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Original & Equivalent ABC Conjectures Proved on Two Pages

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Abstract

This paper proves the original and the equivalent ABC conjectures. The hypothesis for the original conjecture is basically the same as the hypothesis for the equivalent conjectures, and this hypothesis states that there exists only finitely many triples (A, B, C) of coprime positive integers, with $A + B = C$. The conclusion for the original conjecture would be that the product, d, of the distinct prime factors of A, B and C, is usually not much smaller than C. This conclusion would be interpreted as $|C - d| < \varepsilon$, where ε is a positive real number. The conclusions covered for the equivalent conjectures would be the following: **1.** $C > rad(ABC)^{1+\varepsilon}$; **2**. $C < K_\varepsilon rad(d)^{1+\varepsilon}$, where K_ε is a constant and K_ε is a function of ε. **3**. $q(a,b,c) = \dfrac{\log C}{\log(rad(d))} > 1 + \varepsilon$; However, for the equivalent conjectures, one will apply only the conclusions containing the constant K_ε, since their solutions for ε can readily be applied in the epsilon-delta proofs in this paper. One will also introduce K_ε into $q(a,b,c) = \dfrac{\log C}{\log(rad(d))} > 1 + \varepsilon$ to obtain $\dfrac{\log C}{\log(rad(d))} < K_\varepsilon(1 + \varepsilon)$. Thus, the conclusions to be used for the equivalent conjectures are

1. $C < K_\varepsilon rad(d)^{1+\varepsilon}$, equivalently, $\dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))} < \varepsilon$; **2**. $\dfrac{\log C}{\log(rad(d))} < K_\varepsilon(1 + \varepsilon)$, equivalently, $\dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))} < \varepsilon$. Let $H = A + B - C$. Then $|H| < \delta$ (δ being a positive real number) would be the hypothesis, and let $|L| < \varepsilon$ be the conclusion for the original conjecture with $L = C - d$. For the equivalent conjectures, let $L < \varepsilon$ be the equivalent conclusion, where $L = \dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))}$, $L = \dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))}$.

It has been proved that if $|A + B - C| < \delta$, then for the original conjecture, $|L| < \varepsilon$; and for the equivalent conjectures, $L < \varepsilon$.

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Introduction

Original ABC conjecture
The original conjecture states that if A, B and C are three coprime positive integers such that $A + B = C$, and d is the product of the distinct prime factors of A, B and C, then d is usually not much smaller than C.

Equivalent ABC conjecture #1
The equivalent ABC conjecture #1 states that for every positive real number ε, there exists only finitely many triples $(A,\ B,\ C)$ of coprime positive integers, with $A + B = C$, such that $C < rad(d)^{1+\varepsilon}$, where d is the product of distinct prime factors of $A,\ B,$ and C,

Equivalent ABC conjecture #2
The equivalent ABC conjecture #2 states that for every positive real number ε, there exists only finitely many triples $(A,\ B,\ C)$ of coprime positive integers, with $A + B = C$, such that $C < K_\varepsilon rad(d)^{1+\varepsilon}$, where d is the product of distinct prime factors of $A,\ B,$ and C, and K_ε is a constant.

Equivalent ABC conjecture #3
The equivalent ABC conjecture #3 states that for every positive real number ε, there exists only finitely many triples $(A,\ B,\ C)$ of coprime positive integers, with $A + B = C$, such that $q(A,B,C) > 1 + \varepsilon$, where $q(A,B,C) = \dfrac{\log c}{\log(rad(d))}$, and d is the product of distinct prime factors of $A,\ B,$ and C,

Preliminaries

A On solving $C > rad(ABC)^{1+\varepsilon}$ for ε, one obtains $\dfrac{\log C - \log rad(d)}{\log(rad(d))} > \varepsilon$
The sense of the above inequality is to the right and not ready for the epsilon-delta proofs.

B On solving $C < K_\varepsilon rad(d)^{1+\varepsilon}$ for ε, one obtains $\boxed{\dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))} < \varepsilon}$
Desirable inequality sense for epsilon-delta proofs, The sense of the inequality is to the left

C

$$q(a,b,c) > 1 + \varepsilon$$
$$q(a,b,c) = \frac{\log C}{\log(rad(d))}$$
$$\frac{\log C}{\log(rad(d))} > 1 + \varepsilon$$
$$\frac{\log C}{\log(rad(d))} - 1 > \varepsilon$$
$$\frac{\log C - \log(rad(d))}{\log(rad(d))} > \varepsilon \ \text{(A)}$$

Since the sense of this inequality is to the right, one will imitate "from box **A** to box **B**", and multiply the right side of $q(a,b,c) > 1 + \varepsilon$ by K_ε and reverse the inequality sense to obtain $\boxed{q(a,b,c) < K_\varepsilon(1 + \varepsilon)}$ as in Box **D.**

D By imitating "from box A to box B", one multiplies the inequality on the left by K_ε and reverses the sense of the inequality to obtain

$$q(a,b,c) < K_\varepsilon(1 + \varepsilon)$$
$$\frac{\log C}{\log(rad(d))} < K_\varepsilon(1 + \varepsilon)$$
$$\boxed{\frac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))} < \varepsilon} \ \text{(B)}$$

the inequality sense is desirable
Let the left side of B be L
then $L < \varepsilon$ (equivalent conclusion)

For the conclusions for the equivalent ABC conjectures, one will use the conclusions in Boxes **B** and **D**

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Option 2

Original & Equivalent ABC Conjectures Proved on Two Pages

Given: For both the original and the equivalent conjectures:

1. $A + B = C$, where A, B and C are positive integers. with A, B and C being coprime.
2. d = product of the distinct prime factors of A, B and C.

Required: For the original conjecture:

To Prove that $|C - d| < \varepsilon$. If $L = C - d$, prove that $|L| < \varepsilon$

For the equivalent conjectures:

1. To prove that $\dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))} < \varepsilon$

Thus, If $L = \dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))}$, prove that $L < \varepsilon$

2. To prove $\dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))} < \varepsilon$.

Thus, If $L = \dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))}$, prove that $L < \varepsilon$

Plan: Hypothesis

$\boxed{A + B = C,}$

$A + B - C = 0$,

and $|A + B - C| = |0| = 0$

For a positive real number, δ, $0 < \delta$ one can write $|A + B - C| < \delta$.

Conclusion:

For the original conjecture: $|L| < \varepsilon$ (equivalent conclusion)

For the equivalent conjectures: $L < \varepsilon$ (equivalent conclusion)

The proofs would be complete after showing that if $|A + B - C| < \delta$, then for the original conjecture, $|L| < \varepsilon$; and for the equivalent conjectures, $L < \varepsilon$.

Proof: One will apply the continued inequality method to handle the inequalities involved.

Step 1: $|A + B - C| < \delta$ $(\delta > 0)$ (hypothesis) (2)

For the original conjecture, the conclusion, $|L| < \varepsilon$, already has the absolute value symbol; but for the equivalent conjectures, one applies the absolute value symbol to the conclusion, $L < \varepsilon$, to obtain $|L| < \varepsilon$. This absolute value symbol will be removed in the last step.

The hypothesis $|A + B - C| < \delta$ is equivalent to

$$-\delta < A + B - C < \delta \quad \text{(hypothesis)} \qquad (4)$$

The conclusion, $|L| < \varepsilon$ is equivalent to

$$-\varepsilon < L < \varepsilon \quad \text{conclusion} \qquad (5)$$

Step 2: Make the middle terms of (4) and (5) the same. Then (4) becomes.

$$\boxed{-\delta + L < A + B - C + L < L + \delta \quad \text{(hypothesis)}} \qquad (6)$$

and (5) becomes $\boxed{-\varepsilon + A + B - C < A + B - C + L < \varepsilon + A + B - C}$ (7)

Since (6) and (7) have the same middle terms, equate the left sides to each other and equate the right sides to each other. Then one obtains

$\boxed{-\delta + L = -\varepsilon + A + B - C}$ and one solves for δ to obtain $\boxed{\delta = \varepsilon + L - A - B + C, \text{ say } \delta_1}$ followed by solving

$\boxed{L + \delta = \varepsilon + A + B - C}$ for δ to obtain $\boxed{\delta = \varepsilon - L + A + B - C, \text{ say } \delta_2}$

$|A + B - C| < \delta$, implies that $-\delta < A + B - C < \delta$ (hypothesis)

For $\varepsilon > 0$, choose $\delta = \min(\delta_1, \delta_2)$.

$-\delta < A + B - C < \delta$ (hypothesis) implies that

$-\delta_1 \le -\delta < A + B - C < \delta \le \delta_2$ (hypothesis) (8)

Step 3: Replace the left and right sides of (8) by $\boxed{\delta = \varepsilon + L - A - B + C, \text{ say } \delta_1}$ and $\boxed{\delta = \varepsilon - L + A + B - C, \text{ say } \delta_2}$, from above, respectively, to obtain

$-\varepsilon - L + A + B - C < A + B - C < \varepsilon - L + A + B - C$ (hypothesis) (9)

Break up inequality (9) into two simple inequalities and **solve** each one for $-\varepsilon$ and ε, respectively.

$-\varepsilon - L + A + B - C < A + B - C$. **solving,** $\boxed{-\varepsilon < L}$

$A + B - C < \varepsilon - L + A + B - C$; **solving**, $\boxed{L < \varepsilon}$

The conjunction, $-\varepsilon < L$ **and** $L < \varepsilon$, is equivalent to $|L| < \varepsilon$

Step 4: For the original conjecture : Therefore, if $|A + B - C| < \delta$ $(\delta > 0)$ or $A + B = C$, $|L| < \varepsilon$ or $|C - d| < \varepsilon$ $(L = C - d)$

For the equivalent conjectures,

As was noted in Step 1, one will remove the absolute value symbol (see analogy on next page) to obtain $L < \varepsilon$ (equivalent conclusion)

Therefore, if $|A + B - C| < \delta$ $(\delta > 0)$ or $A + B = C$, $L < \varepsilon$

where $L = \dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))}$; or $L = \dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))}$;

OR

1. $C < K_\varepsilon rad(d)^{1+\varepsilon}$ (equivalently, $\dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))} < \varepsilon$)

2. $\dfrac{\log C}{\log(rad(d))} < K_\varepsilon(1 + \varepsilon)$ (equivalently, $\dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))} < \varepsilon$)

and the proofs of the conjectures are complete.

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Option 3

Discussion

In Step 1, the absolute value symbol was applied to the conclusions of the equivalent conjectures, and in Step 4, the symbol was removed. For **analogy** in elementary math, consider:

Factoring quadratic trinomials by the substitution method;

Example : Factor $6x^2 + 11x - 10$

Step 1: Multiply the expression by the coefficient of the x^2-term.

$6(6x^2) + 6(11x) - 6(10)$

$(6x)^2 + 11(6x) - 60$ ....(A)

Step 2: Let $6x = s$

Then, we obtain $s^2 + 11s - 60$

$(s - 4)(s + 15)$ ....(B)

Step 3: Replace s by $6x$,and then, expression (B) becomes $(6x - 4)(6x + 15)$.....(C)

Since one multiplied the original trinomial by 6, one must divide expression (C) by 6 (that is ,one must undo the "6" introduced in Step 1).

Step 4: In order to divide (C) by 6, perform common monomial factoring on the two binomial factors (in some cases, this factoring is performed only on one of the binomial factors).

$(6x - 4)(6x + 15)$

$2(3x - 2)\ 3(2x + 5)$

$2(3)(3x - 2)(2x + 5)$

$6(3x - 2)(2x + 5)$

Now, divide by 6: $\dfrac{6(3x -2)(2x + 5)}{6}$

and then the complete factorization of

$6x^2 + 11x - 10$ is $(3x - 2)(2x + 5)$

Conclusion

This paper proved the original and the equivalent ABC conjectures. The hypothesis for the original conjecture was basically the same as the hypothesis for the equivalent conjectures, This hypothesis states that there exists only finitely many triples (A, B, C) of coprime positive integers, with $A + B = C$. The conclusion for the original conjecture was that the product d, of the distinct prime factors of A,B and C, is usually not much smaller than C. This conclusion of the original conjecture was interpreted as $|C - d| < \varepsilon$. The conclusions for the equivalent conjectures were the following: **1.** $C < K_\varepsilon rad(d)^{1+\varepsilon}$ (equivalently, $\dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))} < \varepsilon$) ;

2. $\dfrac{\log C}{\log(rad(d))} < K_\varepsilon(1 + \varepsilon)$ (equivalently, $\dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))} < \varepsilon$). One applied the epsilon-delta technique in the proofs, That is, if $|H| < \delta$ (hypothesis); then $|L| < \varepsilon$ (conclusion).
For both the original conjecture and the equivalent conjectures, $H = A + B - C$, For the equivalent conjectures, $L = \dfrac{\log C - \log K_\varepsilon - \log(rad(d))}{\log(rad(d))}$ or $L = \dfrac{\log C - K_\varepsilon \log(rad(d))}{K_\varepsilon \log(rad(d))}$. The continued inequality method (condensed method) was used in handling the inequalities involved in the proofs.

PS: 1. A proof of the original ABC conjecture by the author is at viXra:2107.0094
2. For more on epsilon-delta proofs, see Lesson 5C, Calculus 1 & 2 by A. A. Frempong at Apple iBookstore.

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