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On Factorization of Multivectors in $Cl(3,0)$, $Cl(1,2)$ and $Cl(0,3)$, by Exponentials and Idempotents

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ABSTRACT
In this paper we consider general multivector elements of Clifford algebras $Cl(3,0)$, $Cl(1,2)$ and $Cl(0,3)$, and look for possibilities to factorize multivectors into products of blades, idempotents and exponentials, where the exponents are frequently blades of grades zero (scalar) to n (pseudoscalar).

KEYWORDS
Clifford algebra; factorization; idempotents

1. Introduction

The important role of the *polar representation* of complex numbers and quaternions is widely known. Here[1] we endeavor to *extend* this approach to higher dimensional associative Clifford geometric algebras, which play important roles in geometry, physics and computer science [1,2,10,16,24,28]. Exponentials of hyper complex elements and blades also appear as *kernels* in complex, quaternionic and Clifford Fourier and wavelet transforms [17]. Important *related questions* are the computation of logarithms of multivectors [4], square roots [4,13,15,18,25], inverses [19], transformation rotors [23], and polar decompositions [27], etc. Concrete applications may therefore be to forward and reverse kinematic motions of robot arms, where such factorizations could be useful, or in drone controls.[2] In earlier work the question of factorization into exponential factors, blades and idempotents for Clifford algebras $Cl(p,q)$, $n = p + q = 1, 2$ [22] has been studied. This motivates us to progress by extending [22] to the case $n = 3$. But since the case $Cl(2,1)$ is of particular complexity, we will treat it by itself in a subsequent paper.

Because *subalgebras isomorphic* to the algebra of *hyperbolic numbers* appear frequently, we include the description of hyperbolic planes of [22] again. Furthermore, the subalgebra structure, in particular that of even subalgebras, is seen to play an essential role, therefore we also study the even subalgebra of $Cl(1,2)$ (and similarly of $Cl(2,1)$), both isomorphic to $Cl(2,0)$, split-quaternions or coquaternions. As far as

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[1]The use of this paper is subject to the *Creative Peace License* [12]. We dedicate this paper to the *truth* (Jesus: *I am the way and the truth and the life. No one comes to the Father except through me*, see John 14:6, NIV). *Soli Deo Gloria.*

[2]Private communcation with R. Abłamowicz.

possible we aim at explicit, step by step verifiable proofs. The arrangement of the sections by the Clifford algebra studied is by increasing complexity, which mainly stems from the increasing number of idempotents in the respective algebra. An introduction to Clifford geometric algebras is contained in [14], a concise mathematical definition in [6], and a comprehensive study relevant for mathematics and physics in [10].

A prime example of the type of factorization that we envision can be seen in (4.34), which has four exponential factors with a scalar, vector, bivector and a trivector in the respective exponent. But because all algebras under consideration are not division algebras, we necessarily have non-invertible multivectors and their factorizations are found to include non-invertible idempotents as factors (e.g. in (4.11)) or even their linear combinations (e.g. in (5.19)). Note that we also include the representation (2.11) for elements of a hyperbolic plane in our wider notion of exponential factors.

The paper is *structured* as follows. Section 2 reviews [22] hyperbolic numbers and their factorization in terms of *exponentials* and *idempotents*, and invertibility. Section 3 studies the important *even subalgebras* of $Cl(1,2)$ (and $Cl(2,1)$), providing essential results for the full blown study of $Cl(1,2)$ following later. Section 4 studies the factorization of multivectors in Clifford algebras $Cl(3,0)$ and $Cl(0,3)$, which can be *meaningfully grouped* together, because their even subalgebras $Cl_2(3,0)$ and $Cl_2(0,3)$ are both isomorphic to quaternions.

In order to achieve a factorization of multivectors in $Cl(1,2)$ in terms of exponential factors with *blades as exponentials* and idempotents (for non invertible multivectors), we take a *direct* approach in Section 5, making use of the study of the even subalgebra in Section 3. An alternative factorization in $Cl(1,2)$, based on the *isomorphism* to $Cl(3,0)$ appears in Appendix B. The paper concludes with Section 6, followed by acknowledgments and references.

2. Hyperbolic planes

Since subalgebras isomorphic to the algebra of a hyperbolic plane[3] will occur repeatedly in our analysis, and to establish notation for later use in this paper, we reproduce this short study of hyperbolic planes from [22]. An element $u \neq 1$ that squares to $u^2 = +1$ generates a *hyperbolic plane* $\{b + au\}$, $a, b \in \mathbb{R}$ with basis $\{1, u\}$. A relevant alternative basis $\{id_-, id_+\}$ is given by two not invertible idempotents

$$id_+ = \frac{1+u}{2}, \quad id_- = \frac{1-u}{2}, \quad id_+ + id_- = 1, \quad id_+ - id_- = u,$$
$$id_+^2 = id_+, \quad id_-^2 = id_-, \quad id_+ id_- = id_- id_+ = 0. \tag{2.1}$$

Adopting the definitions

$$x^0 = 1, \qquad 0! = 1, \qquad e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{2.2}$$

[3]The Clifford algebra $Cl(1,0)$ and the even subalgebra $Cl_2(1,1)$ of the two-dimensional space-time algebra are both isomorphic to the hyperbolic plane. Invertible elements of $Cl_2(1,1)$ represent boosts (changes of velocity), of elementary importance in special relativity.

for powers of a general element x and its exponential[4], we obtain for $a \in \mathbb{R}$

$$e^{a\,id_\pm} = 1 + (e^a - 1)id_\pm, \qquad e^{au} = \cosh a + u \sinh a. \tag{2.3}$$

General *nonzero* elements $m = b + au$ of the hyperbolic plane can be classified by whether $|a| = |b|$ (m is not invertible), or $|a| \neq |b|$ (m is invertible). For $|a| = |b|$ we have the four subcases

$$\begin{array}{ll} b = a > 0, & m = 2b\,id_+, \\ b = a < 0, & m = 2b\,id_+ = -2|b|\,id_+, \\ b = -a > 0, & m = 2b\,id_-, \\ b = -a < 0, & m = 2b\,id_- = -2|b|\,id_-. \end{array} \tag{2.4}$$

Examples are for each line of (2.4): $1+u = 2(1+u)/2 = 2id_+$, $-2-2u = -4(1+u)/2 = 4(-id_+)$, $3-3u = 6(1-u)/2 = 6id_-$, $-4+4u = -8(1-u)/2 = 8(-id_-)$. Thus according to (2.4) for $|a| = |b| \neq 0$ we can always represent m as[5]

$$m = 2|b|h^{id}(u), \quad \text{with} \quad h^{id}(u) \in \{\pm id_+, \pm id_-\}, \tag{2.5}$$

and therefore as

$$m = e^{\alpha_0} h^{id}(u), \qquad \alpha_0 = \ln(2|b|). \tag{2.6}$$

Note that $h^{id}(u)^2 = id_\pm$. Geometrically, the four values of $h^{id}(u)$ specify *four bisector* directions, one in each quadrant of the hyperbolic plane. Because idempotents $id_\pm$ are not invertible, all hyperbolic numbers with $|a| = |b|$ cannot be inverted.

For general (evidently *nonzero*) elements $m = b+au$ with $|a| \neq |b|$ we can distinguish four subcases

$$\begin{array}{ll} b > |a| \geq 0, & m = b + au, \\ a > |b| \geq 0, & m = (a + bu)u, \\ b < -|a| \leq 0, & m = -(-b - au), \\ a < -|b| \leq 0, & m = -(-a - bu)u. \end{array} \tag{2.7}$$

Examples for (2.7) are line by line: $4 \pm u$, $\pm 1 + 4u = (4 \pm u)u$, $-4 \mp u = -(4 \pm u)$, $\mp 1 - 4u = -(4 \pm u)u$. Thus according to (2.7) for $|a| \neq |b|$ we can always represent any m as

$$m = (\beta + \alpha u)h(u), \quad \text{with} \quad h(u) \in \{\pm 1, \pm u\}, \tag{2.8}$$

such that $\beta > |\alpha| \geq 0$, and therefore m can be factored as

$$m = e^{\alpha_0} m' = e^{\alpha_0} e^{\alpha_u u} h(u), \quad \alpha_0 = \tfrac{1}{2}\ln(\beta^2 - \alpha^2), \quad \alpha_u = \operatorname{atanh}(\alpha/\beta). \tag{2.9}$$

In the examples for (2.7) we have $\alpha = \pm 1, \beta = 4, \alpha_0 \approx 1.35, \alpha_u \approx \pm 0.255$. Note that $h(u)^2 = 1$ and therefore $h(u)^{-1} = h(u)$. Geometrically, the four possible values of $h(u)$

[4] In this paper we do not make further use of $e^{a\,id_\pm}$. But we note that even though $id_\pm$ is not invertible, $e^{a\,id_\pm}$ has inverse $e^{-a\,id_\pm}$, similar to null-vectors not being invertible, but their exponential functions have a multiplicative inverse.

[5] Note that (2.5) together with (2.4) provides a unique specification for the assignment of $h^{id}(u)$ from the set $\{\pm id_+, \pm id_-\}$, thus effectively *defining* the four-valued function $h^{id}(u)$. Similarly (2.8) together with (2.7) effectively *defines* $h(u)$ uniquely.

Table 1. Multiplication table of $Cl_2(2,1)$.

	1	e_{12}	e_{23}	e_{31}
1	1	e_{12}	e_{23}	e_{31}
e_{12}	e_{12}	-1	$-e_{31}$	e_{23}
e_{23}	e_{23}	e_{31}	$+1$	e_{12}
e_{31}	e_{31}	$-e_{23}$	$-e_{12}$	$+1$

Table 2. Multiplication table of $Cl_2(1,2)$.

	1	e_{12}	e_{23}	e_{31}
1	1	e_{12}	e_{23}	e_{31}
e_{12}	e_{12}	$+1$	e_{31}	e_{23}
e_{23}	e_{23}	$-e_{31}$	-1	e_{12}
e_{31}	e_{31}	$-e_{23}$	$-e_{12}$	$+1$

uniquely specify the four quadrants in the hyperbolic plane, delimited by two straight lines (bisectors) with directions $id_{\pm}$. The inverse of hyperbolic numbers with $|a| \neq |b|$ can *always* be easily computed as

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_u u} h(u). \tag{2.10}$$

In *summary*, any $m = b + au \neq 0$ in the hyperbolic plane can be factorized as

$$m = E(m) = E(a,b,u) = e^{\alpha_0} \begin{cases} h^{id}(u) & \text{for} \quad |a| = |b|, \\ e^{\alpha_u u} h(u) & \text{for} \quad |a| \neq |b|. \end{cases} \tag{2.11}$$

Equation (2.11) provides a first example of what we mean by *exponential factorization*. Note that we introduce the *new notation* $E(m) = E(a,b,u)$ to indicate the factorization (2.11) in terms of one or two exponential functions and eight possible values. The computation of the factorization (2.11) is based on (2.4) to (2.6) for the first four cases involving idempotents, i.e. $h^{id}(u) \in \{+id_+, -id_+, +id_-, -id_i\}$, and on (2.7) to (2.9) for the remaining four cases involving the hyperbolic exponential factor and $h(u) \in \{+1, -1, +u, -u\}$. The hyperbolic number m is invertible if and only if $|a| \neq |b|$.

3. Even subalgebra of $Cl(1,2)$

3.1. Isomorphisms of even subalgebras of $Cl(1,2)$ (and $Cl(2,1)$)

As we will soon see in Section 4, quaternions isomorphic to the even subalgebras of $Cl(3,0)$ and $Cl(0,3)$ play a pivotal role in the factorization of these algebras. Similarly we can expect that the even subalgebra $Cl_2(1,2)$ (and $Cl(2,1)$) with basis $\{1, e_{12}, e_{23}, e_{31}\}$ of $Cl(1,2)$ (and $Cl(2,1)$) might be of high relevance. They have the following multiplication tables: Table 1 and Table 2. These two tables are obviously isomorphic, if we identify $e_{23} = e'_{12}$, $e_{31} = e'_{31}$, $e_{12} = e'_{23}$, where $\{e_{12}, e_{23}, e_{31}\} \subset Cl_2(2,1)$ and $\{e'_{12}, e'_{23}, e'_{31}\} \subset Cl_2(1,2)$. This isomorphism is the reason, why we include the even subalgebra $Cl_2(2,1)$, without extra effort.

Furthermore, the two tables are isomorphic to $Cl(2,0)$ by identifying $e_1 = e'_{23}$, $e_2 = e'_{31}$, $e_{12} = e'_{12}$, where $\{e_1, e_2, e_{12}\} \subset Cl(2,0)$ and $\{e'_{12}, e'_{23}, e'_{31}\} \subset Cl_2(2,1)$. Alternatively, we can identify $e_1 = e'_{12}$, $e_2 = e'_{31}$, $e_{12} = e'_{23}$, where $\{e_1, e_2, e_{12}\} \subset Cl(2,0)$ and $\{e'_{12}, e'_{23}, e'_{31}\} \subset Cl_2(1,2)$.

The isomorphism with $Cl(2,0)$ does allow to utilize the factorization of $Cl(2,0)$ derived in Section 5 of [22]. We recapitulate the result here[6]

$$
\begin{aligned}
m &= m_1e_1 + m_2e_2 + m_0 + m_{12}e_{12} \\
&= \begin{cases} e^{\alpha_0}e^{\alpha_2 e_{12}}, \;\; \alpha_0 = \ln(b), \;\; \alpha_2 = \operatorname{atan2}(m_{12}, m_0) & \text{for } m_1 = m_2 = 0, \\ e^{\alpha_0'}u', \quad \alpha_0' = \ln(a), & \text{for } m_0 = m_{12} = 0, \\ (b + au)\, e^{\alpha_2 e_{12}} = E(a,b,u)\, e^{\alpha_2 e_{12}}, & \text{otherwise,} \end{cases}
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
a &= \sqrt{(m_1e_1 + m_2e_2)^2} = \sqrt{m_1^2 + m_2^2}, \qquad b = \sqrt{m_0^2 + m_{12}^2}, \\
u' &= (m_1e_1 + m_2e_2)/a, \qquad u = e^{\alpha_2 e_{12}}\, u',
\end{aligned}
\tag{3.2}
$$

and $E(a,b,u)$ has been defined in (2.11). Because a and b are positive, the eight possible values of $E(a,b,u)$ reduce to only three, i.e. only the first line of (2.4) and the first two lines of (2.7) are relevant. We further observe about (3.1) that the third line subsumes the first for $a = 0$, and the third line subsumes the second for $b = \alpha_2 = 0$. This means that $m \in Cl(2,0)$, can *always* be factored in the form

$$
m = (b + au)e^{\alpha_2 e_{12}},
\tag{3.3}
$$

with $a \geq 0$ and $b \geq 0$. And m is always invertible, except when $a = b$. In (3.3) u is a vector with positive unit square and e_{12} is a bivector with negative unit square. In Appendix A we discuss an interesting alternative factorization which aims at a single exponential factor with bivector exponent, and explain why we still prefer (3.3) in the rest of this paper.

4. Factorization in $Cl(3,0)$ and $Cl(0,3)$

For the case of $Cl(3,0)$, isomorphic to complexified quaternions (biquaternions) we refer to earlier work in [27]. In order to reveal the *analogies* and *differences* between $Cl(3,0)$ and $Cl(0,3)$, we treat both algebras with the same level of detail.

4.1. Computation and factorization of $m\overline{m}$

Following the approach in [4] and [27] we first compute the central multivector $m\overline{m}$, which we will use in the following Section 4.2 to turn m (invertible, i.e. free of idempotent factors) into a unit norm multivecor M, $M\overline{M} = 1$. As for notation, independent of the choice of vector space basis, unit vectors u, unit bivectors i_2, and the central unit pseudoscalar[7] $i = e_{123}$ in $Cl(3,0)$ square to

$$
u^2 = +1, \qquad i_2^2 = -1, \qquad i^2 = -1.
\tag{4.1}
$$

[6] Note that the meaning of $\operatorname{atan2}(y,x)$ is the mathematically positive angle of the vector $xe_1 + ye_2$ with the x-axis in the Euclidean plane, if the vector is attached to the origin.

[7] With an orthonormal basis of $\mathbb{R}^3$ the expression $i = e_{123}$ is valid. In a general basis we have $I = e_1 \wedge e_2 \wedge e_3$, $|I| = \sqrt{|I^2|}$, $i = I/|I|$. Similarly for $\mathbb{R}^{0,3}$.

While in $Cl(0,3)$ they square to

$$u^2 = -1, \qquad i_2^2 = -1, \qquad i^2 = +1. \tag{4.2}$$

The even subalgebras of both $Cl(3,0)$ and $Cl(0,3)$ are *isomorphic to quaternions* $\mathbb{H}$: $Cl_2(3,0) \cong Cl_2(0,3) \cong \mathbb{H}$. That means general multivectors m in $Cl(3,0)$ and $Cl(0,3)$ can always be represented as complex ($i^2 = -1$) or hyperbolic ($i^2 = +1$) (bi)quaternions[8]:

$$m = p + iq, \tag{4.3}$$

where in both cases p and q are (isomorphic to) quaternions

$$p = a_p e^{\alpha_p i_p}, \quad q = a_q e^{\alpha_q i_q}, \quad a_p, a_q \in \mathbb{R}_0^+, \quad i_p^2 = i_q^2 = -1, \tag{4.4}$$

with bivectors $i_p, i_q \in Cl_2(3,0)$ or $\in Cl_2(0,3)$.

Remark 4.1. Note that for $a_q = 0$ or $a_p = 0$ the factorization is already achieved[9] in the form of

$$\begin{aligned} m = a_p e^{\alpha_p i_p} = e^{\alpha_0} e^{\alpha_p i_p}, \;\; \alpha_0 = \ln a_p, \\ \text{or} \quad m = i a_q e^{\alpha_q i_q} = i e^{\alpha_0'} e^{\alpha_q i_q}, \;\; \alpha_0' = \ln a_q. \end{aligned} \tag{4.5}$$

This also means, that for $a_q = 0$ or $a_p = 0$, the multivector m can always be inverted as

$$m^{-1} = e^{-\alpha_0} e^{-\alpha_p i_p} \qquad \text{or} \qquad m^{-1} = i^{-1} e^{-\alpha_0'} e^{-\alpha_q i_q}, \tag{4.6}$$

respectively.

In the rest of this section, we therefore assume that both $a_p \neq 0$ and $a_q \neq 0$.

Clifford conjugation [10,14,24] maps

$$i_p \to -i_p, \qquad i_q \to -i_q, \qquad i \to i. \tag{4.7}$$

Clifford conjugation applied to (4.4) is equivalent to quaternion conjugation. Therefore we obtain the *central* multivector

$$\begin{aligned} m\overline{m} &= (p + iq)(\overline{p} + i\overline{q}) = p\overline{p} + i^2 q\overline{q} + i2\tfrac{1}{2}(p\overline{q} + q\overline{p}) \\ &= a_p^2 + i^2 a_q^2 + i2 a_p a_q \cos(p,q) = r_0 + i r_3 \in \mathbb{R} + i\mathbb{R}, \end{aligned} \tag{4.8}$$

and $\cos(p,q)$ being the cosine of the four-dimensional (4D) *angle* between quaternions p, q, because $\frac{1}{2}r_3 = \frac{1}{2}(p\overline{q}+q\overline{p})$ expresses the *inner* (or scalar) *product* in four dimensions for quaternions. In $Cl(0,3)$ the scalar part $r_0 = a_p^2 + i^2 a_q^2 = a_p^2 + a_q^2$ will only be zero

[8] Even though we introduce the notation p and q for the two even grade components of the Clifford algebra over a three-dimensional vector space, confusion with the signature variables in $Cl(p,q)$ should be negligible, because for each section we will specify the Clifford algebra under consideration with its explicit signature values, e.g. $Cl(3,0)$, $Cl(0,3)$, etc.

[9] The factor $i \in Cl(3,0)$ can be expressed as $e^{\frac{\pi}{2} i}$ and $i \in Cl(3,0)$ formally as $h(1,0,i)$ to represent $i a_q e^{\alpha_q i_q}$ as *pure* exponential factorization.

if both a_p and a_q are zero, that is if m itself is zero, but then the factorization result 0 would be trivial, and it would be contrary to our assumption for both $a_p \neq 0$ and $a_q \neq 0$.

Considering $Cl(0,3)$, we thus have $|\cos(p,q)| \leq 1$, which means

$$r_0 = a_p^2 + a_q^2 \geq |2a_p a_q| \geq |r_3| = 2|a_p a_q \cos(p,q)|. \tag{4.9}$$

For equality in (4.9) we would need $|2a_p a_q| = 2|a_p a_q \cos(p,q)|$, that is $\cos(p,q) = \pm 1$, and we would further need $a_p^2 + a_q^2 = |2a_p a_q|$, i.e. $a_p = a_q$. Together this would mean that for equality in (4.9)

$$p = \pm q, \qquad m = p(1 \pm i) = 2p\frac{1 \pm i}{2}, \tag{4.10}$$

where $\frac{1\pm i}{2}$ is an *idempotent* in the hyperbolic case (i.e. in $Cl(0,3)$ where $i^2 = +1$).

This means, that in $Cl(0,3)$ when equality in (4.9) holds, the not invertible multivector m can be finally represented as

$$\begin{aligned} &m = 2p\, id_\pm = 2a_p e^{\alpha_p i_p} id_\pm = e^{\alpha_0} e^{\alpha_p i_p} id_\pm = e^{\alpha_0} id_\pm e^{\alpha_p i_p}, \\ &\alpha_0 = \ln(2a_p). \end{aligned} \tag{4.11}$$

We have introduced the idempotent notation of (2.1), by setting there $u = i$, we note that with i being central, $id_\pm$ are also central, and we further note that due to the presence of the idempotent factor m cannot be inverted in this case.

For the rest of this section, we therefore *assume* the case of true inequality $r_0 > |r_3|$ in (4.9) in the algebra $Cl(0,3)$, which means that $r_0 + ir_3$ can then always be represented as exponential $r_0 + ir_3 = e^{2\alpha_0} e^{2\alpha_3 i}$. This assumption on r_0, r_3 for (4.9) is *not* needed in the case of $Cl(3,0)$.

In $Cl(3,0)$ with $r_0 = a_p^2 + i^2 a_q^2 = a_p^2 - a_q^2$ and $r_3 = 2a_p a_q \cos(p,q)$ it is possible that $m\overline{m}$ is zero without m being zero, that is for $a_p = a_q$ and $\cos(p,q) = 0$, that is the angle $\angle(p,q) = \pi/2$, i.e. when $q = p\mathbf{f}$, with any pure quaternion $\mathbf{f}$. Then the factorization of m can easily be given as:

$$\begin{aligned} &m = p(1 + i\mathbf{f}) = 2p\frac{1 + i\mathbf{f}}{2} = e^{\alpha_0} e^{\alpha_2 i_2} \frac{1 + i\mathbf{f}}{2}, \\ &\alpha_0 = \ln(2a_p), \qquad \alpha_2 = \alpha_p, \qquad i_2 = i_p, \qquad \mathbf{f} = p^{-1}q, \end{aligned} \tag{4.12}$$

with idempotent $\frac{1+i\mathbf{f}}{2}$, where $(i\mathbf{f})^2 = i\mathbf{f}i\mathbf{f} = i^2\mathbf{f}^2 = (-1)^2 = +1$. In this case m has no inverse because of the idempotent factor. In the rest of this section, we assume that $m\overline{m}$ will be different from zero.

We can now always factorize $m\overline{m}$ and furthermore compute its square root as

$$m\overline{m} = e^{2\alpha_0} e^{2\alpha_3 i}, \qquad \sqrt{m\overline{m}} = e^{\alpha_0} e^{\alpha_3 i}, \tag{4.13}$$

with

$$e^{\alpha_0} = (r_0^2 - i^2 r_3^2)^{\frac{1}{4}}, \qquad \alpha_0 = \tfrac{1}{4}\ln(r_0^2 - i^2 r_3^2), \tag{4.14}$$

and

$$\alpha_3 = \frac{1}{2}\begin{cases} \operatorname{atan2}(r_3, r_0) & \text{for} \quad m \in Cl(3,0) \\ \operatorname{atanh}(r_3/r_0) & \text{for} \quad m \in Cl(0,3) \end{cases} . \tag{4.15}$$

4.2. Factorization of normed multivector M with $M\overline{M} = 1$

Next, we divide m by the central square root $\sqrt{m\overline{m}}$ and obtain the normed multivector

$$M = \frac{m}{\sqrt{m\overline{m}}} = me^{-\alpha_0}e^{-\alpha_3 i}, \tag{4.16}$$

with unit norm

$$M\,\overline{M} = 1. \tag{4.17}$$

Factorization of M multiplied by $\sqrt{m\overline{m}}$ of (4.13) will then give the final factorization result for invertible m. The resulting form of M will therefore be (similar to (4.3) and (4.4))

$$M = P + Qi = a_P e^{\alpha_P i_P} + i a_Q e^{\alpha_Q i_Q} = e^{\alpha_P i_P}(a_P + i a_Q e^{-\alpha_P i_P} e^{\alpha_Q i_Q}), \tag{4.18}$$

with

$$M\,\overline{M} = 1 = a_P^2 + i^2 a_Q^2. \tag{4.19}$$

The two quaternions P and Q can always be computed explicitly as

$$P = \langle M\rangle_{\text{even}}, \qquad Q = \langle M\rangle_{\text{odd}}\, i^{-1}, \tag{4.20}$$

with amplitudes

$$a_P = \sqrt{P\overline{P}}, \qquad a_Q = \sqrt{Q\overline{Q}}, \tag{4.21}$$

unit bivectors

$$i_P = \frac{\langle P\rangle_2}{|\langle P\rangle_2|} \quad \text{with} \quad |\langle P\rangle_2| = \sqrt{-\langle P\rangle_2^2}, \tag{4.22}$$

$$i_Q = \frac{\langle Q\rangle_2}{|\langle Q\rangle_2|} \quad \text{with} \quad |\langle Q\rangle_2| = \sqrt{-\langle Q\rangle_2^2}, \tag{4.23}$$

and phase angles

$$\alpha_P = \operatorname{atan2}(\langle P\rangle_2 i_P^{-1}, \langle P\rangle_0), \qquad \alpha_Q = \operatorname{atan2}(\langle Q\rangle_2 i_Q^{-1}, \langle Q\rangle_0), \tag{4.24}$$

Computation of $M\overline{M}$ yields

$$\begin{aligned} M\overline{M} &= e^{\alpha_P i_P}(a_P + i a_Q e^{-\alpha_P i_P} e^{\alpha_Q i_Q})(a_P + i a_Q e^{-\alpha_Q i_Q} e^{\alpha_P i_P}) e^{-\alpha_P i_P} \\ &= e^{\alpha_P i_P}\big(a_P^2 + i^2 a_Q^2 + i a_P a_Q (e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + e^{-\alpha_Q i_Q}) e^{\alpha_P i_P}\big) e^{-\alpha_P i_P} \\ &= a_P^2 + i^2 a_Q^2 + i a_P a_Q e^{\alpha_P i_P}\big(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + e^{-\alpha_Q i_Q} e^{\alpha_P i_P}\big) e^{-\alpha_P i_P}\,. \end{aligned} \tag{4.25}$$

Because by construction $M\overline{M} = a_P^2 + i^2 a_Q^2 = 1$ we must have the second term in round brackets of line three of (4.25) to be zero

$$e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + e^{-\alpha_Q i_Q} e^{\alpha_P i_P} = e^{-\alpha_P i_P} e^{\alpha_Q i_Q} + (e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim} = 0. \tag{4.26}$$

Note that the tilde notation $(\ldots)^{\sim}$ for the Clifford algebra reverse [10,14,24] could in this case (when only the even subalgebras are concerned) be replaced by the Clifford conjugation, hence $(\ldots)^{\sim}$ is also equivalent to the application of quaternion conjugation.

We now analyze M further

$$\begin{aligned} M &= a_P e^{\alpha_P i_P} + i a_Q e^{\alpha_Q i_Q} = e^{\alpha_P i_P}(a_P + a_Q i (e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - 0)) \\ &\overset{(4.26)}{=} e^{\alpha_P i_P}\big(a_P + a_Q i (e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - \tfrac{1}{2} e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - \tfrac{1}{2}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim})\big) \\ &= e^{\alpha_P i_P}\big(a_P + a_Q i \tfrac{1}{2}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - (e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim})\big), \end{aligned} \tag{4.27}$$

where the term

$$\tfrac{1}{2}(e^{-\alpha_P i_P} e^{\alpha_Q i_Q} - (e^{-\alpha_P i_P} e^{\alpha_Q i_Q})^{\sim}) = \langle e^{-\alpha_P i_P} e^{\alpha_Q i_Q} \rangle_2 \tag{4.28}$$

is a pure bivector. Multiplied with trivector i we get a vector with length ω and unit direction u, $u^2 = 1 = -i^2$ for $Cl(3,0)$, and $u^2 = -1 = -i^2$ for $Cl(0,3)$,

$$\omega u = i \langle e^{-\alpha_P i_P} e^{\alpha_Q i_Q} \rangle_2 = i \left\langle \left(\frac{P}{a_P}\right)^{-1} \frac{Q}{a_Q} \right\rangle_2 = i\,\frac{a_P}{a_Q}\,\langle P^{-1} Q \rangle_2. \tag{4.29}$$

Thus in full generality, the normed invertible multivector M can be represented by

$$M = e^{\alpha_P i_P}(a_P + a_Q\,\omega u) = (a_P + a_Q\,\omega u') e^{\alpha_P i_P}, \;\; u' = e^{\alpha_P i_P} u e^{-\alpha_P i_P}. \tag{4.30}$$

Note that unit vector u', is simply a rotated version of u. Computing

$$\begin{aligned} M\overline{M} &= e^{\alpha_P i_P}(a_P + a_Q \omega u)(a_P - a_Q \omega u) e^{-\alpha_P i_P} \\ &= \ldots = a_P^2 - u^2 a_Q^2 \omega^2 = a_P^2 + i^2 a_Q^2 \omega^2, \end{aligned} \tag{4.31}$$

shows by comparison with (4.19), that $\omega^2 = 1$, i.e. $\omega = 1$. Without restriction of generality, we can therefore express

$$\begin{aligned} M &= e^{\alpha_P i_P}(a_P + a_Q u) = (a_P + a_Q u') e^{\alpha_P i_P}, \\ M\overline{M} &= a_P^2 - u^2 a_Q^2 = a_P^2 - u'^2 a_Q^2 = 1, \end{aligned} \tag{4.32}$$

We thus end up with

$$M = e^{\alpha_2 i_2} e^{\alpha_1 u} = e^{\alpha_1 u'} e^{\alpha_2 i_2},$$
$$\alpha_1 = \left\{ \begin{array}{l} \operatorname{atanh}(a_Q/a_P) \text{ for } Cl(3,0) \\ \operatorname{atan2}(a_Q, a_P) \text{ for } Cl(0,3) \end{array} \right\}, \quad \alpha_2 = \alpha_P, \qquad i_2 = i_P. \tag{4.33}$$

We note that in $Cl(3,0)$ due to $M\overline{M} = a_P^2 - u^2 a_Q^2 = a_P^2 - a_Q^2 = 1$, which should be compared with $\cosh(\alpha_1) - \sinh(\alpha_1) = 1$, the hyperbolic plane case distinctions in (2.11) are not needed here.

And we finally have under the above assumptions (invertibility of m) the exponential factorization

$$\begin{aligned} m = M\sqrt{m\overline{m}} &= e^{\alpha_2 i_2} e^{\alpha_1 u} e^{\alpha_0} e^{\alpha_3 i} \\ &= e^{\alpha_0} e^{\alpha_2 i_2} e^{\alpha_1 u} e^{\alpha_3 i} = e^{\alpha_0} e^{\alpha_1 u'} e^{\alpha_2 i_2} e^{\alpha_3 i}. \end{aligned} \tag{4.34}$$

4.3. Factorization result for $Cl(3,0)$ and $Cl(0,3)$ and inverse

Summarizing all cases we end up with

$$m = p + iq = \begin{cases} e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad q = 0, \\ i^{-1} e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad p = 0, \\ e^{\alpha_0} e^{\alpha_2 i_2} id_{\pm} & \text{for} \quad q = \pm p \quad \text{in} \quad Cl(0,3), \\ e^{\alpha_0} e^{\alpha_2 i_2} \frac{1+i\mathbf{f}}{2} & \text{for} \quad q = p\mathbf{f} \quad \text{in} \quad Cl(3,0), \\ e^{\alpha_0} e^{\alpha_1 u'} e^{\alpha_2 i_2} e^{\alpha_3 i} & \text{otherwise.} \end{cases} \tag{4.35}$$

where we have set in line three: $\alpha_2 = \alpha_p$, $i_2 = i_p$, and idempotents $id_{\pm} = (1 \pm i)/2$, and in line four we refer to (4.12) for the unit bivector $\mathbf{f}$. The value of $i_2 = i_p$ in lines one, and three to five, while in line two we have $i_2 = i_Q$. We note that line one is a special case of line five for $\alpha_1 = \alpha_3 = 0$. For $Cl(3,0)$ line two is a special case of line five for $\alpha_1 = 0$ and $\alpha_3 = -\pi/2$. We may also interpret in $Cl(0,3)$ line two is a special case of line five for $\alpha_1 = 0$ and by replacing $e^{\alpha_3 i} \to h(1,0,i) = i = i^{-1}$. So essentially only lines three to five of (4.35) matter, and in both $Cl(3,0)$ and $Cl(0,3)$ we have one special case with idempotent factor (central in $Cl(3,0)$) and one general case (line five) with full exponential factorization.

Except for $q = \pm p$ in $Cl(0,3)$ and for $a_p = a_q$, $\cos(p,q) = 0$ in $Cl(3,0)$ (equivalent to $q = p\mathbf{f}$, $\mathbf{f}$ a pure quaternion, respectively via isomorphism in $Cl(3,0)$ a unit bivector), we can always invert m and obtain

$$m^{-1} = \begin{cases} e^{-\alpha_0} e^{-\alpha_2 i_2} & \text{for} \quad q = 0, \\ i e^{-\alpha_0} e^{-\alpha_2 i_2} & \text{for} \quad p = 0, \\ \text{none} & \text{for} \quad q = \pm p \quad \text{in} \quad Cl(0,3), \\ \text{none} & \text{for} \quad q = p\mathbf{f} \quad \text{in} \quad Cl(3,0), \\ e^{-\alpha_0} e^{-\alpha_1 u} e^{-\alpha_2 i_2} e^{-\alpha_3 i} & \text{otherwise.} \end{cases} \tag{4.36}$$

where we have used in line five that according to (4.30) and (4.33)

$$u' = e^{\alpha_P i_P} u e^{-\alpha_P i_P} = e^{\alpha_2 i_2} u e^{-\alpha_2 i_2}. \tag{4.37}$$

5. Direct Factorization of $Cl(1,2)$

In Appendix B we discuss factorization of $Cl(1,2)$ using an isomorphism to $Cl(3,0)$, but because in the resulting exponential factorization the exponentials are not pure blades[10], we aim here for a direct factorization with pure blade exponentials, even though at first sight this appears more laborious. The procedure we use is similar to Section 4, but as may be expected there is a more delicate idempotent structure, which leads to more intricate results. To keep the computations elementary, we again work with an explicit orthonormal basis. This does not exclude the possibility to lift computations and results to an invariant basis independent level after gaining deeper algebraic understanding in the future, for which the present explicit results may then still serve as a kind of reference case. Furthermore, many Clifford multivector computer algebra systems begin with the definition of an orthonormal vector space basis.

5.1. Computation and factorization of $m\overline{m}$

With the intent to normalize m we first compute $m\overline{m}$. In $Cl(1,2)$ the central unit pseudoscalar squares to $i^2 = -1$ and

$$\begin{aligned} &e_1^2 = -e_2^2 = -e_3^2 = e_{12}^2 = e_{31}^2 = -e_{23}^2 = 1, \\ &e_1 = -ie_{23}, \quad e_2 = ie_{31}, \quad e_3 = ie_{12}. \end{aligned} \tag{5.1}$$

It allows us to rewrite a general multivector as

$$\begin{aligned} m &= m_0 + m_1e_1 + m_2e_2 + m_3e_3 + m_{12}e_{12} + m_{31}e_{31} + m_{23}e_{23} + m_{123}i \\ &= m_0 + m_{23}e_{23} + m_{12}e_{12} + m_{31}e_{31} + i(m_{123} - m_1e_{23} + m_2e_{31} + m_3e_{12}) \\ &= p_0 + p_{23}e_{23} + p_{12}e_{12} + p_{31}e_{31} + i(q_0 + q_{23}e_{23} + q_{12}e_{12} + q_{31}e_{31}) \\ &= p + iq \end{aligned} \tag{5.2}$$

with suitable identifications of the eight coefficients of m with four coefficients of p and four coefficients of q, where both $p, q \in Cl_2(1,2) \cong Cl(2,0)$. We can therefore represent both p and q as

$$\begin{aligned} &p = (b_p + a_pu_p)e^{\alpha_{2p}e_{23}}, \quad b_p = \sqrt{p_0^2 + p_{23}^2}, \quad a_p = \sqrt{p_{12}^2 + p_{31}^2}, \quad u_p^2 = 1, \\ &q = (b_q + a_qu_q)e^{\alpha_{2q}e_{23}}, \quad b_q = \sqrt{q_0^2 + q_{23}^2}, \quad a_q = \sqrt{q_{12}^2 + q_{31}^2}, \quad u_q^2 = 1, \end{aligned} \tag{5.3}$$

following (3.1). The unit bivectors u_p, u_q with positive square are linear combinations of e_{12} and e_{31}.

If $a_p = b_p = 0$ or $a_q = b_q = 0$, then the final factorization[11] is given by

$$m = iq = i(b_q + a_qu_q)e^{\alpha_{2q}e_{23}} \tag{5.4}$$

[10] With pure blade we mean a simple blade that can be factored as an outer product of vectors.

[11] We clearly can still replace $i(b_q + a_qu_q)$ by $h(a_q, b_q, u_q)e^{\frac{\pi}{2}i}$, and $b_p + a_pu_p$ by $h(a_p, b_p, u_p)$ for full exponential factorization.

or by

$$m = p = (b_p + a_p u_p) e^{\alpha_{2p} e_{23}}, \tag{5.5}$$

respectively. In the rest of this section we can therefore *assume* that both p and q are nonzero.

Note that p is not invertible iff $a_p = b_p$, and likewise q is not invertible iff $a_q = b_q$. For later use we compute

$$p\overline{p} = b_p^2 - a_p^2, \quad q\overline{q} = b_q^2 - a_q^2,$$
$$\frac{1}{2}(q\overline{p} + p\overline{q}) = p_0 q_0 + p_{23} q_{23} - (p_{12} q_{12} + p_{31} q_{31}). \tag{5.6}$$

Let us also compute

$$m\overline{m} = (p + iq)(\overline{p} + i\overline{q}) = p\overline{p} + i^2 q\overline{q} + i(q\overline{p} + p\overline{q}) = p\overline{p} - q\overline{q} + i(q\overline{p} + p\overline{q}). \tag{5.7}$$

$m\overline{m}$ is zero if (1) both hyperbolic type norms $p\overline{p}$ and $q\overline{q}$ agree

$$p\overline{p} - q\overline{q} = b_p^2 - a_p^2 - (b_q^2 - a_q^2) = b_p^2 - b_q^2 - (a_p^2 - a_q^2) = 0, \tag{5.8}$$

and (2) a sort of four-dimensional hyperbolic orthogonality condition is met

$$\frac{1}{2}(q\overline{p} + p\overline{q}) = p_0 q_0 + p_{23} q_{23} - (p_{12} q_{12} + p_{31} q_{31}) = 0. \tag{5.9}$$

If p is not invertible it can be written as

$$p = 2a_p \frac{1 + u_p}{2} e^{\alpha_{2p} e_{23}}, \tag{5.10}$$

where $\frac{1+u_p}{2}$ is an idempotent. Similarly, if q is not invertible it can be written as

$$q = 2a_q \frac{1 + u_q}{2} e^{\alpha_{2q} e_{23}}, \tag{5.11}$$

where $\frac{1+u_q}{2}$ is an idempotent. Before we compute the normed M in Section 5.2, we discuss the cases of non-invertible even subalgebra components p, q, and of invertible ones, respectively.

5.1.1. Noninvertible even subalgebra components

If both p and q are not invertible, then m takes by (5.10) and (5.11) the form

$$m = 2a_p \frac{1 + u_p}{2} e^{\alpha_{2p} e_{23}} + i 2a_q \frac{1 + u_q}{2} e^{\alpha_{2q} e_{23}}. \tag{5.12}$$

Now we want to find out under these assumptions for p and q, when $m\overline{m}$ is invertible and when not. In this case we can compute

$$\begin{aligned}
m\overline{m} &= 0 + i^2 0 + i(q\overline{p} + p\overline{q}) \\
&= ia_p a_q[(1+u_p)e^{\alpha_{2p}e_{23}}e^{-\alpha_{2q}e_{23}}(1-u_q) \\
&\qquad + (1+u_q)e^{\alpha_{2q}e_{23}}e^{-\alpha_{2p}e_{23}}(1-u_p)],
\end{aligned} \tag{5.13}$$

with $\Delta = \alpha_{2p} - \alpha_{2q}$ and $e^{\pm\Delta e_{23}} = \cos\Delta \pm e_{23}\sin\Delta$, this becomes

$$\begin{aligned}
m\overline{m} &= ia_p a_q\{(1+u_p)\cos\Delta(1-u_q) + (1+u_q)\cos\Delta(1-u_p) \\
&\qquad + \sin\Delta[(1+u_p)e_{23}(1-u_q) - (1+u_q)e_{23}(1-u_p)]\} \\
&= ia_p a_q\{\cos\Delta[(1+u_p)(1-u_q) + (1+u_q)(1-u_p)] \\
&\qquad + \sin\Delta e_{23}[(1-u_p)(1-u_q) - (1-u_q)(1-u_p)]\} \\
&= ia_p a_q\{\cos\Delta[2 - u_p u_q - u_q u_p] + \sin\Delta e_{23}[u_p u_q - u_q u_p]\}.
\end{aligned} \tag{5.14}$$

Now the product of the unit bivectors is minus the product of two negative definite vectors $\vec{u}_p, \vec{u}_q$ in the e_{23}-plane with mutual angle ϑ

$$u_p u_q = e_1\vec{u}_p e_1\vec{u}_q = -\vec{u}_p\vec{u}_q. \tag{5.15}$$

Therefore[12]

$$\begin{aligned}
u_p u_q + u_q u_p &= -2\vec{u}_p \cdot \vec{u}_q = 2\cos\vartheta, \\
u_p u_q - u_q u_p &= -2\vec{u}_p \wedge \vec{u}_q = -2e_{23}\sin\vartheta.
\end{aligned} \tag{5.16}$$

Hence

$$\begin{aligned}
m\overline{m} &= ia_p a_q\{\cos\Delta[2 - 2\cos\vartheta] + 2\sin\Delta\sin\vartheta(-e_{23}^2)\} \\
&= 2ia_p a_q\{\cos\Delta - \cos\Delta\cos\vartheta + \sin\Delta\sin\vartheta\} \\
&= 2ia_p a_q\{\cos\Delta - \cos(\Delta+\vartheta)\}
\end{aligned} \tag{5.17}$$

So the product $m\overline{m} = 0$ for the following combinations of Δ and ϑ

$$\begin{array}{ll}
\vartheta = 0, & \text{any } \ 0 \le \Delta < 2\pi, \\
\vartheta = \pi, & \Delta = \frac{\pi}{2}, \frac{3\pi}{2}, \\
0 \le \vartheta < 2\pi, & \Delta = \pi - \frac{\vartheta}{2}.
\end{array} \tag{5.18}$$

Note that the second line is a special case of the third line for $\vartheta = \pm\pi$. In all other cases $m\overline{m} \neq 0$ and m will be invertible, *even* under the assumption that p and q are not invertible. This means that for $m\overline{m} = 0$ the non-invertible multivector m will take one of these three forms

$$m = \left\{ \begin{array}{l}
(1+u_p)[a_p e^{\Delta e_{23}} + ia_q] \\
[a_p(1+u_p)(\pm e_{23}) + ia_q(1-u_p)] \\
[a_p(1+u_p)e^{(\pi-\vartheta/2)e_{23}} + ia_q(1+u_q)]
\end{array} \right\} e^{\alpha_{2q}e_{23}}. \tag{5.19}$$

[12]The equations (5.16) for sine and cosine of ϑ can also simply be interpreted as defining ϑ.

Note that in the third line the angle ϑ is the angle between u_p and u_q, as determined above by (5.16). The three cases in (5.19) have on the right side idempotents (or linear combinations of idempotents) as factors, followed by the common bivector exponential $e^{\alpha_{2q}e_{23}}$. This seems to be the closest we currently can get to a full exponential factorization in this case.

The factorization of an invertible m with non-zero $m\overline{m}$ as in (5.17) when both p and q are not invertible, is discussed at the end of this section.

5.1.2. Invertible even subalgebra component

We first assume that p is invertible, i.e. $a_p \neq b_p$, without assuming q to be invertible. After discussing this case, we will discuss the analogous case for which q is assumed to be invertible, but not p. We can therefore compute the left quotient

$$s = p^{-1}q = (b_s + a_s u_s)e^{\alpha_{2s}e_{23}}, \tag{5.20}$$

which must also be an element of the subalgebra $Cl_2(1,2)$ and can therefore be represented in this form, where a_s, b_s are real non-negative numbers, bivector u_s has square $u_s^2 = +1$, and $0 \leq \alpha_{2s} < 2\pi$. This allows us to rewrite m as

$$m = p(1 + ip^{-1}q) = p(1 + is). \tag{5.21}$$

For this form of m we compute

$$\begin{aligned} m\overline{m} &= p(1+is)(1-i\overline{s})\overline{p} = p[1 + i^2 s\overline{s} + i(s+\overline{s})]\overline{p} \\ &= p[1 + i^2(b_s^2 - a_s^2) + i(2b_s\cos\alpha_{2s} + a_s\sin\alpha_{2s}u_s e_{23} + a_s\sin\alpha_{2s}e_{23}u_s)]\overline{p} \\ &= p\overline{p}[1 + i^2(b_s^2 - a_s^2) + i2b_s\cos\alpha_{2s}] \\ &= (b_p^2 - a_p^2)[1 - (b_s^2 - a_s^2) + i2b_s\cos\alpha_{2s}], \end{aligned} \tag{5.22}$$

where we have used for the fourth equality that $u_s e_{23} = -e_{23}u_s$. By assumption the factor $(b_p^2 - a_p^2) \neq 0$, so for $m\overline{m}$ to be zero we must have

$$b_s^2 - a_s^2 = 1, \tag{5.23}$$

which implies that $b_s > 0$, and we must have

$$\cos\alpha_{2s} = 0 \quad \Leftrightarrow \quad \alpha_{2s} = \frac{\pi}{2}, \frac{3\pi}{2} \quad \Leftrightarrow \quad e^{\alpha_{2s}e_{23}} = \pm e_{23}. \tag{5.24}$$

The relationship $b_s^2 - a_s^2 = 1$ is that of hyperbolic cosine and sine for some angle φ_s. We conclude that $m\overline{m}$ will be zero for this form of quotient s

$$s = (b_s + a_s u_s)e^{\alpha_{2s}e_{23}} = (\cosh\varphi_s + u_s\sinh\varphi_s) \pm e_{23} = \pm e^{\varphi_s u_s}e_{23}, \tag{5.25}$$

and therefore

$$q = ps = \pm p\, e^{\varphi_s u_s}e_{23}, \tag{5.26}$$

and

$$m = p + iq = 2p\frac{1+is}{2}. \tag{5.27}$$

Furthermore, we compute the square of s as

$$s^2 = (\pm e^{\varphi_s u_s} e_{23})^2 = e^{\varphi_s u_s} e_{23} e^{\varphi_s u_s} e_{23} = e^{\varphi_s u_s} e^{-\varphi_s u_s} e_{23}^2 = e_{23}^2 = -1, \tag{5.28}$$

where we used in the third equality that $e_{23} u_s = -e_{23} u_s$. This means that

$$(is)^2 = i^2 s^2 = (-1)^2 = +1, \tag{5.29}$$

and therefore $\frac{1+is}{2}$ is an *idempotent*. So assuming that p is invertible and m is not invertible we obtain the factorization of m as

$$m = 2p\frac{1+is}{2} = 2(b_p + a_p u_p) e^{\alpha_{2p} e_{23}} \frac{1+is}{2}, \tag{5.30}$$

where the factor $2(b_p + a_p u_p)$ can further be put into exponential form using $E(2a_p, 2b_p, u_p)$ in (2.11). The idempotent factor $\frac{1+is}{2}$ means that m is manifestly (obviously) not invertible.

Now let us instead assume, that q is invertible. We can multiply m with i^{-1}

$$\begin{aligned} m' &= i^{-1}m = i^{-1}p + i^{-1}iq = q + i^{-1}p = p' + iq', \\ p' &= q, \qquad q' = i^{-2}p = i^2 p, \end{aligned} \tag{5.31}$$

where for $Cl(1,2)$ we will have $i^{-2} = i^2 = -1$, for $Cl(2,1)$ we would have $i^{-2} = i^2 = -1$. We can now apply the above analysis of m with p invertible to m' with p' invertible, and in the end multiply the result with i to get the expression for $m = im' = ii^{-1}m$. We also notice that

$$m'\overline{m'} = i^{-2} m\overline{m} = \pm m\overline{m}, \tag{5.32}$$

which means that $m'\overline{m'} = 0$, iff $m\overline{m} = 0$, and if we factorize $m'\overline{m'} \neq 0$ and compute its square root $\sqrt{m'\overline{m'}}$, then $\sqrt{m\overline{m}} = i\sqrt{m'\overline{m'}}$, which means to add to $\sqrt{m'\overline{m'}}$ a phase factor $e^{i\frac{\pi}{2}}$ in the case of $i^2 = -1$.

Doing this we get that

$$s' = p'^{-1} q' = q^{-1}(i^2 p) = i^2 q^{-1} p. \tag{5.33}$$

Following the analogous steps above we obtain, if we assume that p' is invertible and m' (and therefore m) is not invertible, then the factorization of m' (and m) will be

$$\begin{aligned} m' &= 2p'\frac{1+is'}{2} = 2(b_{p'} + a_{p'} u_{p'}) e^{\alpha_{2p'} e_{23}} \frac{1+is'}{2} \\ &\overset{(p'=q)}{=} 2q\frac{1+is'}{2} = 2(b_q + a_q u_q) e^{\alpha_{2q} e_{23}} \frac{1+is'}{2} \\ m &= im' = i2(b_q + a_q u_q) e^{\alpha_{2q} e_{23}} \frac{1+is'}{2}, \end{aligned} \tag{5.34}$$

where the factor $2(b_q + a_q u_q)$ can further be put into exponential form using $E(2a_q, 2b_q, u_q)$ in (2.11). The idempotent factor $\frac{1+is'}{2}$ means that m' (and therefore m) is again manifestly not invertible.

If the central value $m\overline{m} \neq 0$, then m is invertible as

$$m^{-1} = \frac{\overline{m}}{m\overline{m}}. \tag{5.35}$$

Because $m\overline{m}$ is then given as a non-zero sum of scalar and trivector, and $i^2 = -1$, we can always represent it as

$$m\overline{m} = e^{2\alpha_0} e^{2\alpha_3 i}, \tag{5.36}$$

and its square root as

$$\sqrt{m\overline{m}} = e^{\alpha_0} e^{\alpha_3 i}, \tag{5.37}$$

and we can divide m by this square root to get a new normed multivector

$$M = \frac{m}{\sqrt{m\overline{m}}} = m e^{-\alpha_0} e^{-\alpha_3 i}, \qquad M\overline{M} = 1. \tag{5.38}$$

5.2. Factorization of normed multivector M, $M\overline{M} = 1$

We represent M again as a sum of two elements from the even subalgebra $Cl_2(1,2)$

$$\begin{aligned} M = P + iQ, \quad P = \langle M \rangle_{even} = (b_P + a_P) e^{\alpha_{2P} e_{23}}, \\ Q = \langle M \rangle_{odd} i^{-1} = (b_Q + a_Q) e^{\alpha_{2Q} e_{23}}. \end{aligned} \tag{5.39}$$

and compute

$$M\overline{M} = (P + iQ)(\overline{P} + i\overline{Q}) = P\overline{P} + i^2 Q\overline{Q} + i(Q\overline{P} + P\overline{Q}) = 1, \tag{5.40}$$

and therefore we must have

$$Q\overline{P} + P\overline{Q} = 0 \Leftrightarrow Q\overline{P} = -P\overline{Q}, \quad M\overline{M} = P\overline{P} + i^2 Q\overline{Q} = 1. \tag{5.41}$$

We now ask what happens when P or Q or both are not invertible. If P is not invertible, then we have $b_P = a_P$, and if Q is not invertible we have $b_Q = a_Q$. If we assume both P and Q not invertible then we have $P\overline{P} = Q\overline{Q} = 0$ and consequently

$$M\overline{M} = P\overline{P} + i^2 Q\overline{Q} = 0 - i^2 0 = 0 \neq 1, \tag{5.42}$$

which is a contradiction. Therefore either P or Q or both must be invertible, which we assume in the next subsection.

5.2.1. One even subalgebra component of M being invertible

We first assume P to be invertible, which allows us to compute

$$Q\overline{P} + P\overline{Q} = 0 \Leftrightarrow P(P^{-1}Q + \overline{Q}\,\overline{P}^{-1})\overline{P} = 0 \Leftrightarrow P^{-1}Q + \overline{Q}\,\overline{P}^{-1} = 0. \tag{5.43}$$

Then M can be rewritten as

$$\begin{aligned} M &= P + iQ = P(1 + iP^{-1}Q) = P(1 + i(P^{-1}Q - 0)) \\ &= P(1 + i(P^{-1}Q - \frac{1}{2}P^{-1}Q - \frac{1}{2}\overline{Q}\,\overline{P}^{-1}) \\ &= P(1 + i\frac{1}{2}(P^{-1}Q - \overline{Q}\,\overline{P}^{-1})), \end{aligned} \tag{5.44}$$

where we observe that

$$\frac{1}{2}(P^{-1}Q - \overline{Q}\,\overline{P}^{-1}) = \langle P^{-1}Q\rangle_2 \tag{5.45}$$

is a pure bivector and therefore

$$i\langle P^{-1}Q\rangle_2 = \vec{\omega} \tag{5.46}$$

a vector. Therefore

$$\begin{aligned} M &= P(1 + \vec{\omega}), \\ M\overline{M} &= P(1 + \vec{\omega})(1 - \vec{\omega})\overline{P} = P(1 - \vec{\omega}^2)\overline{P} = P\overline{P} - P\overline{P}\vec{\omega}^2 \\ &\overset{(5.41)}{=} P\overline{P} + i^2 Q\overline{Q}. \end{aligned} \tag{5.47}$$

From the last equality of (5.47) we conclude

$$-P\overline{P}\vec{\omega}^2 = +i^2 Q\overline{Q}, \tag{5.48}$$

that is

$$\vec{\omega}^2 = \frac{-i^2 Q\overline{Q}}{P\overline{P}} \overset{i^2=-1}{=} \frac{Q\overline{Q}}{P\overline{P}} \begin{cases} > 0 & \text{for } \frac{Q\overline{Q}}{P\overline{P}} > 0, \\ = 0 & \text{for } Q\overline{Q} = 0, \\ < 0 & \text{for } \frac{Q\overline{Q}}{P\overline{P}} < 0. \end{cases} \tag{5.49}$$

This leads to the following factorization of M

$$M = P(1 + \vec{\omega}) = (b_P + a_P u_P)e^{\alpha_{2P}e_{23}} \begin{cases} E(\omega, 1, \frac{\vec{\omega}}{\omega}), & \omega = \sqrt{\vec{\omega}^2}, \\ 1 + \vec{\omega} = e^{\vec{\omega}}, & \vec{\omega}^2 = 0, \\ e^{\alpha'_0}e^{\alpha_1\frac{\vec{\omega}}{\omega}}, & \omega = \sqrt{-\vec{\omega}^2}, \end{cases} \tag{5.50}$$

with

$$\alpha_1 = \operatorname{atan2}(\omega, 1), \qquad \alpha'_0 = \ln(\sqrt{1 + \omega^2}). \tag{5.51}$$

Now let us instead assume that Q is invertible (and therefore $\overline{Q}$ as well), without specifying the invertibility of P.

$$Q\overline{P} + P\overline{Q} = 0 \Leftrightarrow Q(\overline{P}\,\overline{Q}^{-1} + Q^{-1}P)\overline{Q} \Leftrightarrow \overline{P}\,\overline{Q}^{-1} + Q^{-1}P = 0. \tag{5.52}$$

We can therefore express

$$\begin{aligned}\overline{M} &= \overline{P} + i\overline{Q} = (\overline{P}\,\overline{Q}^{-1} + i)\overline{Q} = (\overline{P}\,\overline{Q}^{-1} - 0 + i)\overline{Q} \\ &= (\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}Q^{-1}P + i)\overline{Q} \\ &= (\tfrac{1}{2}\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}Q^{-1}P + i)\overline{Q}\end{aligned} \tag{5.53}$$

with pure bivector

$$\overline{B} = \tfrac{1}{2}\overline{P}\,\overline{Q}^{-1} - \tfrac{1}{2}Q^{-1}P = -\tfrac{1}{2}\langle Q^{-1}P\rangle_2 = \tfrac{1}{2}\langle \overline{P}\,\overline{Q}^{-1}\rangle_2 = \tfrac{1}{2}\langle \overline{Q^{-1}P}\rangle_2. \tag{5.54}$$

So we get

$$\overline{M} = (i + \overline{B})\overline{Q}, \tag{5.55}$$

and hence

$$M = Q(i + B) = iQ(1 + i^{-1}B) = iQ(1 + \vec{\mu}), \quad \vec{\mu} = i^{-1}B. \tag{5.56}$$

We further compute

$$\begin{aligned}M\overline{M} &= iQ(1+\vec{\mu})i(1-\vec{\mu})\overline{Q} = i^2Q(1-\vec{\mu}^2)\overline{Q} = i^2Q\overline{Q} - i^2\vec{\mu}^2Q\overline{Q} \\ &\overset{(5.41)}{=} P\overline{P} + i^2Q\overline{Q}\end{aligned} \tag{5.57}$$

which implies that for $i^2 = -1$

$$P\overline{P} = -i^2\vec{\mu}^2Q\overline{Q} \Leftrightarrow \vec{\mu}^2 = -i^2\frac{P\overline{P}}{Q\overline{Q}} = \frac{P\overline{P}}{Q\overline{Q}}\begin{cases} > 0 & \text{for } \frac{P\overline{P}}{Q\overline{Q}} > 0, \\ = 0 & \text{for } P\overline{P} = 0, \\ < 0 & \text{for } \frac{P\overline{P}}{Q\overline{Q}} < 0. \end{cases} \tag{5.58}$$

This leads to the following factorization of M

$$M = iQ(1+\vec{\mu}) = i(b_Q + a_Qu_Q)e^{\alpha_{2Q}e_{23}}\begin{cases} E(\mu, 1, \frac{\vec{\mu}}{\mu}), & \mu = \sqrt{\vec{\mu}^2}, \\ 1 + \vec{\mu} = e^{\vec{\mu}}, & \vec{\mu}^2 = 0, \\ e^{\alpha_0''}e^{\alpha_1\frac{\vec{\mu}}{\mu}}, & \mu = \sqrt{-\vec{\mu}^2}, \end{cases} \tag{5.59}$$

with

$$\alpha_1 = \operatorname{atan2}(\mu, 1), \qquad \alpha_0'' = \ln(\sqrt{1+\mu^2}). \tag{5.60}$$

We note that the two factorizations (5.50) or (5.59) have a nearly identical form. We obtain (5.59) by exchanging P and Q in (5.50) and by multiplying with i, which can formally be replaced by $i = e^{\frac{\pi}{2}i}$.

5.3. Result of factorization in $Cl(1,2)$

For better overview, we summarize the results of Section 5. If only one of the two subalgebra components p, q of $m = p + iq$ is non-zero, then the final factorizations are directly given by the factorization of iq in (5.4) or p in (5.5).

Factorizations of non-invertible $m \in Cl(1,2)$ were stated in (5.19) for non-invertible p and q, in (5.30) for invertible p and in (5.34) for invertible q.

Finally for either P invertible or Q invertible we obtain

$$m = \sqrt{m\overline{m}}M = e^{\alpha_0} e^{\alpha_3 i} M \tag{5.61}$$

assuming the factorized forms (5.50) or (5.59) for M.

If both p and q are not invertible, then $m\overline{m}$ will have the form (5.17) of i times a real scalar S. If this real scalar is not zero, then $m\overline{m}$ can be factorized as

$$m\overline{m} = e^{2\alpha_0}(\pm i) = e^{2\alpha_0} e^{2(\pm\frac{\pi}{4})i}, \quad \alpha_0 = \tfrac{1}{2}\ln(|S|), \quad \sqrt{m\overline{m}} = e^{\alpha_0} e^{(\pm\frac{\pi}{4})i}, \tag{5.62}$$

the sign will be identical with that of S. We can again divide by $\sqrt{m\overline{m}}$ with the result

$$M = m e^{-\alpha_0} e^{(\mp\frac{\pi}{4})i}, \qquad M\overline{M} = 1. \tag{5.63}$$

The very same analysis applied in (5.39) to (5.61) with the factorized forms (5.50) (for P invertible) or (5.59) (for Q invertible) can then be applied again. In (5.61) we will have $\alpha_0 = \frac{1}{2}\ln(|S|)$ and $\alpha_3 = \pm\frac{\pi}{4}$.

6. Conclusion

In this paper we have considered general elements of the three Clifford algebras $Cl(3,0)$, $Cl(1,2)$ and $Cl(0,3)$, and studied multivector factorization into products of exponentials, idempotents and blades, where the exponents are frequently blades of grades zero (scalar) to n (pseudoscalar). Depending on the algebra, we used methods of direct computation or applied several isomorphisms, to simplify the computation at hand or make use of known results in isomorphic representations. Furthermore, all results of this work could be implemented in Clifford algebra software like [26].

It may be possible in the future to extend this approach to even higher dimensional Clifford algebras, but simple products of exponentials and idempotents may, due to the dimensionality of the k-vector spaces, have to include multiple non-commuting exponential factors with k-vectors of the same grade in the exponents. Of particular interest would be to apply our methods to conformal geometric algebra $Cl(4,1)$ widely used in computer graphics and robotics [5,16]. Furthermore a complete factorization study of $Cl(1,3)$ and $Cl(3,1)$ that are both of great importance in special relativity and relativistic physics [3,10,11,20] may be of considerable interest. The present work can e.g. be applied in the study of Lipschitz versors, see e.g. E.4.2 in [28], pinor and spinor groups, and in the development of Clifford Fourier and wavelet transformations [17,20], compare also the third paragraph on motivation for this research in the introduction Section 1.

It might also be of interest to represent Clifford algebras $Cl(3,0)$, $Cl(1,2)$ and $Cl(0,3)$, in terms of tensor products of quaternions and their subalgebras, and re-express the results we have obtained above, or even further develop them, compare

[8,9].

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References

[1] Buchholz S, Tachibana K, Hitzer E. Optimal Learning Rates for Clifford Neurons. In: de Sá JM, Alexandre LA, Duch W, Mandic D., editors. Artificial Neural Networks – ICANN 2007. Lecture Notes in Computer Science, Berlin, Heidelberg:Springer; 2007;4668:864–873.

[2] Buchholz S, Hitzer E, Tachibana K. Coordinate independent update formulas for versor Clifford neurons. Proc. Joint 4th Int. Conf. on Soft Comp. and Intel. Sys. and 9th Int. Symp. on Adv. Intel. Sys., 17–21 Sep. 2008, Nagoya, Japan, 2008;814–819.

[3] Doran C, Lasenby A. Geometric Algebra for Physicists. Cambridge (UK): Cambridge University Press; 2007.

[4] Dorst L, Valkenburg R. Square Root and Logarithm of Rotors in 3D Conformal Geometric Algebra Using Polar Decomposition. in Dorst L, Lasenby J, editors. Guide to Geometric Algebra in Practice. London:Springer; 2011;81–104.

[5] Dorst L, Lasenby J, editors. Guide to Geometric Algebra in Practice. London:Springer: 2011.

[6] Falcao MI, Malonek HR. Generalized Exponentials through Appell sets in $\mathbb{R}^{n+1}$ and Bessel functions. AIP Conference Proceedings, 2007;936:738–741.

[7] Genesis chapter 1 verse 1, in The Holy Bible, English Standard Version, Wheaton, Illinois:Crossway Bibles, Good News Publishers; 2001.

[8] Girard PR, et al. Hyperquaternions: An Efficient Mathematical Formalism for Geometry. in Nielsen F, Barbaresco F, editors. Geometric Science of Information 2019, LNCS. 2019;11712:116–125.

[9] Girard PR, et al. Hyperquaternions: A New Tool for Physics. Adv. Appl. Clifford Algebras. 2018;28(68):1–14

[10] Hestenes D, Sobczyk G. Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics. Heidelberg:Springer; 1984.

[11] Hestenes D. Space-Time Algebra. Basel:Birkhäuser; 2015.

[12] Hitzer E. Creative Peace License. `http://gaupdate.wordpress.com/2011/12/14/the-creative-peace-license-14-dec-2011/`

[13] Hitzer E, Abłamowicz R. Geometric Roots of -1 in Clifford Algebras $Cl(p, q)$ with $p+q \leq 4$. Adv. Appl. Clifford Algebras. 2011;21(1):121–144.

[14] Hitzer E. Introduction to Clifford's Geometric Algebra, Journal of the Society of Instrument and Control Engineers. 2012 Apr.;51(4):338–350

[15] Hitzer E, Helmstetter J, Abłamowicz R. Maple worksheets created with `CLIFFORD` for a verification of results in [18]. `http://math.tntech.edu/rafal/publications.html` (©2012).

[16] Hitzer E, Nitta T, Kuroe Y. Applications of Clifford's Geometric Algebra. Adv. Appl. Clifford Algebras. 2013 Mar.;23:377–404.

[17] Hitzer E, Sangwine SJ, editors. Quaternion and Clifford Fourier transforms and wavelets. Trends in Mathematics, Basel:Birkhäuser 2013;27.

[18] Hitzer E, Helmstetter J, Abłamowicz R. Square Roots of −1 in Real Clifford Algebras. in: Hitzer E, Sangwine SJ, editors. Quaternion and Clifford Fourier transforms and wavelets. Trends in Mathematics, Basel:Birkhäuser 2013;27:123–153

[19] Hitzer E, Sangwine SJ. Multivector and multivector matrix inverses in real Clifford algebras. Appl. Math. and Comp. 2017 Oct.;311(C):375–389.

[20] Hitzer E. Special relativistic Fourier transformation and convolutions. Math. Meth. Appl. Sci. 2019 Mar.;42(7):2244–2255.

[21] Hitzer E, Sangwine SJ. Foundations of Conic Conformal Geometric Algebra and Compact Versors for Rotation, Translation and Scaling. Adv. Appl. Clifford Algebras. 2019;29(96):1–16.

[22] Hitzer E. Exponential Factorization of Multivectors in $Cl(p,q)$, $p+q<3$. Math. Meth. Appl. Sci. 2020 Jul.;1–15. DOI: `https://doi.org/10.1002/mma.6629`.

[23] Lasenby J, Hadfield H, Lasenby A. Calculating the Rotor Between Conformal Objects. Adv. Appl. Clifford Algebras. 2019;29(102):1–9.

[24] Lounesto P. Clifford Algebras and Spinors. Cambridge (UK):Cambridge University Press; 2001.

[25] Sangwine SJ. Biquaternion (Complexified Quaternion) Roots of −1. Adv. Appl. Clifford Algebras. 2006;16(1):63–68.

[26] Sangwine SJ, Hitzer E. Clifford Multivector Toolbox (for MATLAB), 2015–2021, Software library available at: `http://clifford-multivector-toolbox.sourceforge.net/`

[27] Sangwine SJ, Hitzer E. Polar decomposition of complexified quaternions and octonions. Adv. Appl. Clifford Algebras. 2020;30(23):1–12.

[28] Xambó-Descamps S. Real Spinorial Groups – A Short Mathematical Introduction. SBMAC SpringerBriefs in Mathematics. Cham: Springer Nature Switzerland AG; 2018.

Appendix A. Alternative factorization of $Cl_2(1,2)$

An alternative factorization of $Cl_2(1,2)$ (or $Cl_2(2,1)$) can be obtained in the following way. In the following we focus only on $Cl_2(2,1)$, but the case for $Cl_2(1,2)$, due to its *isomorphic* structure, works analogous.

$$m = m_0 + m_{23}e_{23} + m_{31}e_{31} + m_{12}e_{12}. \tag{A1}$$

We distinguish five cases. First $m_0 \neq 0$, $\langle m\rangle_2 = m_{23}e_{23} + m_{31}e_{31} + m_{12}e_{12} = 0$:

$$m = m_0 = \frac{m_0}{|m_0|}e^{\alpha_0} = \pm\, e^{\alpha_0}, \qquad \alpha_0 = \ln(|m_0|). \tag{A2}$$

Second, $\langle m\rangle_2^2 < 0$:

$$\begin{aligned} m &= m_0 + |\langle m\rangle_2|\frac{\langle m\rangle_2}{|\langle m\rangle_2|} = a_m e^{\alpha_2 i_2} = e^{\alpha_0}e^{\alpha_2 i_2}, \qquad |\langle m\rangle_2| = \sqrt{-\langle m\rangle_2^2}, \\ i_2 &= \frac{\langle m\rangle_2}{|\langle m\rangle_2|}, \qquad i_2^2 = -1, \qquad \alpha_2 = \operatorname{atan2}(|\langle m\rangle_2|, m_0), \\ a_m &= \sqrt{m_0^2 + |\langle m\rangle_2|^2} = \sqrt{m_0^2 - \langle m\rangle_2^2}, \qquad \alpha_0 = \ln(a_m). \end{aligned} \tag{A3}$$

We observe that the second case subsumes the first case for $\alpha_2 \in \{0, \pi\}$. Third, $m_0 = 0$, $m = \langle m \rangle_2 \neq 0$, $m^2 = \langle m \rangle_2^2 = 0$:

$$m = \langle m \rangle_2 = e^{\alpha_0} i_2, \quad \alpha_0 = \ln(\sqrt{2}|m_{12}|), \quad i_2 = \frac{\langle m \rangle_2}{\sqrt{2}|m_{12}|}, \quad m^2 = i_2^2 = 0. \tag{A4}$$

We observe that in the third case m is a not invertible null-bivector. As an *example*[13] for the third case we consider the following example.

Example A.1.

$$m = \langle m \rangle_2 = 3e_{12} + 3e_{23} \approx e^{1.45} \frac{e_{12} + e_{23}}{\sqrt{2}}, \quad m_{12} = |m_{12}| = m_{23} = 3,$$
$$\alpha_0 = \ln(\sqrt{2}\,3) \approx 1.45, \quad i_2 = \frac{e_{12} + e_{23}}{\sqrt{2}}. \tag{A5}$$

Fourth, $m_0 \neq 0$, $\langle m \rangle_2 \neq 0$, $\langle m \rangle_2^2 = 0$:

$$m = m_0 + \langle m \rangle_2 = m_0(1 + \frac{1}{m_0}\langle m \rangle_2) = \frac{m_0}{|m_0|} e^{\alpha_0}(1 + \alpha_2 i_2) = \pm e^{\alpha_0} e^{\alpha_2 i_2},$$
$$\alpha_0 = \ln(|m_0|), \quad i_2 = \frac{\langle m \rangle_2}{\sqrt{2}|m_{12}|}, \quad \alpha_2 = \frac{\sqrt{2}|m_{12}|}{m_0}, \tag{A6}$$

where the sign factor is determined by $\frac{m_0}{|m_0|} = \pm 1$. Fifth, $\langle m \rangle_2^2 > 0$:

$$m = m_0 + \langle m \rangle_2 = m_0 + |\langle m \rangle_2| i_2 = E(|\langle m \rangle_2|, m_0, i_2),$$
$$|\langle m \rangle_2| = \sqrt{\langle m \rangle_2^2}, \qquad i_2 = \frac{\langle m \rangle_2}{|\langle m \rangle_2|}, \qquad i_2^2 = +1, \tag{A7}$$

where $E(|\langle m \rangle_2|, m_0, i_2)$ is determined by (2.11), with $a = |\langle m \rangle_2|$, $b = m_0$, $u = i_2$. In the fifth case m is not invertible for $|m_0| = |\langle m \rangle_2|$. We finally summarize all five cases[14]

$$m = \begin{cases} \pm e^{\alpha_0} & \text{for} \quad \langle m \rangle_2 = 0, \\ e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad \langle m \rangle_2^2 < 0, \\ e^{\alpha_0} i_2 & \text{for} \quad m_0 = 0, \quad \langle m \rangle_2 \neq 0, \quad \langle m \rangle_2^2 = 0, \\ \pm e^{\alpha_0} e^{\alpha_2 i_2} & \text{for} \quad m_0 \neq 0, \quad \langle m \rangle_2 \neq 0, \quad \langle m \rangle_2^2 = 0, \\ E(|\langle m \rangle_2|, m_0, i_2) & \text{for} \quad \langle m \rangle_2^2 > 0. \end{cases} \tag{A8}$$

Let us compare the factorizations (3.3) and (A8): (A8) always has only one bivector exponential (except for the third line $e^{\alpha_0} i_2$), but it is *more complicated* (more case distinctions) than (3.3). Because following (3.3) all cases can be accommodated in the *single* expression $m = (b + au)e^{\alpha_2 e_{12}}$, with $a \geq 0$ and $b \geq 0$, which is always invertible

[13]In conformal geometric algebra $Cl(4,1)$ two null-vectors are defined for the origin and for infinity. Conventionally they are $e_0 = (e_5 - e_4)/2$, $e_\infty = e_5 + e_4$, such that $e_0 \cdot e_\infty = -1$. In certain contexts it has proven to be of advantage to instead choose a symmetric definition $e_0 = (e_5 - e_4)/\sqrt{2}$, $e_\infty = (e_5 + e_4)/\sqrt{2}$, see e.g. [21]. By analogy, this motivates our introduction of $\sqrt{2}$ in the denominator of the null bivector i_2 above.

[14]Note that in lines two to five of (A8) the bivectors i_2 are specific to each line, as defined in (A3), (A4), (A6), and (A7), respectively.

Table B1. Multiplication table of $Cl(3,0)$.

	1	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
1	1	e_1	e_2	e_3	e_{12}	e_{23}	e_{31}	e_{123}
e_1	e_1	1	e_{12}	$-e_{31}$	e_2	e_{123}	$-e_3$	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	$-e_1$	e_3	e_{123}	e_{31}
e_3	e_3	e_{31}	$-e_{23}$	1	e_{123}	$-e_2$	e_1	e_{12}
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	-1	$-e_{31}$	e_{23}	$-e_3$
e_{23}	e_{23}	e_{123}	$-e_3$	e_2	e_{31}	-1	$-e_{12}$	$-e_1$
e_{31}	e_{31}	e_3	e_{123}	$-e_1$	$-e_{23}$	e_{12}	-1	$-e_2$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	$-e_3$	$-e_1$	$-e_2$	-1

Table B2. Multiplication table of $Cl(1,2) \cong Cl(3,0)$.

	1	E_1	E_{12}	E_{31}	E_2	E_{23}	E_3	E_{123}
1	1	E_1	E_{12}	E_{31}	E_2	E_{23}	E_3	E_{123}
E_1	E_1	1	E_2	$-E_3$	e_2	E_{123}	$-E_{31}$	E_{23}
E_{12}	E_{12}	$-E_2$	1	E_{23}	$-E_1$	E_{31}	E_{123}	E_3
E_{31}	E_{31}	E_3	$-E_{23}$	1	E_{123}	$-E_{12}$	E_1	E_2
E_2	E_2	$-E_{12}$	E_1	E_{123}	-1	$-E_3$	E_{23}	$-E_{31}$
E_{23}	E_{23}	E_{123}	$-E_{31}$	E_{12}	E_3	-1	$-E_2$	$-E_1$
E_3	E_3	E_{31}	E_{123}	$-E_1$	$-E_{23}$	E_2	-1	$-E_{12}$
E_{123}	E_{123}	E_{23}	E_3	E_2	$-E_{31}$	$-E_1$	$-E_{12}$	-1

except when $a = b$ (presence of an idempotent factor for $a = b \neq 0$). The inverse is given by

$$m^{-1} = e^{-\alpha_2 e_{12}}(b + au)^{-1} = e^{-\alpha_2 e_{12}} \frac{b - au}{b^2 - a^2}, \tag{A9}$$

whenever $a \neq b$, compare (3.2). By these reasons, we prefer to generally use (3.3) in this paper.

Appendix B. Factorization of $Cl(1,2) \cong Cl(3,0)$

The results of the Section 4 lend themselves to factorize multivectors in $Cl(1,2) \cong Cl(3,0)$, based on the isomorphism $Cl(1,2) \cong Cl(3,0)$. We list the multiplication tables, Table B1 for $Cl(3,0)$ and Table B2 for $Cl(1,2)$. $Cl(1,2) \cong Cl(3,0)$ can be verified from Tables B1 and B2, which can be brought into agreement by identifying

$$\begin{aligned} 1 = 1, E_1 = e_1, E_2 = e_{12}, E_3 = e_{31}, \\ E_{12} = e_2, E_{23} = e_{23}, E_{31} = e_3, E_{123} = e_{123}, \end{aligned} \tag{B1}$$

where $\{E_1, E_2, E_3\}$ is the orthonormal vector basis of $\mathbb{R}^{1,2}$ generating $Cl(1,2)$, and $\{e_1, e_2, e_3\}$ is the orthonormal vector basis of $\mathbb{R}^3$ generating $Cl(3,0)$.

Factorization of multivectors $m \in Cl(1,2)$ can be achieved by mapping m via the isomorphism (B1) to its isomorphic counterpart $m' \in Cl(3,0)$, then factorize m' in $Cl(3,0)$, and finally map the factorized form back with applying (B1) again in reverse. In particular the unit vector $u \in Cl(3,0)$ and the unit bivector i_2 in (4.34) become

$$u = u_1 e_1 + u_2 e_2 + u_3 e_3 = u_1 E_1 + u_2 E_{12} + u_3 E_{31}, \tag{B2}$$

$$i_2 = b_{12} e_{12} + b_{23} e_{23} + b_{31} e_{31} = b_{12} E_2 + b_{23} E_{23} + b_{31} E_3, \tag{B3}$$

with $u_1^2 + u_2^2 + u_3^2 = 1$, and $b_{12}^2 + b_{23}^2 + b_{31}^2 = 1$.

Viewed strictly in $Cl(1,2)$, the exponentials corresponding to $e^{\alpha_1 u}$ and $e^{\alpha_2 i_2}$ will therefore no longer have a single grade one vector or a single grade two bivector as respective arguments, but in both cases a sum of vector plus bivector will appear as arguments.

E. Hitzer, On Factorization of Multivectors in Cl(3,0), Cl(1,2) and Cl(0,3), by Exponentials and Idempotents, accepted by Complex Variables and Elliptic Equations, 25 pages, Oct. 2021, DOI: https://doi.org/10.1080/17476933.2021.2001462