ASSUMING $c < R^2$ HOLDS IMPLIES THE abc CONJECTURE IS FALSE

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To my wife Wahida, my daughter Sinda and my son Mohamed Mazen

Abstract. In this paper about the *abc* conjecture, assuming the conjecture $c < R^2$ is true, we give an elementary proof that the *abc* conjecture is false.

1. Introduction and notations

Let a positive integer $a = \prod_i a_i^{\alpha_i}$, a_i prime integers and $\alpha_i \geq 1$ positive integers. We call *radical* of a the integer $\prod_i a_i$ noted by $rad(a)$. Then a is written as :

$$a = \prod_i a_i^{\alpha_i} = rad(a).\prod_i a_i^{\alpha_i - 1} \tag{1.1}$$

We note:

$$\mu_a = \prod_i a_i^{\alpha_i - 1} \Longrightarrow a = \mu_a.rad(a) \tag{1.2}$$

The *abc* conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) [1]. It describes the distribution of the prime factors of two integers with those of its sum. The definition of the *abc* conjecture is given below:

Conjecture 1.1. ***(abc Conjecture)****: For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :*

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{1.3}$$

where K is a constant depending only of ϵ.

The idea to try to write a paper about this conjecture was born after the publication of an article in Quanta magazine about the remarks of professors Peter Scholze of the University of Bonn and Jakob Stix of Goethe University Frankfurt concerning the proof of Shinichi Mochizuki [2] in November 2018. The difficulty to find a proof of the *abc* conjecture is due to the incomprehensibility how the prime factors are organized in c giving a, b with $c = a + b$. Since 2018, I have studied the question, attacked the problem from diverse angles and tried some methods to resolve it using different choices of the constant $K(\epsilon)$.

Numerically we know that $\dfrac{Logc}{Log(rad(abc))} \leq 1.629912$ [3]. A conjecture was proposed that $c < rad^2(abc)$ [4]. One proof of the last conjecture helps us to resolve

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the *abc* conjecture. In my paper, I assume that $c < rad^2(abc)$ holds, then I give the proof that the *abc* conjecture is false in the case $R < c$ and $\forall \epsilon$, $0 < \epsilon < 1$.

2. Preliminaries

Let a, b, c (respectively a, c) positive integers relatively prime with $c = a + b, a > b, b \geq 2$ (respectively $c = a + 1$, $a \geq 2$). We denote $R = rad(abc)$ in the case $c = a + b$ or $R = rad(ac)$ in the case $c = a + 1$.

We assume in the following of the paper that $c < R^2$ holds. We recall the following proposition [5]:

Proposition 2.1. *Let $\epsilon \longrightarrow K(\epsilon)$ the application verifying the abc conjecture, then:*

$$lim_{\epsilon \to 0} K(\epsilon) = +\infty \tag{2.1}$$

I have arrived to conclude that, assuming that $c < rad^2(abc)$ is true, the *abc* conjecture does not hold when $0 < \epsilon < 1$ and $c > R$, it follows that the *abc* conjecture as it was defined is false.

3. The Proof of the *abc* Conjecture is false

Proof. - We recall the definition of the *abc* conjecture:

For each $\epsilon > 0$, there exists $K(\epsilon)$ such that if a, b, c positive integers relatively prime with $c = a + b$, then :

$$c < K(\epsilon).rad^{1+\epsilon}(abc) \tag{3.1}$$

where K is a constant depending only of ϵ.

From the equation (3.1) above, $K(\epsilon) > 0$.

Case-1- For $\epsilon = 1$, we can take $K(1) = 1$ and let a, b, c positive integers relatively prime with $c = a + b$, then we can write:

$$c < R^2 \Longrightarrow c < 1.R^{1+1} \tag{3.2}$$

and the *abc* conjecture is true for $\epsilon = 1$.

Case-2- For $\epsilon > 1$, we can choose $K(\epsilon) = \epsilon$. From $c < R^2$, we obtain :

$$c < R^2 < \epsilon.R^{1+1} < K(\epsilon).R^{1+\epsilon} \tag{3.3}$$

and the *abc* conjecture is true for $\epsilon > 1$.

Case-3- From the above equation (3.3), we can write:

$$c < \epsilon.R^{1+\epsilon}, \quad \epsilon > 1 \Longrightarrow$$

$$c < K\left(\frac{1}{\epsilon}\right).R^{1+\frac{1}{\epsilon}}.\left[\frac{\epsilon.R^{1+\epsilon}}{K\left(\frac{1}{\epsilon}\right).R^{1+\frac{1}{\epsilon}}}\right], \quad \epsilon' = \frac{1}{\epsilon} \Longrightarrow$$

$$0 < \epsilon' < 1, \; c < K(\epsilon').R^{1+\epsilon'} \quad \text{if} \quad \left[\frac{\epsilon.R^{1+\epsilon}}{K\left(\frac{1}{\epsilon}\right).R^{1+\frac{1}{\epsilon}}}\right] < 1 \tag{3.4}$$

We have the case $0 < \epsilon' < 1$, the abc conjecture is true for $\epsilon' \in]0,1[$ if as it is written above:

$$\frac{\epsilon.R^{1+\epsilon}}{K\left(\frac{1}{\epsilon}\right).R^{1+\frac{1}{\epsilon}}} < 1 \Longrightarrow R^{\frac{1}{\epsilon'}-\epsilon'} < \epsilon'.K(\epsilon'), \quad \forall\epsilon', \forall(a,b,c)$$

But there is an obstruction to the inequality above when we choose the triplet (a,b,c) so that $c = a+b$ is a large integer $\Longrightarrow R$ is also. As $\frac{1}{\epsilon'} - \epsilon' > 1 - \epsilon' > 0$ and $\epsilon' K(\epsilon')$ is bound, it follows the contradiction. This case is equivalent to suppose that the abc conjecture is true with the condition $c < R^2 < K(\epsilon')R^{1+\epsilon'}$, it gives $R^{1-\epsilon'} < K(\epsilon')$.

Now, we suppose that the abc conjecture is true. We choose $\epsilon' \in]0,1[$ and there is a constant $K(\epsilon')$ so that :

$$c < K(\epsilon')R^{1+\epsilon'} < R^2 \Longrightarrow K(\epsilon') < R^{1-\epsilon'}$$

If $\epsilon' \longrightarrow 0^+$, and using the proposition 2.1 above, we obtain again a contradiction.

Finally, we consider the case $R^2 = K(\epsilon')R^{1+\epsilon'} \Longrightarrow K(\epsilon') = R^{1-\epsilon'}$. Then the contradiction with $K(\epsilon')$ is a constant depending only of ϵ'.

Hence the abc conjecture is not true for $\epsilon' \in]0,1[$.
We conclude that the abc conjecture is false and the proof is finished. $\square$

4. Conclusion

Assuming that the conjecture $c < R^2$ holds, we have given an elementary proof that the abc conjecture is false. We can announce the important theorem:

Theorem 4.1. *Assuming that the conjecture $c < R^2$ holds, then the abc conjecture is false.*

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