sin(n) = \sqrt{2}\cos\left(n - \frac{1}{4}\pi\right) = \sqrt{2}\sin\left(n + \frac{1}{4}\pi\right)$.

With a = -1, b = -1, R = $\sqrt{2}$; $-\cos(n) - \sin(n) = \sqrt{2}\cos\left(n + \frac{3}{4}\pi\right) = \sqrt{2}\sin\left(n - \frac{3}{4}\pi\right)$.

$\int f(n)dn = F(n) + C$ with $F'(n) = f(n)$. With |a| = 1 and |b| = 1, consider single-term [simple] trigonometric functions: $f(n)$ = a$cos(n)$ which belongs to an even function and $f(n)$ = b$sin(n)$ which belongs to an odd function. Whereas all linear combination of [simple] $cos(n)$ and [simple] $sin(n)$ as sum or difference such as $f(n)$ = cos(n) + sin(n) and $f(n)$ = cos(n) - sin(n) belong to neither even nor odd functions, then their corresponding $F(n)$ being linear combination of [simple] $cos(n)$ and [simple] $sin(n)$ as sum or difference must also belong to neither even nor odd functions. With both f(n) and corresponding F(n) considered as [simple] functions and relevant trigonometric identities being applied, they can intrinsically and arbitrarily be expressed as either [simple] single cosine wave or [simple] single sine wave containing a phase shift $\frac{1}{4}\pi$ or $\frac{3}{4}\pi$ and a scaled amplitude $\sqrt{2}$ [= $2^{\frac{1}{2}}$ which is base 2 endowed with exponent $\frac{1}{2}$]. Respectively, F(n) and f(n) have an infinite number of x-axis intercept points called Pseudo-zeroes and Zeroes but nil Origin intercept points.

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