The Digits of Infinity

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Abstract: An original method for calculating the numerical digits of infinity is presented, based on the symmetry of the regular infinity-sided polygon (known as the regular apeirogon). The first actual infinite number is found to be an even number, a power of two, and the last few digits are calculated to be ...432948736. The last thousand digits of infinity are presented.

Introduction: What is infinity?

For as long as humans have contemplated infinity, it has been filled with mystery and paradox. But this does not need to be the case. We can come to understand the properties of infinity in a way that is coherent and consistent without contradictory definitions and without features that are impossible for other numbers.

Early on, Aristotle made the vital distinction between the never ending, unlimited *potential infinity*, and a total, completed *actual infinity* [1]. We will be dealing with actual infinity, an actual number, a total amount, a number with an exact value, not one less or one more than this exact value. By infinity, we mean the first infinite number. Cantor's *transfinite numbers* introduced the notion of a first infinite number, followed by several higher orders of infinity, each infinitely larger than the previous one, distinct from some highest *absolute infinity* which is not subject to further increase [2] However, the transfinite numbers of Cantor, and later, Zermelo and Fraenkel, have contradictory foundations with very few if any actual number theoretic properties and give no explicit explanation of some kind of transition from the finite to the infinite. John H. Conway and Donald Knuth's *surreal numbers* improved upon Cantor's infinities and expanded number systems like the *hyperreal numbers* and the *superreals*, making the distinction between ∞, $\infty+1$, $\sqrt{\infty}$, and so on (with "∞" written as lower case omega, "ω") [3]. Surreal numbers like $\sqrt{\infty}$ may be seen as a kind of stepping stone between the finite and the infinite.

But what *is* infinity? These symbols do not tell us what infinity actually is. What are the specific numerical properties of infinity besides its ordering and comparatively high magnitude? What number theoretic properties does infinity have? Is infinity odd or even? Is infinity a prime number or a composite number? What is the prime factorization of infinity? Is there any sense in which we can say there are digits of the number infinity, and if so, what are they?

How could we ever calculate the digits of infinity?

1 The Parity of Infinity: Is Infinity Odd or Even?

1.1 Coxeter's infinity sided polygon

We begin our investigation of infinity by looking at the infinity sided polygon known as *the regular apeirogon* [4].

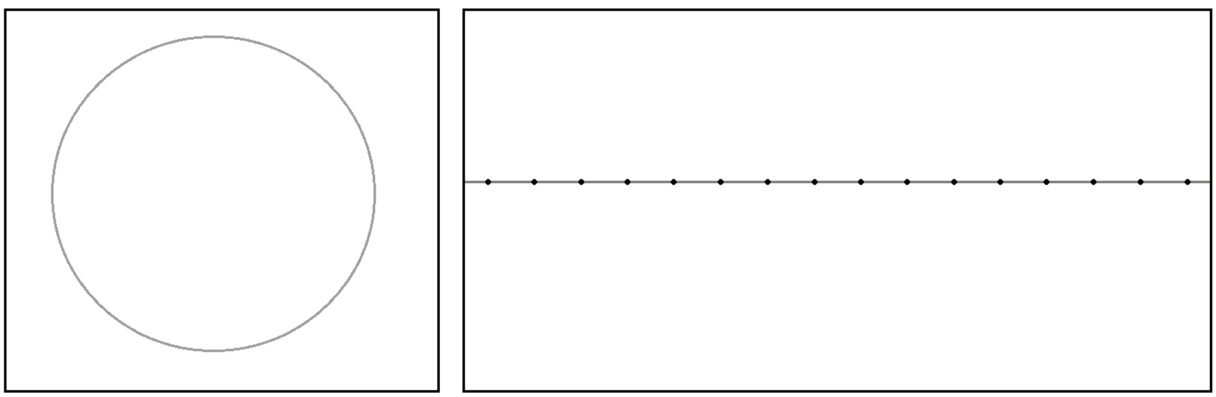

fig. 1.1.1 Two views of the regular apeirogon

Definition 1.1. *A **regular apeirogon** has equal-length sides and equal corner angles, just like any regular polygon. Its Schläfli symbol is {∞}. If the corner angles are 180°, the overall form of the apeirogon resembles a straight line, This line may be considered as a circle of infinite radius, by analogy with regular polygons with great number of edges, which resemble a circle. A **regular apeirogon** can be defined as a partition of the Euclidean line E^1 into infinitely many equal-length segments, generalizing the regular n-gon, which can be defined as a partition of the circle S^1 into finitely many equal-length segments.*

The regular apeirogon is the generalised infinite case of a regular polygon. From this we can see that, in the up close zoomed in view, the apeirogon looks like a segmented line, extending infinitely in both directions, but modularly connected, such that, moving to the right will eventually get back to the left side after some exact number of step; and being the infinite case of a partitioned circle, in a zoomed out view or for a shrunken down apeirogon, the apeirogon appears as a circle. This shared characteristic of infinite lines and circles is sometimes called a *generalized circle*.

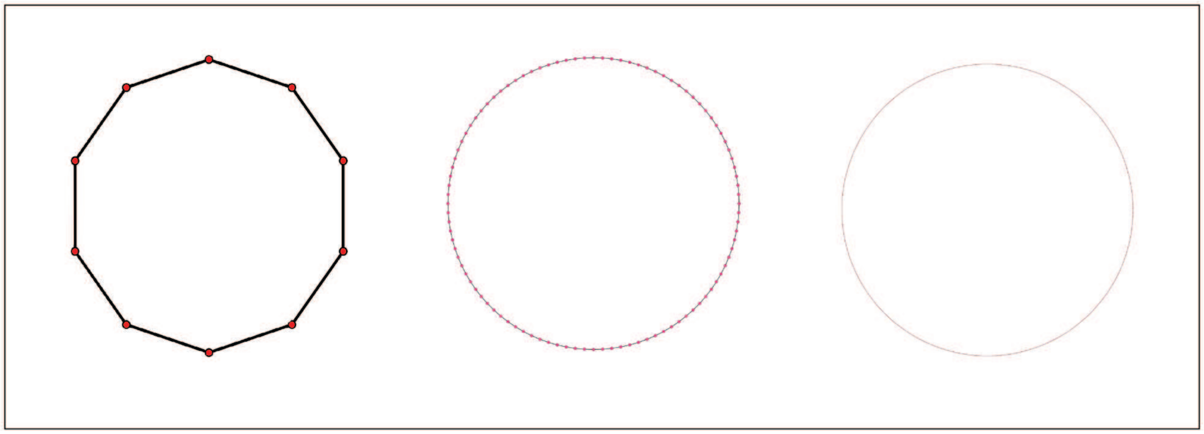

fig. 1.1.2 The sequence of regular polygons converges to a circle as more and more edges are added[5], while the angle between two edges of a regular polygon converges to 180°. From left to right, the regular 10-gon, 100-gon and 1000-gon.

This connection between infinite lines and circles, and the modular properties of an infinite line can also be seen in a series of circles with increasing circumference. As the circumference increases, the curvature decreases. A circle with an infinite circumference becomes a modularly connected straight line.

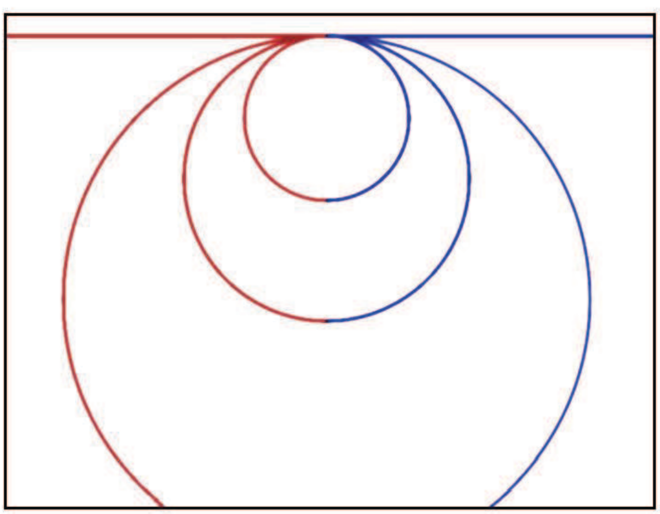

fig. 1.1.3 As the circumference of a circle increases, its curvature decreases. A circle with an infinite circumference is a modularly connected infinite line [6].

Unlike a continuous circle, the apeirogon is a discrete object, segmented into distinct units. So, in a sense, the regular apeirogon is the maximally partitioned discrete circle, and a maximally partitioned infinite line, both at the same time. There are symmetric properties that a continuous mathematical object may have that a discrete object may not have. Here we want to explore the properties that are shared in common with the discrete circle and the continuous circle, the properties that a regular apeirogon shares with a continuous circle.

Axiom 1. *The numerical properties of infinity can be deduced from the apeirogon, which shares properties with a modularly connected infinite line and that of a perfect circle.*

1.2 The shared properties of the symmetry of a circle

The apeirogon is a discrete object, sharing properties with a perfect circle, a circle being a continuous object. The apeirogon is identical in every way to a segmented infinite line. They are the exact same object. By examining the properties of these objects, we may deduce information about the symmetry of the apeirogon, and therefore we may deduce the numerical properties of infinity.

First off, a circle must have reflective symmetry, also known as mirror symmetry. This is a property that discrete objects and continuous objects can both have, so the apeirogon must have this property. The top of a circle should look like the bottom of a circle. The left side must look like the right side. The same should be true diagonally, and in the spaces between the diagonals, and the spaces between those spaces, and so on. This is true for the circle and for the apeirogon.

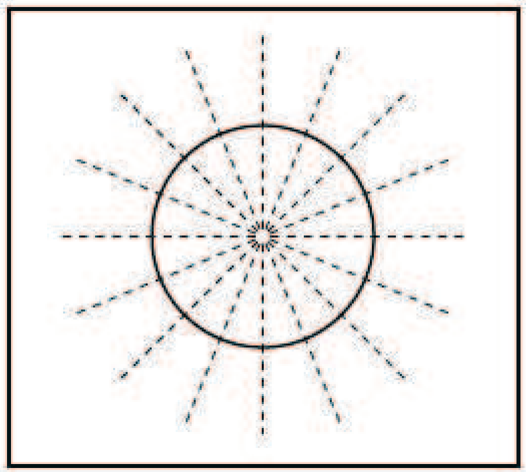

fig. 1.2.1 The primary axes of symmetry of a circle

There are some properties that a continuous circle has which a discrete apeirogon can not have all at the same time. For example, being a continuous object, the circle can have 3-fold radial symmetry, 4-fold symmetry, 5-fold symmetry, and so on, all at the same time, while still maintaining reflective symmetry, diagonal axial symmetry and symmetry in between these areas, all at the same time. This is not possible for a discrete object.

For example, consider a polygon with 362,880 sides, which is

$$(9!) = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9$$

This polygon has 9-fold symmetry, 8-fold symmetry, 7-fold symmetry, etc, for all numbers up to 9, but it can only be symmetrically dissected 7 times, at which point the same symmetry does not hold up between the remaining 2,835 edges. So we can not expect the apeirogon to have all manners of symmetry that a continuous object can have. Therefore, we will not rely on the multitude of various radial symmetries of the continuous circle to inform us about the properties of the discrete apeirogon and we will not attempt to deduce information about the numerical properties of infinity from the radial symmetry of the circle.

Axiom 1.2 *Since the continuous circle is a continuous object and the regular apeirogon is a discrete object, we are not be able to deduce numerical properties of infinity from the n-fold radial symmetry of a circle since there are combinations of symmetries that continuous objects can have all at once that discrete objects cannot have at the same time. All other symmetry properties that apeirogons and circles share in common will be used to deduce information about the properties of infinity.*

Soon we will explore the properties of the regular apeirogon in depth, but first, let's examine some regular polygons with a small, finite number of sides. If we line up the first few regular polygons in order, with one vertex at the top, called an *apex*, we see that every even sided polygon has a single vertex at the bottom, while every odd sided polygon has a single edge directly below the apex. For future reference, we will call this the *standard orientation*.

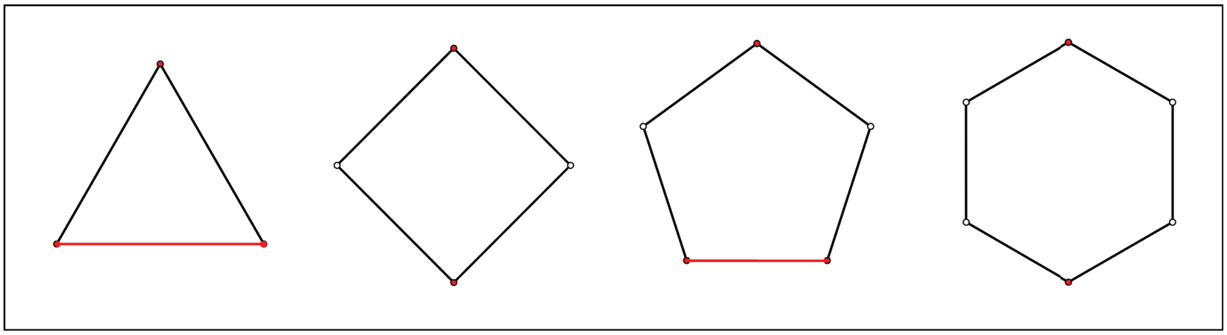

fig. 1.2.2 The top of an odd-sided regular polygon in standard orientation does not match the bottom, while the top of any even-sided regular polygon matches the bottom.

From this we can deduce that, in order to maintain the symmetry shared with a circle, having horizontal mirror symmetry, an apeirogon oriented with one vertex at its apex in standard position should have a single corresponding vertex below. Since regular polygons with this feature always have an even number of sides, we can conclude that infinity must be an even number.

Theorem 1. *Since an apeirogon shares symmetric properties with a circle, having horizontal mirror symmetry, a top most vertex must correspond to a single vertex below, a feature found in regular polygons with an even number of sides and not found in polygons with an odd number of sides. Therefore, if numerical properties can be deduced from the apeirogon, infinity must be an even number.*

Conclusion 1

It has been shown that if infinity is an actual number, and if the numerical properties of infinity can be deduced from the regular apeirogon, that infinity must be an even number, having a single vertex below its apex when oriented with a single vertex at top.

At this point it should not come as a surprise that infinity is even if we expect infinity to be highly symmetrical. Being an even number above 2, infinity cannot be prime.

2 Calculating the digits of infinity

2.1 Ruling out certain digits

Now that we have shown that infinity is an even number, we already have a good start to figuring out the digits of infinity.

All even numbers must end in 0, 2, 4, 6, or 8.

Theorem 2.1.1 *Since infinity is an even number, the last digit of infinity must be either 0, 2, 4, 6, or 8.*

How can we determine which of these five digits is correct? Let's proceed by considering the vertical symmetry of the apeirogon:

Similar to our previous example, in order to have vertical reflective symmetry, the left and right side must match. The horizontal components should also match the vertical components. So this means that we must find a vertex on the left side corresponding to a vertex on the right.

Polygons with this property, having a vertex at the apex, and one directly below, and a vertex to the far left and far right, without having edges in these places, have the numerical property of being a multiple of 4. That is, the number of sides is some whole number multiplied by 4.

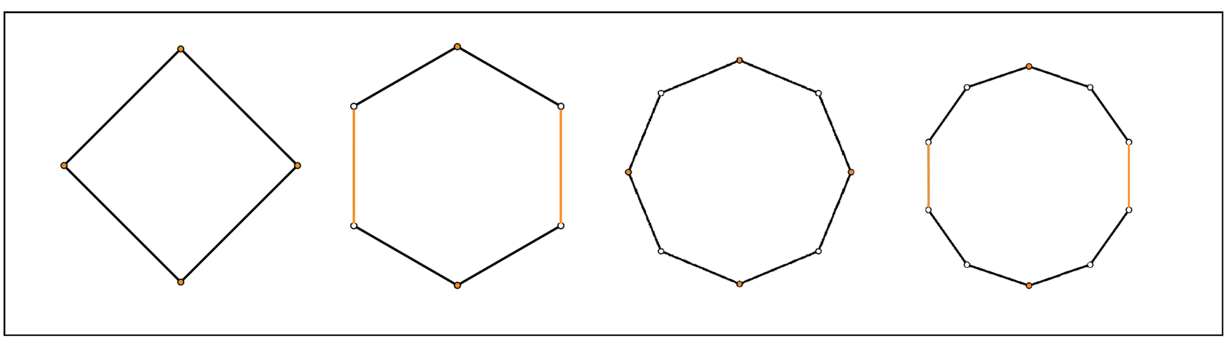

fig. 2.1.1 The top, bottom, left and right vertices of a regular 4-gon and 8-gon all match, while the top and bottom do not match the left and right for a regular 6-gon and 10-gon

Therefore, as we can see, to maintain this symmetry, the number of sides of an apeirogon must be a multiple of 4.

Theorem 2.1.2 *Having matching horizontal and vertical mirror symmetry, the number of sides of an apeirogon must be a multiple of 4.*

Now we can fill in the gaps between the four vertices we found in the earlier steps. In order for an apeirogon to have what we will call *maximal symmetry*, there must be four additional diagonally symmetric vertices between the other four vertices we have established. Similarly, there must be 8 vertices between those 8 vertices, and 16 more in between the next, and so on, following the sequence

$$2,\ 4,\ 8,\ 16,\ 32,\ 64\ \text{etc.}$$

Thus, the number of sides of an apeirogon must be a power of 2.

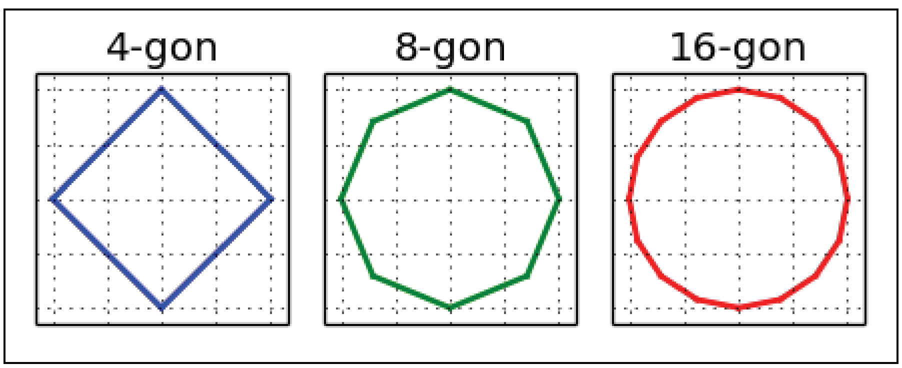

fig. 2.1.2 The regular 4-gon, 8-gon, and 16-gon, the first three highly symmetrical polygons having a number of sides equal to a power of 2.

Theorem 2.1.3 *Having maximal symmetry, we must find single vertices exactly between the vertices established in step 2.1.2, and there must be vertices between these new vertices, following the sequence 2, 4, 8, 16, etc. therefore infinity must be a power of 2.*

2.2 Infinity is a power of 2

We have shown that infinity is an even number, and now we have shown that infinity is a power of 2.

$$\begin{aligned}
2^1 &= 2\\
2^2 &= 4\\
2^3 &= 8\\
2^4 &= 16\\
2^5 &= 32\\
2^6 &= 64\\
2^7 &= 128\\
2^8 &= 256
\end{aligned}$$

Fig. 2.2 The last digit of powers of 2

Since the last digits of powers of 2 always follow a regular periodic pattern, we know that the final digit of infinity must end in:

2, 4, 8 or 6

Theorem 2.2.1 *Since infinity is a power of 2, and the last digit of a power of 2 must end in either 2, 4, 8, or 6, the last digit of infinity must end in either 2, 4, 8, or 6.*

2.3 The last digit of infinity

We have begun filling in the vertices of the apeirogon according to the maximal symmetry shared with a circle, using powers of 2, following the sequence:

$$2 \text{ x } 2 \text{ x } 2 \text{ x } 2 \ldots$$

A faster way to achieve this maximal symmetry is by adding vertices according to the exponential sequence:

$$2,\ 2\text{^}2 = 4,\ 4\text{^}2 = 16,\ 16\text{^}2 = 256... \text{ and so on.}$$

A001146 as a simple table

n	a(n)
0	2
1	4
2	16
3	256
4	65536
5	4294967296
6	18446744073709551616
7	340282366920938463463374607431768211456
8	115792089237316195423570985008687907853269984665640564039457584007913129639936

Fig. 2.3 the sequence *A001146* from the Online Encyclopedia of Integer Sequences [7]

After the first two terms, numbers of this sequence always end in 6.

Theorem 2.3.1 *Filling in the gaps between vertices of an apeirogon with the exponential series 2, 2^2, 4^2, 16^2, etc, satisfies the symmetry of a circle faster than a sequence based on powers of 2. Since after the first two terms, the last digit of numbers of this sequence is always a 6, the last digit of infinity must be 6.*

2.4 The infinity sequence

We have shown that infinity is an even number, that infinity is a power of two, and now we have shown that the last digit of infinity is 6. How can we find more digits of infinity?

We have started with even numbers, powers of 2, and now exponents of powers of 2. To achieve maximal symmetry for our apeirogon, there is another sequence we can follow:

2 , 2^2 , 2^2^2 , 2^2^2^2, … and so on.

By convention, 2^2^2^2 is grouped together as 2^(2^(2^2)), not (((2^2)^2)^2). Grouping exponents this way results in a much, much larger number.

We can now look at the numbers of this sequence, which we will call the *infinity sequence.*

1) 2 = 2
2) 2^2 = 4
3) 2^2^2 = 2^4 = 16
4) 2^2^2^2= 2^16 = 65536

The 5th number in this sequence already has over 19,000 digits!

The first few digits are 2003529... and the last are ... 19156736.

5) 2003529...19156736 (19,729 digits)

At this point, we can only keep track of the last few digits! The 6th number has more than 10^19,729 digits!

6) ...7437428736

It should be pointed out that a pattern is emerging after the second term in the sequence. First in the last digit, then in the last two digits, then in the last three digits, and so on:

1**6**
655**36**
...156**736**
...742**8736**

The last digits are converging on some kind of pattern:

...6
...36
...736
...8736

Here are the next six terms in the sequence:

7) ...9621748736
8) ...9960948736
9) ...7112948736
10) ...4232948736
11) ...1432948736
12) ...3432948736

So far, we have calculated the last ten digits of infinity:

...3432948736

The infinity sequence we have discovered corresponds to the sequence *A206636* in the Online Encyclopedia of Integer Sequences [8]:

n	**a(n)**
1	6
2	36
3	736
4	8736
5	48736
6	948736
7	2948736
8	32948736
9	432948736
10	3432948736
11	53432948736
12	353432948736
13	5353432948736
14	75353432948736
15	075353432948736
16	5075353432948736
17	15075353432948736
18	615075353432948736
19	8615075353432948736
20	98615075353432948736
21	098615075353432948736
22	8098615075353432948736

So, we have found the prime factorization of infinity, and the last 22 digits of infinity are:

...8098615075353432948736

Q.E.D.

Here are the last 1,000 digits of ∞:

...64525341261610654078857424007827232078152975436059
46147650094389896258515692489298997259791546185349312
85138512464633571289181307555258568621874838557279070
23570230004779008324816646646396293811943626972369
73858100812558516375764382111286275124759698067406126
52990369670891836977939109796553466025385692124601766
41463288843282377635382755535319460748837224000537
39215628121237852888212099277956138800258050803647824
297753686050106977780463952098075388313636151792213
94726153872010887240847382700999490250338825869846280
56890842439279478584734963628655567202472361038218281
751326535178118742346244692953213255086748088283196
36559141189838590864032603899870131537628336658436847
48619141444039780186719944644055340240243583302228
1121378836687865256680756780796573298891689086155684
83959555365409162609960429926969192718906250362357442
47694204953632210807979617634839681451051151735754592
43654693520017384422568887190225094017989855389270
91212128284162862457014552869687260015985333809861507
5353432948736

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