# Foundations for strip adjustment of airborne laserscanning data with conformal geometric algebra

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**Abstract.** Typically, airborne laserscanning includes a laser mounted on an airplane or drone (its pulsed beam direction can scan in flight direction and perpendicular to it) an intertial positioning system of gyroscopes, and a global navigation satellite system. The data, relative orientation and relative distance of these three systems are combined in computing strips of ground surface point locations in an earth fixed coordinate system. Finally, all laserscanning strips are combined via iterative closes point methods to an interactive three-dimensional terrain map. In this work we describe the mathematical framework for how to use the iterative closest point method for the adjustment of the airborne laserscanning data strips in the framework of conformal geometric algebra.

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## 1. Introduction

Over the last decades, the classical airborne laser scanning (ALS) has developed into a well established standard survey tool to acquire highly detailed and accurate spatial data of the Earth's topography [29]. Furthermore, new application fields emerged due to rapid sensor developments in recent years, namely the so-called topo-bathymetric LiDAR (Light Detection and Ranging) respectively airborne laser bathymetry [23]. LiDAR utilizes a laser beam emitted from a sensor mounted on/in an airborne vehicle (aircraft, helicopter or drone), that is reflected at the Earth's surface, and received by the sensor. The result of such measurements is a dense 3D point cloud capturing the terrain surface above (topography) and below (bathymetry) water and objects including buildings, vegetation, power lines, cars etc. However, one major outcome of LiDAR data processing is in most cases the extraction of a digital terrain model [19] for the purpose of monitoring (e.g. morphology [2, 6]) and modelling (e.g. hydraulics, feature extraction [3, 25]).

Certain steps are mandatory in LiDAR processing including among others for example the filtering of noise points, point classification [21], and strip adjustment. As summarized by Steinbacher et al. [28], available software packages for ALS/ALB data processing as provided by sensor manufacturers and independent software tools have their pros and cons in terms of maintaining a continuous processing chain. The independent software development on HydroVISH and the usage of an open-source data format HDF5 ,as described by Steinbacher et al. [28], allows for a flexible data processing in the daily business, and at the same time it supports the implementation and testing of new analysis approaches and algorithms. In this paper, we describe the mathematical framework of a new approach to expand the essential ALS/ALB data processing step of the strip adjustment.

Usually a certain observational region is scanned two or three times in order to improve data quality, both in terms of increased point density and accuracy with respect to non-systematic data acquisition errors. Each of such scans ("flight strip") overlaps on the same physical domain and objects, but will not fit precisely after transformation from scanner to world coordinates due to the limited precision of recording the GPS coordinates for the flight trajectory, recording of roll-pitch-yaw values (orientation of the airborne vehicle), and accuracy in the LiDAR signal data acquisition within the scanner device itself, i.e. angular direction and distance computed from measured light runtime (compare Fig. 1). Moreover, also the lever arm and boresight misalignment parameters – instrumental properties that remain constant for all points and strips – are only known with limited precision.

Therefore, those multiple strips need to be corrected for these discrepancies. These corrections consist of rigid rotations and translations of the $n$ strips in order to minimize their difference relative to a specific reference strip or actual ground truth data, constituting a $6n$-dimensional parameter space in addition to the 12 systematic errors from lever arm, misalignment and airplane/helicopter/drone orientation. The "conventional" approach to model rotations employs matrices, a tool commonly used in engineering. A superior approach utilizes quaternions, which come with a reduced parameter space as compared to matrices: A quaternion consists of four parameters, whereas a rotation matrix in 3D space consists of nine parameters. Reducing the number of parameters thus improves computational performance and numerical stability.

In the case of adjusting flight strips also translation must be considered, which cannot be covered by quaternions. Homogeneous coordinates may be used via $4 \times 4$ matrixes, such as known from projective geometry and computer graphics. An alternative approach allowing to avoid matrices is offered by geometric algebra, which is a generalization of quaternions to arbitrary dimensions. In particular, conformal geometric algebra (CGA) is suitable here: this formalism allows to unite rotations and translations within one mathematical operator, the so-called “motor”. As a five-dimensional version of quaternions it is expected to provide the same improvements of robustness and computational efficiency that quaternions provide over rotation matrices. Furthermore, geometric algebra completes the usual vector-space operations by defining an invertible product of vectors, the so-called *geometric product*. Empowered by such an invertible product of vectors, now also the division by a vector is defined, and consequently the derivation by a direction, whereas the conventional approach only allows derivation by a one-dimensional parameter. This more powerful mathematical framework therefore allows to formulate minimization algorithms in a novel way, leading to a more robust solution (see Remark 1).

For an introduction to conformal geometric algebra we refer to [16]. Regarding the conventional method of strip adjustment of ALS data with the iterative closest point (ICP) method, we refer to [8,9]. Our paper is structured as follows: Section 2 provides background on geometric algebra including the notation used in this paper. Next, Section 3 gives an overview of how round and flat objects can be described by certain multivectors in CGA. Then, Section 4 outlines how the conventional ALS data adjustment formalism can be reformulated in CGA, and Section 5 indicates how time-dependent corrections of trajectory errors can be achieved. In CGA all geometric transformations become motor operations, and their multivector argument differentiation is studied in Section 6. Section 7 continues with the directional derivatives of squared distance point-to-plane terms in the correspondence cost function $\Omega$ for overlapping ALS adjustment. This enables us to sketch in Section 8, how Newton’s method can be applied in geometric algebra for ALS adjustment. Essential for Newton’s method is the computation of the design matrix to which Section 9 is devoted.

## 2. Preliminaries of geometric algebra

The co-creator of calculus W. Leibniz (1646–1716) dreamed of a new type of mathematics in which every number, every operation and every relation would have a clear geometric counterpart. Subsequently the inventor of the concept of vector space and our modern notion of algebra H. Grassmann (1809–1877) was officially credited to fulfill Leibniz’s vision. Contemporary to Grassmann was W. Hamilton (1805–1865), who took great pride in establishing the algebra of rotation generators in three dimensions, which he himself called quaternion algebra. About 30 years later W. Clifford (1845–1876) successfully fused Grassmann's and Hamilton's work together in what he called geometric algebra. Geometric algebra can be understood as an algebra of a vector space and all its subspaces equipped with an associative and invertible geometric product of vectors.

Transformation groups generated by products of reflections in geometric algebra are known as Clifford (or Lipschitz or versor) groups [10, 22]. Versors (Clifford group or Lipschitz elements) are simply the geometric products of the normal vectors to the (hyper) planes of reflection.

### 2.1. Basic notions of geometric algebra

**Definition 1.** *Clifford geometric algebra. A Clifford geometric algebra $Cl(p, q)$ is defined by the associative geometric product of elements of a quadratic vector space $\mathbb{R}^{p,q}$, their linear combination and closure. $Cl(p, q)$ includes the field of real numbers $\mathbb{R}$ and the vector space $\mathbb{R}^{p,q}$ as subspaces. The geometric product of two vectors is defined as*

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}, \tag{1}$$

*where $\mathbf{a} \cdot \mathbf{b}$ indicates the standard inner product and the bivector $\mathbf{a} \wedge \mathbf{b}$ indicates Grassmann's antisymmetric outer product. $\mathbf{a} \wedge \mathbf{b}$ can be geometrically interpreted as the oriented parallelogram area spanned by the vectors $\mathbf{a}$ and $\mathbf{b}$. Geometric algebras are graded, with grades (subspace dimensions) ranging from zero (scalars) to $n = p + q$ (pseudoscalars, $n$-volumes).*

For example geometric algebra $Cl_3 = Cl(3, 0)$ of three-dimensional Euclidean space $\mathbb{R}^3 = \mathbb{R}^{3,0}$ has an eight-dimensional basis[1] of scalars (grade 0), vectors (grade 1), bivectors (grade 2) and trivectors (grade 3). Trivectors in $Cl_3$ are also referred to as oriented volumes or pseudoscalars. Using an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$ we can write the basis of $Cl_3$ as

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_3\mathbf{e}_1, \mathbf{e}_1\mathbf{e}_2, I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\}. \tag{2}$$

In (2) $I_3$ is the unit trivector (normed pseudoscalar), i.e. the oriented volume of a unit cube. The even subalgebra $Cl_3^+$ of $Cl_3$ is isomorphic to the quaternions $\mathbb{H}$ of Hamilton. We therefore call elements of $Cl_3^+$ rotors, because they rotate all elements of $Cl_3$ . The role of complex (and quaternion) conjugation is naturally taken by reversion ($\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_s \in \mathbb{R}^{p,q}$, $s \in \mathbb{N}$)

$$(\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_s)^\sim = \mathbf{a}_s \ldots \mathbf{a}_2\mathbf{a}_1. \tag{3}$$

The inverse of a non-null vector $\mathbf{a} \in \mathbb{R}^{p,q}$ is

$$\mathbf{a}^{-1} = \frac{\mathbf{a}}{\mathbf{a}^2}, \qquad \mathbf{a}^{-1}\mathbf{a} = \mathbf{a}\mathbf{a}^{-1} = 1. \tag{4}$$

A reflection at a hyperplane normal to $\mathbf{a}$ is

$$\mathbf{x}' = -\mathbf{a}^{-1}\mathbf{x}\mathbf{a}. \tag{5}$$

[1] Notation: In geometric algebra $Cl^k(p, q), 0 \leq k \leq n = p + q$, denotes the vector space of grade-$k$ elements, e.g. $Cl^2(3, 0)$ is the three-dimensional space of bivectors in $Cl(3, 0)$.

A rotation by the angle $\theta$ in the plane of a unit bivector $\mathbf{i}$ can thus be given as the product $R = \mathbf{ab}$ of two vectors $\mathbf{a}$, $\mathbf{b}$ from the $\mathbf{i}$-plane (i.e. geometrically as a sequence of two reflections) with angle $\theta/2$,

$$\mathbb{R}^{p,q} \ni \mathbf{x} \to R^{-1}\mathbf{x}R \in \mathbb{R}^{p,q}, \tag{6}$$

where the vectors $\mathbf{a}$, $\mathbf{b}$ are in the plane of the unit bivector $\mathbf{i} \in Cl(p,q)$ if and only if $\mathbf{a} \wedge \mathbf{i} = \mathbf{b} \wedge \mathbf{i} = 0$. The rotor $R$ can also be expanded as

$$R = \mathbf{ab} = |\mathbf{a}||\mathbf{b}| \exp(\theta\mathbf{i}/2) = |\mathbf{a}||\mathbf{b}|(\cos\frac{\theta}{2} + \mathbf{i}\sin\frac{\theta}{2}), \tag{7}$$

where $|\mathbf{a}|$, and $|\mathbf{b}|$, are the lengths of $\mathbf{a}$, $\mathbf{b}$. This description corresponds exactly to using quaternions.

Blades of grade $k$, $0 \leq k \leq n = p+q$ are the outer products of $k$ vectors $\mathbf{a}_l$ $(1 \leq l \leq k)$ and directly represent the $k$-dimensional vector subspaces $V$ spanned by the set of vectors $\mathbf{a}_l$ $(1 \leq l \leq k)$. This is also called the outer product null space (OPNS) representation.

$$\mathbf{x} \in V = \mathrm{span}[\mathbf{a}_1, \ldots \mathbf{a}_k] \;\Leftrightarrow\; \mathbf{x} \wedge \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_k = 0. \tag{8}$$

Extracting a certain grade part from the geometric product of two blades $A_k$ and $B_l$ has a deep geometric meaning.

One example is the grade $l-k$ part (left contraction [4]) and the $k-l$ part (right contraction [4]) of the geometric product $A_kB_l$, that represents the orthogonal complement of a $k$-blade $A_k$ in an $l$-blade $B_l$, provided that $A_k$ is contained in $B_l$, and vice versa,

$$A_k \rfloor B_l = \langle A_k B_l \rangle_{l-k}, \qquad A_k \lfloor B_l = \langle A_k B_l \rangle_{k-l}. \tag{9}$$

Another important grade part of the geometric product of $A_k$ and $B_l$ is the maximum grade $l+k$ part, also called the outer product part

$$A_k \wedge B_l = \langle A_k B_l \rangle_{l+k}. \tag{10}$$

If $A_k \wedge B_l$ is non-zero it represents the union of the disjoint (except for the zero vector) subspaces represented by $A_k$ and $B_l$.

The *dual* of a multivector $A$ is defined by geometric division with the pseudoscalar $I = A^* = A\,I^{-1}$, which maps $k$-blades into $(n-k)$-blades, where $n = p+q$. Duality transforms inner products (contractions) to outer products and vice versa. The outer product null space representation (OPNS) of (8) is therefore transformed by duality into the socalled inner product null space (IPNS) representation

$$x \wedge A = 0 \Longleftrightarrow x \cdot A^* = 0. \tag{11}$$

Note that in geometric algebra expressions the inner product, contractions and the outer product have priority over the full geometric product. For instance, $\mathbf{a} \wedge \mathbf{b} I = (\mathbf{a} \wedge \mathbf{b}) I$. The algebraic equations in this and the following section can be either computed by hand, expanding all blades in terms of basis vectors, or they can be computed with software, like The Clifford Toolbox for MATLAB [26].

### 2.2. Conformal geometric algebra

This section introduces conformal geometric algebra (CGA) in slightly modified form. We specify its basis vectors and show important blade computations.

CGA $Cl(4,1)$ is defined over a real 5-dimensional vector space $\mathbb{R}^{4,1}$. The basis vectors of this space are divided into two groups: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (corresponding to the Euclidean vectors of $\mathbb{R}^3$), and $\{\mathbf{e}_0, \mathbf{e}_\infty\}$. The inner products between them are defined in Table 1.

TABLE 1. Inner product between CGA basis vectors.

| | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_0$ | $\mathbf{e}_\infty$ |
|---|---|---|---|---|---|
| $\mathbf{e}_1$ | 1 | 0 | 0 | · | · |
| $\mathbf{e}_2$ | 0 | 1 | 0 | · | · |
| $\mathbf{e}_3$ | 0 | 0 | 1 | · | · |
| $\mathbf{e}_0$ | · | · | · | 0 | −1 |
| $\mathbf{e}_\infty$ | · | · | · | −1 | 0 |

For efficient computation, a diagonal metric matrix may be useful. The algebra $Cl(4,1)$ generated by the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and basis vector $\mathbf{e}_{+1}$ squaring to $+1$ along with one other basis vector $\mathbf{e}_{-1}$, squaring to $-1$ would correspond to a diagonal metric matrix. The transformation from the diagonal metric basis to that of Table 1 can be defined as follows[2]

$$\mathbf{e}_\infty = \frac{1}{\sqrt{2}}(\mathbf{e}_{+1} + \mathbf{e}_{-1}), \qquad \mathbf{e}_0 = \frac{1}{\sqrt{2}}(\mathbf{e}_{-1} - \mathbf{e}_{+1}). \tag{12}$$

The following inner products follow readily from Table 1:

$$\mathbf{e}_\infty \cdot \mathbf{e}_0 = -1, \qquad \mathbf{e}_0^2 = \mathbf{e}_\infty^2 = 0, \tag{13}$$

We further define the bivector $E$, as

$$E = \mathbf{e}_\infty \wedge \mathbf{e}_0 = \mathbf{e}_{+1}\mathbf{e}_{-1}, \tag{14}$$

and obtain the following products

$$E^2 = 1, \qquad \mathbf{e}_0 E = -E\mathbf{e}_0 = -\mathbf{e}_0, \qquad \mathbf{e}_\infty E = -E\mathbf{e}_\infty = \mathbf{e}_\infty. \tag{15}$$

We define the pseudo-scalar $I_3$ in $\mathbb{R}^3$:

$$I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3, \quad I_3^2 = -1, \quad I_3^{-1} = -I_3, \tag{16}$$

[2]Traditionally, null basis vectors $\mathbf{e}_\infty = \mathbf{e}_{+1} + \mathbf{e}_{-1}$, $\mathbf{e}_0 = \frac{1}{2}(\mathbf{e}_{-1} - \mathbf{e}_{+1})$, are defined, as in [4, 14]. But in general any factor $\lambda \in \mathbb{R} \setminus \{0\}$, could be fixed and define $\mathbf{e}_\infty = \frac{1}{\lambda\sqrt{2}}(\mathbf{e}_{+1}+\mathbf{e}_{-1})$, $\mathbf{e}_0 = \frac{\lambda}{\sqrt{2}}(\mathbf{e}_{-1}-\mathbf{e}_{+1})$, while preserving the scalar products of Table 1. This freedom to operate with a continuously parametrized basis (equivalent to a continuously parametrized set of horospheres) has e.g. been used advantageously by El Mir et al for elegant algebraic view point change representation in [5]. On the other hand [17] showed that for the modelling of quadrics $\lambda = 1$ is of advantage.

and the conformal pseudo-scalar $I_5$ and its inverse $I_5^{-1}$ (used for dualization in CGA) are:

$$I_5 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,\mathbf{e}_\infty \wedge \mathbf{e}_0 = I_3 E, \qquad I_5^2 = -1, \qquad I_5^{-1} = -I_5. \tag{17}$$

The dual of a multivector indicates division by the pseudo-scalar, e.g., $A^* = -AI_5$, $A = A^* I_5$. From eq. (1.19) in [14], we generally have the useful duality between outer and inner products of non-scalar blades $A, B$ in geometric algebra:

$$(A \wedge B)^* = A \cdot B^*, \quad A \wedge (B^*) = (A \cdot B)^* \;\Leftrightarrow\; A \wedge (B\,I) = (A \cdot B)\,I, \tag{18}$$

which indicates that

$$A \wedge B = 0 \;\Leftrightarrow\; A \cdot B^* = 0, \qquad A \cdot B = 0 \;\Leftrightarrow\; A \wedge B^* = 0. \tag{19}$$

## 3. CGA objects

The following subsections introduce the important definition of a general point in CGA, and show next how all round and flat geometric objects (point pairs, flat points, circles, lines, spheres, planes) are defined in CGA.

### 3.1. Point in CGA

The point $X$ of CGA corresponding to the Euclidean point $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \in \mathbb{R}^3$, is defined as

$$X = \mathbf{x} + \tfrac{1}{2}\mathbf{x}^2\mathbf{e}_\infty + \mathbf{e}_0 = \mathbf{x} + \frac{1}{2}|\mathbf{x}|^2\mathbf{e}_\infty + \mathbf{e}_0. \tag{20}$$

It is a convenient property of CGA points that the inner product between two points is identical with the squared distance between them. Let $X_1$ and $X_2$ be two points, their inner product is

$$X_1 \cdot X_2 = (\mathbf{x}_1 + \tfrac{1}{2}\mathbf{x}_1^2\mathbf{e}_\infty + \mathbf{e}_0) \cdot (\mathbf{x}_2 + \tfrac{1}{2}\mathbf{x}_2^2\mathbf{e}_\infty + \mathbf{e}_0), \tag{21}$$

which, together with Table 1, implies

$$X_1 \cdot X_2 = \mathbf{x}_1 \cdot \mathbf{x}_2 - \tfrac{1}{2}(\mathbf{x}_1^2 + \mathbf{x}_2^2) = -\tfrac{1}{2}(\mathbf{x}_1 - \mathbf{x}_2)^2. \tag{22}$$

We see that the inner product is equivalent to minus half the squared Euclidean distance between $X_1$ and $X_2$.

### 3.2. Round and flat objects in CGA

By round objects, we mean points, point pairs, circles and spheres with uniform curvature. In CGA, these can be defined by the outer product of one to four points. Their center $C$, radius $r$ and Euclidean carrier blade $D$ can be easily extracted. Alternatively, they can be directly constructed from their center $C$, radius $r$ and Euclidean carrier $D$.

Wedging any round object with the point at infinity $\mathbf{e}_\infty$, gives the corresponding flat object multivector. From it the orthogonal distance to the origin $\mathbf{c}_\perp$ and the Euclidean carrier $D$ can easily be extracted.

We now briefly review the CGA description of round and flat objects. The round objects are

$$P = X, \tag{23}$$
$$Pp = X_1 \wedge X_2, \tag{24}$$
$$Circle = X_1 \wedge X_2 \wedge X_3, \tag{25}$$
$$Sphere = X_1 \wedge X_2 \wedge X_3 \wedge X_4. \tag{26}$$

The corresponding flat objects are flat point, line, plane and the whole three-dimensional space

$$Flatp = P \wedge \mathbf{e}_\infty, \tag{27}$$
$$Line = Pp \wedge \mathbf{e}_\infty = X_1 \wedge X_2 \wedge \mathbf{e}_\infty, \tag{28}$$
$$Plane = Circle \wedge \mathbf{e}_\infty = X_1 \wedge X_2 \wedge X_3 \wedge \mathbf{e}_\infty, \tag{29}$$
$$Space = Sphere \wedge \mathbf{e}_\infty = X_1 \wedge X_2 \wedge X_3 \wedge X_4 \wedge \mathbf{e}_\infty \propto I_5. \tag{30}$$

The above are standard CGA results found in [14]. All round entities have one common multivector form[3]

$$S = D \wedge \mathbf{c} + \left[\tfrac{1}{2}(\mathbf{c}^2 + r^2)D - \mathbf{c}(\mathbf{c}\rfloor D)\right]\mathbf{e}_\infty + D\mathbf{e}_0 + (D\lfloor \mathbf{c})E. \tag{31}$$

The Euclidean carriers $D$, blades of grade zero to three, for the rounds are as follows:

$$D = \begin{cases} 1, & \text{point } X \\ \mathbf{d}, & \text{point pair } Pp \\ \mathbf{i}_c, & \text{circle } Circle \\ I_3, & \text{sphere } Sphere, \end{cases} \tag{32}$$

where the unit point pair connection direction vector is $\mathbf{d} = (\mathbf{x}_1 - \mathbf{x}_2)/2r$ and the Euclidean circle plane bivector $\mathbf{i}_c$. The radius $r$ of a round object and its center $C$ are generally determined by

$$r^2 = \frac{S\widetilde{S}}{(S \wedge \mathbf{e}_\infty)(S \wedge \mathbf{e}_\infty)^\sim}, \qquad C = S\,\mathbf{e}_\infty\,S. \tag{33}$$

where $\widetilde{S}$ indicates the reverse of $S$.

All embedded flat entities have one common multivector form

$$F = S \wedge \mathbf{e}_\infty = D \wedge \mathbf{c}\mathbf{e}_\infty - DE = D\mathbf{c}_\perp \mathbf{e}_\infty - DE \tag{34}$$

where the orthogonal Euclidean distances of the flat objects from the origin are

$$\mathbf{c}_\perp = \begin{cases} \mathbf{x}, & \text{finite-infinite point pair } Flatp \\ \mathbf{c}_\perp, & \text{line } Line \\ \mathbf{c}_\perp, & \text{plane } Plane \\ 0, & \text{3D space } Space. \end{cases} \tag{35}$$

[3]Note that the left- and right contraction $\rfloor$ and $\lfloor$, respectively, are needed essentially.

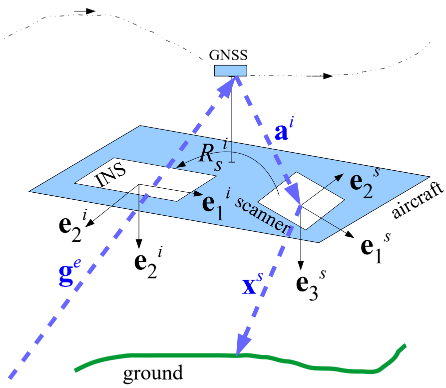


FIGURE 1. System components and coordinates for ALS geodata. Compare also Fig. 2 of [8].

The Euclidean carrier blade $D$, and the orthogonal Euclidean distance vector of $F$ from the origin, can both be directly determined from the flat object multivector as

$$D = F \lfloor E\,, \qquad \mathbf{c}_{\perp} = D^{-1}(F \wedge \mathbf{e}_0)\lfloor E\,. \tag{36}$$

## 4. Conventional ALS data adjustment formalism and its reformulation in CGA

To help the reader understand the notation, we refer to Fig. 1 (compare also Fig. 2 of [8]), that features the scanner coordinate system ($s$-system), an inertial navigation system INS ($i$-system), a global navigation satellite system GNSS ($n$-system of north, east, nadir = opposite of zenith), and an earth-centered, earth-fixed coordinate system ($e$-system).

The three-dimensional vector $\mathbf{x}^s(t) \in \mathbb{R}^3$ of the laser point in the $s$-system is specified by its range $\rho$, and two angles $\alpha$ and $\beta$. The scanner has the orthonormal coordinate vectors $\{\mathbf{e}_1^s, \mathbf{e}_2^s, \mathbf{e}_3^s\}$, where $\mathbf{e}_1^s, \mathbf{e}_2^s$ represents the horizontal $x^s, y^s$ plane of the scanner and $\mathbf{e}_3^s$ the vertical direction to earth (looking down from the airplane). We can think of $\mathbf{e}_1^s$ describing the forward flight direction, and $\mathbf{e}_2^s$ the horizontal direction orthogonal to the flight direction. Usually $\beta$ expresses a rotation in the $\mathbf{e}_1^s, \mathbf{e}_3^s$ plane, and $\alpha$ a rotation in the $\mathbf{e}_2^s, \mathbf{e}_3^s$ plane orthogonal to the flight direction. The scanned points in the $s$-system are (see (8) of [8])

$$\mathbf{x}^s(t) = \rho(t)\widetilde{R}_{\beta(t)}\widetilde{R}_{\alpha(t)}\mathbf{e}_3^s R_{\alpha(t)} R_{\beta(t)}, \quad R_\alpha = e^{\frac{1}{2}\alpha \mathbf{e}_{23}^s}, \quad R_\beta = e^{\frac{1}{2}\beta \mathbf{e}_{31}^s}, \tag{37}$$

with unit plane bivectors $\mathbf{e}^s_{23}$, $(\mathbf{e}^s_{23})^2 = -1$, describing the $\mathbf{e}^s_2, \mathbf{e}^s_3$ plane and[4] $\mathbf{e}^s_{31}$, $(\mathbf{e}^s_{31})^2 = -1$, the $\mathbf{e}^s_1, \mathbf{e}^s_3$ plane. $\rho(t)$ represents the scanner range, and $\alpha(t)$ may represent the beam deflection (across the flight track), whereas for a linear scanner we may have $\beta(t) = 0$.

The rotation from the scanner to the INS (known as boresight misalignment) is conventionally described by three Euler angles $\omega, \phi, \kappa$. But we prefer the rotor formulation (plane of rotation bivector and angle) in CGA [31] :

$$R^i_s = e^{\frac{1}{2}\varphi \mathbf{e}_{si}} = e^{\frac{1}{2}\mathbf{e}^s_{23}\omega} e^{\frac{1}{2}\mathbf{e}^s_{31}\phi} e^{\frac{1}{2}\mathbf{e}^s_{12}\kappa}, \tag{38}$$

where the unit bivector $\mathbf{e}_{si}$, $\mathbf{e}^2_{si} = -1$, shows the oriented plane of the boresight misalignment rotation[5] with misalignment angle $\varphi$.

The positional offset (lever arm) between GNSS antenna and scanner origin is $\mathbf{a}^i = a^i_x \mathbf{e}^i_1 + a^i_y \mathbf{e}^i_2 + a^i_z \mathbf{e}^i_3$. The translation from GNSS to the scanner is effected via the translation operator[6] (translator)

$$T(\mathbf{a}^i) = 1 + \frac{1}{2}\mathbf{a}^i \mathbf{e}_\infty = e^{\frac{1}{2}\mathbf{a}^i \mathbf{e}_\infty}. \tag{39}$$

For the application[7] of the translator (39) all points $\mathbf{x} \in \mathbb{R}^3$ must be extended to conformal points $X \in \mathbb{R}^{4,1}$, see (20). Note that dropping the two extra components in $\mathbf{e}_\infty$, $\mathbf{e}_0$, returns the conventional three-dimensional position $\mathbf{x} \in \mathbb{R}^3$.

The rotation from the aircraft $i$-system (INS) to the $n$-system is parametrized with three Euler angels of roll $\phi(t)$, pitch $\theta(t)$, and yaw $\psi(t)$. The rotor form [31] is

$$R^n_i = e^{\frac{1}{2}\mathbf{e}^i_{23}\psi} e^{\frac{1}{2}\mathbf{e}^i_{31}\theta} e^{\frac{1}{2}\mathbf{e}^i_{12}\phi} = e^{\frac{1}{2}\phi_{in} e_{in}}, \tag{40}$$

with three orthonormal basis vectors $\{\mathbf{e}^i_1, \mathbf{e}^i_2, \mathbf{e}^i_3\}$ of the $i$-system. The last unified rotor expression in (40) describes the rotation between from the $i$ to the $n$-system with a single bivector angle argument $\phi_{in} e_{in}$ in the exponent.

The rotor from the GNSS $n$-system to the earth-fixed $e$-system is given by longitude[8] $\lambda$ and latitude $\varphi$ [32] that correspond to the direction of the position $\mathbf{g}^e(t)$ of the GNSS antenna in the $e$-system relative to earth center

$$R^e_n(t) = e^{\frac{1}{2}\mathbf{e}^e_{31}\varphi} e^{\frac{1}{2}\mathbf{e}^e_{12}\lambda}, \tag{41}$$

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[4]We use the notation $\mathbf{e}_{31}$ for the bivector of the $\mathbf{e}_1, \mathbf{e}_3$-plane, since then we can conveniently compute $\mathbf{e}_{31} = \mathbf{e}_2 \mathbf{e}_{123}$, preserving the cyclic order of the indexes $31|2$ on the left and right. And cyclic index interchange gives the other two bivector and normal vector relationships without need for sign considerations. Software implementations of GA, like GAALOP [11], may rather use a lexicographical order, that is $\mathbf{e}_{13} = -\mathbf{e}_{31}$, which may need to be taken care of when implementing an algorithm.

[5]In the readjustment of the boresight misalignment in CGA one can simply directly optimize with respect to the Euclidean bivector $\varphi \mathbf{e}_{si}$.

[6]Product of two parallel planes (29), perpendicular to $\mathbf{a}^i$ and at distance $\frac{1}{2}|\mathbf{a}^i|$.

[7]Note that the *same* rotation operators (rotors), e.g. of (37), are used in the geometric algebra $Cl(3,0)$ of three-dimensional space $\mathbb{R}^3$ and in conformal geometric algebra $Cl(4,1)$.

[8]The notation longitude $\lambda$ and latitude $\varphi$ simply follows equ. (5) of [8].

where $\mathbf{e}_3^e$ points along earth axis to the north pole, $\mathbf{e}_1^e$ points in the equator plane from earth center to the Greenwich meridian, and $\mathbf{e}_2^e$ is pointing east in the equator plane orthogonal to $\mathbf{e}_1^e$.

Translation from earth center to the onboard GNSS antenna is effected by the translator

$$T(\mathbf{g}^e(t)) = 1 + \frac{1}{2}\mathbf{g}^e(t)\mathbf{e}_\infty = e^{\frac{1}{2}\mathbf{g}^e(t)\mathbf{e}_\infty}, \tag{42}$$

Combining the measurements of the scanner, the aircraft trajectory, and the mounting calibration parameter the terrain point position at time $t$ is (see (1) in [8])

$$X^e(t) = \widetilde{T}(\mathbf{g}^e(t))\widetilde{R}_n^e(t)\widetilde{R}_i^n\widetilde{T}(\mathbf{a}^i)\widetilde{R}_s^i\, X^s(t)R_s^iT(\mathbf{a}^i)R_i^nR_n^e(t)T(\mathbf{g}^e(t)). \tag{43}$$

The combination of two rotors gives a third rotor. The combination of rotor and translator in CGA is called motor. For example, $M_i^e = R_i^nR_n^e(t)T(\mathbf{g}^e(t))$, and $M_s^i = R_s^iT(\mathbf{a}^i)$. We can therefore rewrite (43) as

$$X^e(t) = \widetilde{M}_i^e\widetilde{M}_s^iX^s(t)M_s^iM_i^e = \widetilde{M}_s^eX^s(t)M_s^e, \tag{44}$$

where the motor $M_s^i$ describes the geometric transformation from scanner $s$-system to aircraft INS $i$-system, and the second motor $M_i^e$ the transformation from aircraft INS $i$-system to earth centered $e$-system. The total motor $M_s^e = M_s^iM_i^e$ transforms the scanner data $X^s(t)$ in a single step from the $s$-system to the earth centered $e$-system.

The conventional readjustment of airborne laserscanning data introduces offsets ($\Delta$) and scale parameters ($\varepsilon$). For the scanner points these are

$$\begin{aligned}\rho(t) &= \Delta\rho + \rho_0(t)(1+\varepsilon_\rho),\\ \alpha(t) &= \Delta\alpha + \alpha_0(t)(1+\varepsilon_\alpha),\\ \beta(t) &= \Delta\beta + \beta_0(t)(1+\varepsilon_\beta),\end{aligned} \tag{45}$$

with original scanner measurements $\rho_0(t), \alpha_0(t), \beta_0(t)$.

Trajectory correction parameters for each strip $i$ are for aircraft roll, pitch and yaw, respectively,

$$\begin{aligned}\phi(t) &= \phi_0(t) + \Delta\phi_i,\\ \theta(t) &= \theta_0(t) + \Delta\theta_i,\\ \psi(t) &= \psi_0(t) + \Delta\psi_i,\end{aligned} \tag{46}$$

with original values $(\phi_0(t), \theta_0(t), \psi_0(t))$, angle corrections $(\Delta\phi_i, \Delta\theta_i, \Delta\psi_i)$ and ALS strip index $i$. The position corrections for the GNSS position $\mathbf{g}^e(t)$ are

$$\begin{aligned}\mathbf{g}^e(t) &= \mathbf{g}_0^e(t) + \Delta\mathbf{g}^e\\ &= (g_{x0}(t) + \Delta g_{xi})\mathbf{e}_1^e + (g_{y0}(t) + \Delta g_{yi})\mathbf{e}_2^e + (g_{z0}(t) + \Delta g_{zi})\mathbf{e}_3^e,\end{aligned} \tag{47}$$

with original values $(g_{x0}(t), g_{y0}(t), g_{z0}(t))$, and position corrections $(\Delta g_{xi}$, $\Delta g_{yi}$, $\Delta g_{zi})$, $i$ being again the ALS strip index. This gives a total of six parameters for each strip $(\Delta\phi_i, \Delta\theta_i, \Delta\psi_i)$ and $(\Delta g_{xi}, \Delta g_{yi}, \Delta g_{zi})$.

As mounting calibration parameters serve the (boresight) misalignment angles $(\omega, \phi, \kappa)$ and the lever-arm components $a_x, a_y, a_z$.

In summary, there are $12 + 6n$ parameters for a set of $n$ ALS strips.

A *correspondence* is defined between pairs of points from overlapping strips and from their normal vectors (conventionally) or local strip plane bivectors (our choice) fitted to the neighboring points (in the same plane in the same strip). Note that the normal vectors are dual to the plane bivectors via multiplication with the three-dimensional space pseudoscalar $I_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, where it does not matter which set of orthonormal coordinate vectors is employed since the unit oriented pseudoscalar is unique. Two corresponding points in overlapping strips should belong to the same plane but need not to be identical.

We skip the question of point sampling and of rejection of unreliable or false correspondences treated in [8], but add some hints for implementation in CGA in footnotes.

In CGA it is straightforward to fit a plane to three points [14]. One simply takes the outer product of three conformal points on the plane

$$P_k = \mathbf{p}_k + \frac{1}{2}\mathbf{p}_k^2 + \mathbf{e}_0, \qquad k = 1, 2, 3, \tag{48}$$

with $\mathbf{p}_k \in \mathbb{R}^3$ the three-dimensional positions of the points. The plane (29) is then described by

$$Plane = P_1 \wedge P_2 \wedge P_3 \wedge \mathbf{e}_\infty, \tag{49}$$

or by its dual vector using the CGA pseudoscalar of $Cl(4, 1)$:

$$Plane^* = Plane\, I_5^{-1} = \mathbf{n} + d\mathbf{e}_\infty, \tag{50}$$

with $\mathbf{n} \in \mathbb{R}^3$ being the three-dimensional unit normal vector to the plane and $d$ its signed (oriented) distance from the origin measured along[9] $\mathbf{n}$. Equation (49) can easily be generalized to fit a plane through more than three points[10].

The oriented Euclidean distance of a conformal point $Q$ from the plane is then

$$d = Plane^* \cdot Q. \tag{51}$$

This is exactly the point-to-plane distance computed in (18) of [8] for a point $Q_i$ from a plane $Plane_i$ fitted through $P_i$ and some neighboring points (see also Fig. 5 in [8])[11]. The distance (51) can therefore be plugged into (17) of [8]

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[9] Taking the conformal vector representations of $Plane_i^*$ from one ALS strip and $Plane_j^*$ from a second overlapping ALS strip allows to compute their angle as $\alpha_{ij} = \cos^{-1}(Plane_i^* \cdot Plane_j^*)$ and thus to decide on correspondence rejection beyond a certain angular threshold, e.g. 5° (see p. 77 of [8]).

[10] This also allows to implement based on CGA the roughness limit criterion for the rejection of correspondences (p. 77 of [8]).

[11] Equation (51) also allows to directly compute the point-to-plane standard deviation estimator (16) of [8].

to compute the weighted sum of squared point-to-plane distances between overlapping ALS data strips

$$\Omega = \sum_{i=1}^{n} w_i d_i^2. \tag{52}$$

The weight $w_i$ is the inverse of the square of the median value of all (non-rejected) point to plane distances belonging to the ALS data strip to which the points belong that made up plane $Plane_i$ (equation (16) and (19) in [8]).

We have therefore formulated all necessary geometric constructions in CGA for optimizing the objective function $\Omega$ with respect to the $12 + 6n$ correction, scale and misalignment parameters for a set of $n$ ALS strips as described in the end of Section 4.4 of [8].

It is now possible to program this readjustment algorithm in GAALOP [11], and benefit from its inbuilt expression optimization to reach superior computational speed and precision.

## 5. Time-dependent correction of trajectory errors

In [9] it is suggested to correct time-dependent trajectory shifts ($\Delta g_{xi}(t)$, $\Delta g_{yi}(t)$, $\Delta g_{zi}(t)$), $i$ indexing the ALS strips, see (47), including time-dependent changes of the orientation angles roll, pitch and yaw ($\Delta\phi_i(t), \Delta\theta_i(t), \Delta\psi_i(t)$), see (46), by modelling these changes with cubic order polynomials in time over a constant segment length $\Delta t$. The length of $\Delta t$ (choice of 10s in [9]) is said to be important to avoid over fitting. Furthermore, a relatively large number of well-distributed ground truth points seems to be important in order to avoid deformations of the terrain. An example of the cubic spline interpolation in one segment for one of the time dependent correction parameters ((8) in [9]) is

$$\begin{aligned}\Delta\phi_{[j,k]}(t) = a_{0[j,k]} - a_{1[j,k]}(t - t_{s[j,k]}) + a_{2[j,k]}(t - t_{s[j,k]})^2 \\ + a_{3[j,k]}(t - t_{s[j,k]})^3,\end{aligned} \tag{53}$$

where the index $k = 1, \ldots, n_{[j]}$ counts the time intervals of the $j$th strip and $t_{s[j,k]} = t_{s[j]} + (k-1)\Delta t$ is the start time of the $k$th time interval of the $j$th strip. At the interval boundaries constraints ensure continuity of the polynomials as well as of their first and second derivatives with respect to time. At the beginning and end of each strip the first and second derivatives are constrained to be zero (boundary conditions). In case of lack of overlap between strips, additional fictional observations are introduced ((19) and (20) of [9]) in order to avoid the resulting system of equations to become singular.

**Remark 1.** *Instead of optimizing the time dependence of the Euler angles* $\phi_i(t), \theta_i(t), \psi_i(t)$, *for roll, pitch and yaw, also the (three components of the) corresponding rotor argument bivector* $[\phi_{in} e_{in}](t)$ *in* $Cl^2(3,0)$, *see* (40), *can be optimized directly. This means to optimize in the Lie algebra of the versor group of rotors in three dimensions. In geometric calculus (built on geometric algebra) one can simply multivector differientiate with respect to the bivector* *argument of the exponential (of a rotor expression), compare slides 15 ff. of [20], and Section* 6 *of this paper, and do something akin to Kalman filtering [27]. Automatic multivector differentiation is also available [30]. Geometric algebra algorithm code optimization is described in [11].*

**Remark 2.** *In CGA one can define point pairs, lines, and their Euclidean carriers, see Section* 3*. Furthermore, on can extract positioned carriers (Th. 2.8 in [14]), relative translators (see (3.6) in [14]), and relative rotors (see (3.1) in [14]). Relative translators and rotors permit to extract relative central (or orthogonal) distances $d$ and relative angles $\alpha$ of corresponding point pairs and lines, which can be used in a cost function to be minimized by iterative parameter adjustment.*

# 6. Multivector argument differentiation of motor transformations

## 6.1. Preliminaries for bivector angle differentiation

Regarding the concept of multivector differentiation, we refer to [13]. As a first step we want to differentiate the rotation (6) of vectors[12] $\vec{x} \in \mathbb{R}^3$, assuming the exponential representation of rotors (7)

$$R(A) = e^{A/2} = \sum_{k=0}^{\infty} \frac{(A/2)^k}{k!} = e^{\mathbf{i}\theta/2} = \cos\frac{\theta}{2} + \mathbf{i}\sin\frac{\theta}{2},$$
$$R^{-1}(A) = \widetilde{R}(A) = e^{-A/2} = e^{-\mathbf{i}\theta/2} = \cos\frac{\theta}{2} - \mathbf{i}\sin\frac{\theta}{2}, \tag{54}$$

with respect to the bivector angle argument $A \in Cl^2(3,0)$

$$A = A_1e_{23} + A_2e_{31} + A_3e_{12}, \qquad \theta = |A| = \sqrt{-A^2} = \sqrt{A_1^2 + A_2^2 + A_3^2},$$
$$\mathbf{i} = A/\theta, \qquad \mathbf{i}^2 = -1, \tag{55}$$

where $\mathbf{i} \in Cl^2(3,0)$ expresses the unit bivector of the rotation plane, and $\theta$ the angle of rotation in that plane. We will use the bivector derivative

$$\partial_A = e^{23}e_{23} * \partial_A + e^{31}e_{31} * \partial_A + e^{12}e_{12} * \partial_A, \tag{56}$$

where $*$ expresses the scalar product (the scalar part of the geometric product), and the reciprocal bivectors $e^J$ are given by

$$e^{23} = -e_{23}, \quad e^{31} = -e_{31}, \quad e^{12} = -e_{12}, \tag{57}$$

such that the inner products

$$e^J \cdot e_K = \delta_{JK}, \qquad J, K \in \{23, 31, 12\}, \tag{58}$$

where $\delta_{JK}$ is the Kronecker delta function. We can therefore express the bivector derivative as

$$\partial_A = -e_{23}\partial_{A_1} - e_{31}\partial_{A_2} - e_{12}\partial_{A_3}, \tag{59}$$

[12] In order to make it easier to identify a variable as a vector, we now switch for the rest of this work to the conventional notation with an arrow on top.

where we used the identities with partial derivatives with respect to the coordinates of $A$: $e_{23} * \partial_A = \partial_{A_1}$, $e_{31} * \partial_A = \partial_{A_2}$, and $e_{12} * \partial_A = \partial_{A_3}$.

We remind the reader of the anti-commutativity of basis bivectors of $Cl(3,0)$

$$e_J e_K = -e_K e_J, \quad \text{for } J \neq K. \tag{60}$$

It will also be useful to remember that bivectors and orthogonal vectors commute, whereas bivectors and parallel vectors anti-commute, e.g.,

$$\begin{aligned} \vec{e}_1 e_{23} &= e_{23}\vec{e}_1 = \vec{e}_1 \wedge e_{23} = e_{23} \wedge \vec{e}_1, \\ \vec{e}_1 e_{12} &= -e_{12}\vec{e}_1 = \vec{e}_1 \cdot e_{12} = -e_{12} \cdot \vec{e}_1, \text{ etc.} \end{aligned} \tag{61}$$

This leads to the following set of useful propositions.

**Proposition 1.** *For all* $\vec{v} \in \mathbb{R}^3$

$$e_1 \vec{v} e_1 + e_2 \vec{v} e_2 + e_3 \vec{v} e_3 = -\vec{v}. \tag{62}$$

*Proof.* Assume $\vec{v} \in \mathbb{R}^3$, $\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2 + \gamma\vec{e}_3$, $\alpha, \beta, \gamma \in \mathbb{R}$, and use $e_i e_k e_i = e_i$ for $k = i$, $e_i e_k e_i = -e_i$ for $k \neq i$, $i, k \in \{1, 2, 3\}$. □

**Proposition 2.** *For all* $B \in Cl^2(3,0)$

$$e_1 B e_1 + e_2 B e_2 + e_3 B e_3 = -B. \tag{63}$$

*Proof.* Assume $B \in Cl^2(3,0)$, with dual vector $\vec{b} = Bi^{-1} = -Bi$, and central unit three-dimensional pseudoscalar $i = e_1 e_2 e_3$. Multiplying both sides of (62) with $-i$ for $\vec{v} = \vec{b}$, gives the result. □

**Proposition 3.** *For all* $\vec{v} \in \mathbb{R}^3$

$$-e_{23}\,\vec{v}\,e_{23} - e_{31}\,\vec{v}\,e_{31} - e_{12}\,\vec{v}\,e_{12} = -\,\vec{v}. \tag{64}$$

*Proof.* Assume $\vec{v} \in \mathbb{R}^3$, $\vec{v} = \alpha\vec{e}_1 + \beta\vec{e}_2 + \gamma\vec{e}_3$, $\alpha, \beta, \gamma \in \mathbb{R}$:

$$\begin{aligned} &- e_{23}\,\vec{v}\,e_{23} - e_{31}\,\vec{v}\,e_{31} - e_{12}\,\vec{v}\,e_{12} \\ &= -e_{23}(\alpha\vec{e}_1 + \beta\vec{e}_2 + \gamma\vec{e}_3)e_{23} - e_{31}(\alpha\vec{e}_1 + \beta\vec{e}_2 + \gamma\vec{e}_3)e_{31} \\ &\quad - e_{12}(\alpha\vec{e}_1 + \beta\vec{e}_2 + \gamma\vec{e}_3)e_{12} \\ &= \alpha\vec{e}_1 - \beta\vec{e}_2 - \gamma\vec{e}_3 - \alpha\vec{e}_1 + \beta\vec{e}_2 - \gamma\vec{e}_3 - \alpha\vec{e}_1 - \beta\vec{e}_2 + \gamma\vec{e}_3 \\ &= -\alpha\vec{e}_1 - \beta\vec{e}_2 - \gamma\vec{e}_3 = -\vec{v}, \end{aligned} \tag{65}$$

where we used (61) and (60) for computation. □

Similarly, we obtain based on (60),

**Proposition 4.** *For all bivectors* $B \in Cl^2(3,0)$ *:*

$$-e_{23} B e_{23} - e_{31} B e_{31} - e_{12} B e_{12} = -B. \tag{66}$$

Because the trivector pseudoscalar $i \in Cl(3,0)$ is central we obtain

**Proposition 5.** *For all trivectors* $T \in Cl(3,0)$ *:*

$$-e_{23} T e_{23} - e_{31} T e_{31} - e_{12} T e_{12} = 3\,T. \tag{67}$$

We also note the following useful general parallel to $\mathbf{i}$ and orthogonal to $\mathbf{i}$ vector and bivector product relationships

$$\vec{x}\mathbf{i} = \vec{x}\cdot\mathbf{i} + \vec{x}\wedge\mathbf{i}, \quad \vec{x}\cdot\mathbf{i} = -\mathbf{i}\cdot\vec{x} = \vec{x}_{||}\mathbf{i} = -\mathbf{i}\vec{x}_{||},$$
$$\vec{x}\wedge\mathbf{i} = \mathbf{i}\wedge\vec{x} = \vec{x}_{\perp}\mathbf{i} = \mathbf{i}\vec{x}_{\perp}, \quad \mathbf{i}\vec{x}\mathbf{i} = \vec{x}_{||} - \vec{x}_{\perp}, \quad \vec{x} + \mathbf{i}\vec{x}\mathbf{i} = 2\vec{x}_{||}, \tag{68}$$

and as a result

$$\vec{x}R = (\vec{x}_{||} + \vec{x}_{\perp})R = R^{-1}\vec{x}_{||} + R\vec{x}_{\perp}. \tag{69}$$

Useful for our computations will be the partial derivatives of $A$, $\theta$, and $\mathbf{i}$ :

$$\partial_{A_1}A = e_{23}, \text{ etc.}, \qquad \partial_{A_1}\theta = \frac{A_1}{\theta}, \text{ etc.},$$
$$\partial_{A_1}\mathbf{i} = \partial_{A_1}\frac{A}{\theta} = \frac{\partial_{A_1}A}{\theta} - \frac{A}{\theta^2}\partial_{A_1}\theta = (\frac{e_{23}}{\theta} - \frac{A}{\theta^2}\frac{A_1}{\theta}) = \frac{1}{\theta}(e_{23} - \frac{A_1}{\theta}\mathbf{i}), \text{ etc.}, \tag{70}$$

The partial derivatives above can be used to compute versatile bivector angle derivatives.

**Proposition 6.** *Assume bivector variable* $A = A_1e_{23} + A_2e_{31} + A_3e_{12}$, $A_1, A_2, A_3 \in \mathbb{R}$, $\theta = |A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$, *rotation plane unit bivector* $\mathbf{i} = A/\theta$, $\vec{x}_{||} = (\vec{x}\cdot\mathbf{i})\mathbf{i}^{-1}$, $\vec{x}_{\perp} = (\vec{x}\wedge\mathbf{i})\mathbf{i}^{-1}$, *and* $\vec{x} \in \mathbb{R}^3$, $\vec{x}$ . *We obtain the following elementary bivector derivatives*

$$\partial_A\mathbf{i} = \frac{2}{\theta}, \quad \partial_A\vec{x}\mathbf{i} = -\frac{2}{\theta}\vec{x}_{\perp}, \quad \partial_A\mathbf{i}\vec{x}\mathbf{i} = \frac{2}{\theta}(\vec{x}\mathbf{i} + \vec{x}\wedge\mathbf{i}). \tag{71}$$

*Proof.* We first compute

$$\begin{aligned}\partial_A\mathbf{i} &= -e_{23}\partial_{A_1}\mathbf{i} - e_{31}\partial_{A_2}\mathbf{i} - e_{12}\partial_{A_3}\mathbf{i}\\ &= -e_{23}\frac{1}{\theta}(e_{23} - \frac{A_1}{\theta}\mathbf{i}) - e_{31}\frac{1}{\theta}(e_{31} - \frac{A_2}{\theta}\mathbf{i}) - e_{12}\frac{1}{\theta}(e_{12} - \frac{A_3}{\theta}\mathbf{i})\\ &= \frac{1}{\theta}(3 + \mathbf{i}^2) = \frac{1}{\theta}(3-1) = \frac{2}{\theta}.\end{aligned} \tag{72}$$

The second equation can be shown as follows.

$$\begin{aligned}\partial_A\vec{x}\mathbf{i} &= -e_{23}\vec{x}\partial_{A_1}\mathbf{i} - e_{31}\vec{x}\partial_{A_2}\mathbf{i} - e_{12}\vec{x}\partial_{A_3}\mathbf{i}\\ &= -e_{23}\vec{x}\frac{1}{\theta}(e_{23} - \frac{A_1}{\theta}\mathbf{i}) - e_{31}\vec{x}\frac{1}{\theta}(e_{31} - \frac{A_2}{\theta}\mathbf{i}) - e_{12}\vec{x}\frac{1}{\theta}(e_{12} - \frac{A_3}{\theta}\mathbf{i})\\ &\overset{\text{Prop.3}}{=} -\frac{\vec{x}}{\theta} + \frac{1}{\theta}e_{23}\vec{x}\frac{A_1}{\theta}\mathbf{i} + \frac{1}{\theta}e_{31}\vec{x}\frac{A_2}{\theta}\mathbf{i} + \frac{1}{\theta}e_{12}\vec{x}\frac{A_3}{\theta}\mathbf{i}\\ &= -\frac{\vec{x}}{\theta} + \mathbf{i}\vec{x}\mathbf{i} = \frac{1}{\theta}(-\vec{x} + \mathbf{i}\vec{x}\mathbf{i})\\ &= \frac{1}{\theta}(-\vec{x}_{||} - \vec{x}_{\perp} + \vec{x}_{||} - \vec{x}_{\perp}) = -\frac{2}{\theta}\vec{x}_{\perp}.\end{aligned} \tag{73}$$

For the third equation we compute with $\mathbf{i}\vec{x} = \mathbf{i}\cdot\vec{x} + \mathbf{i}\wedge\vec{x}$ (vector plus bivector)

$$\begin{aligned}
\partial_A \mathbf{i}\vec{x}\mathbf{i} &= (\partial_A \mathbf{i})\vec{x}\mathbf{i} + \dot{\partial}_A \mathbf{i}\vec{x}\dot{\mathbf{i}} \\
&= \frac{2}{\theta}\vec{x}\mathbf{i} - e_{23}\mathbf{i}\vec{x}\frac{1}{\theta}(e_{23} - \frac{A_1}{\theta}\mathbf{i}) - e_{31}\mathbf{i}\vec{x}\frac{1}{\theta}(e_{31} - \frac{A_2}{\theta}\mathbf{i}) - e_{12}\mathbf{i}\vec{x}\frac{1}{\theta}(e_{12} - \frac{A_3}{\theta}\mathbf{i}) \\
&\overset{\text{Props.3,5}}{=} \frac{2}{\theta}\vec{x}\mathbf{i} - \frac{1}{\theta}\mathbf{i}\cdot\vec{x} + \frac{3}{\theta}\mathbf{i}\wedge\vec{x} + \mathbf{i}\mathbf{i}\frac{1}{\theta}\vec{x}\mathbf{i} \\
&= \frac{2}{\theta}\vec{x}\mathbf{i} - \frac{1}{\theta}\vec{x}\mathbf{i} + \frac{1}{\theta}\vec{x}\cdot\mathbf{i} + \frac{3}{\theta}\vec{x}\wedge\mathbf{i} \\
&= \frac{1}{\theta}(\vec{x}\cdot\mathbf{i} + \vec{x}\wedge\mathbf{i} + \vec{x}\cdot\mathbf{i} + 3\vec{x}\wedge\mathbf{i}) = \frac{2}{\theta}(\vec{x}\cdot\mathbf{i} + 2\vec{x}\wedge\mathbf{i}) \\
&= \frac{2}{\theta}(\vec{x}\mathbf{i} + \vec{x}\wedge\mathbf{i}), \qquad (74)
\end{aligned}$$

and the overdot notation indicates the application of $\partial_A$ only to the last factor $\mathbf{i}$ in the second term of the first line. □

### 6.2. Derivative wrt. bivector angle argument of rotor rotation

Now we want to compute the bivector derivative

$$\partial_A R^{-1}\vec{x}R = \partial_A(R^{-1})\vec{x}R + \dot{\partial}_A R^{-1}\vec{x}\dot{R}, \qquad (75)$$

where we use the independence of $\vec{x}$ from $A$. We therefore first compute the partial derivatives of $R$ and $R^{-1}$ as

$$\begin{aligned}
\partial_{A_1} R &= \partial_{A_1}(\cos\frac{\theta}{2} + \mathbf{i}\sin\frac{\theta}{2}) \\
&= -\frac{1}{2}\frac{A_1}{\theta}\sin\frac{\theta}{2} + \frac{1}{2}\frac{A_1}{\theta}\mathbf{i}\cos\frac{\theta}{2} + \frac{1}{\theta}\sin\frac{\theta}{2}(e_{23} - \frac{A_1}{\theta}\mathbf{i}) \\
&= \frac{1}{2}\frac{A_1}{\theta}(\cos\frac{\theta}{2} + \mathbf{i}\sin\frac{\theta}{2})\mathbf{i} + \frac{1}{\theta}\sin\frac{\theta}{2}(e_{23} - \frac{A_1}{\theta}\mathbf{i}) \\
&= \frac{1}{2}\frac{A_1}{\theta}R\,\mathbf{i} + \frac{1}{\theta}\sin\frac{\theta}{2}(e_{23} - \frac{A_1}{\theta}\mathbf{i}), \\
\partial_{A_2} R &= \frac{1}{2}\frac{A_2}{\theta}R\,\mathbf{i} + \frac{1}{\theta}\sin\frac{\theta}{2}(e_{31} - \frac{A_2}{\theta}\mathbf{i}), \\
\partial_{A_3} R &= \frac{1}{2}\frac{A_3}{\theta}R\,\mathbf{i} + \frac{1}{\theta}\sin\frac{\theta}{2}(e_{12} - \frac{A_3}{\theta}\mathbf{i}), \qquad (76)
\end{aligned}$$

and similarly

$$\begin{aligned}
\partial_{A_1} R^{-1} &= \partial_{A_1}(\cos\frac{\theta}{2} - \mathbf{i}\sin\frac{\theta}{2}) \\
&= \frac{1}{2}\frac{A_1}{\theta}R^{-1}(-\mathbf{i}) - \frac{1}{\theta}\sin\frac{\theta}{2}(e_{23} - \frac{A_1}{\theta}\mathbf{i}), \\
\partial_{A_2} R^{-1} &= \frac{1}{2}\frac{A_2}{\theta}R^{-1}(-\mathbf{i}) - \frac{1}{\theta}\sin\frac{\theta}{2}(e_{31} - \frac{A_2}{\theta}\mathbf{i}), \\
\partial_{A_3} R^{-1} &= \frac{1}{2}\frac{A_3}{\theta}R^{-1}(-\mathbf{i}) - \frac{1}{\theta}\sin\frac{\theta}{2}(e_{12} - \frac{A_3}{\theta}\mathbf{i}). \qquad (77)
\end{aligned}$$

We note, that the second term factor $\frac{1}{\theta}\sin\frac{\theta}{2}$ is indicative for the differentiation of the unit plane bivector $\mathbf{i}$ of the rotation plane, i.e. it shows differential change of the orientation of the rotation plane.

We can now compute

$$\begin{aligned}
\partial_A R^{-1} &= -e_{23}\partial_{A_1}R^{-1} - e_{31}\partial_{A_2}R^{-1} - e_{12}\partial_{A_3}R^{-1} \\
&= \frac{1}{2}\frac{A}{\theta}\sin\frac{\theta}{2} + \frac{1}{2}\frac{A}{\theta}\mathbf{i}\cos\frac{\theta}{2} - \frac{1}{\theta}\sin\frac{\theta}{2}(3+\frac{A}{\theta}\mathbf{i}) \\
&\overset{\mathbf{i}=\frac{A}{\theta}}{=} \frac{1}{2}\mathbf{i}\sin\frac{\theta}{2} + \frac{1}{2}\mathbf{i}^2\cos\frac{\theta}{2} - \frac{1}{\theta}\sin\frac{\theta}{2}(3+\mathbf{i}^2) \\
&\overset{\mathbf{i}^2=-1}{=} -\frac{1}{2}(\cos\frac{\theta}{2} - \mathbf{i}\sin\frac{\theta}{2}) - \frac{2}{\theta}\sin\frac{\theta}{2} \\
&= -\frac{1}{2}R^{-1} - \frac{2}{\theta}\sin\frac{\theta}{2}.
\end{aligned} \tag{78}$$

We finally compute

$$\begin{aligned}
&\partial_A R^{-1}\vec{x}R = (\partial_A R^{-1})\vec{x}R \\
&\quad - e_{23}R^{-1}\vec{x}\partial_{A_1}R - e_{31}R^{-1}\vec{x}\partial_{A_2}R - e_{12}R^{-1}\vec{x}\partial_{A_3}R \\
&= -\frac{1}{2}R^{-1}\vec{x}R - \frac{2}{\theta}\sin\frac{\theta}{2}\vec{x}R - \frac{1}{2}\mathbf{i}R^{-1}\vec{x}R\mathbf{i} + \frac{1}{\theta}\sin\frac{\theta}{2}\mathbf{i}R^{-1}\vec{x}\mathbf{i} \\
&\quad + \frac{1}{\theta}\sin\frac{\theta}{2}(-e_{23}R^{-1}\vec{x}e_{23} - e_{31}R^{-1}\vec{x}e_{31} - e_{12}R^{-1}\vec{x}e_{12}) \\
&\overset{Prps.3,5}{=} -\frac{1}{2}R^{-1}(\vec{x}+\mathbf{i}\vec{x}\mathbf{i})R - \frac{2}{\theta}\sin\frac{\theta}{2}\vec{x}R + \frac{1}{\theta}\sin\frac{\theta}{2}R^{-1}(\mathbf{i}\vec{x}\mathbf{i}) \\
&\quad + \frac{1}{\theta}\sin\frac{\theta}{2}(-\cos\frac{\theta}{2}\vec{x} + \sin\frac{\theta}{2}\mathbf{i}\cdot\vec{x} - 3\sin\frac{\theta}{2}\mathbf{i}\wedge\vec{x}) \\
&= -R^{-1}\vec{x}_{||}R - \frac{2}{\theta}\sin\frac{\theta}{2}R^{-1}\vec{x}_{||} - \frac{2}{\theta}\sin\frac{\theta}{2}R\vec{x}_{\perp} \\
&\quad + \frac{1}{\theta}\sin\frac{\theta}{2}R^{-1}\vec{x}_{||} - \frac{1}{\theta}\sin\frac{\theta}{2}R^{-1}\vec{x}_{\perp} \\
&\quad + \frac{1}{\theta}\sin\frac{\theta}{2}[-R^{-1}\vec{x}_{||} - \cos\frac{\theta}{2}\vec{x}_{\perp} - 3\sin\frac{\theta}{2}\mathbf{i}\vec{x}_{\perp}] \\
&= -R^{-1}\vec{x}_{||}R - \frac{2}{\theta}\sin\frac{\theta}{2}R^{-1}\vec{x}_{||} \\
&\quad - \frac{2}{\theta}\sin\frac{\theta}{2}[R\vec{x}_{\perp} + \frac{1}{2}R^{-1}\vec{x}_{\perp} + \frac{3}{2}\sin\frac{\theta}{2}\mathbf{i}\vec{x}_{\perp} + \frac{1}{2}\cos\frac{\theta}{2}\vec{x}_{\perp}] \\
&= -R^{-1}\vec{x}_{||}R - \frac{2}{\theta}\sin\frac{\theta}{2}[R^{-1}\vec{x}_{||} + 2R\vec{x}_{\perp}] \\
&= -R^{-1}\vec{x}_{||}R - \frac{2}{\theta}\sin\frac{\theta}{2}[\vec{x}R + R\vec{x}_{\perp}],
\end{aligned} \tag{79}$$

where in the third equality we apply Propositions 3 and 5 to the vector plus trivector combination $R^{-1}\vec{x} = (\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\mathbf{i})\vec{x} = \cos\frac{\theta}{2}\vec{x}+\sin\frac{\theta}{2}\mathbf{i}\cdot\vec{x}+\sin\frac{\theta}{2}\mathbf{i}\wedge\vec{x}$. In the fourth equality we applied (68), (69), $\vec{x} = \vec{x}_{||} + \vec{x}_{\perp}$, and $-R^{-1}\vec{x}_{||} = -\cos\frac{\theta}{2}\vec{x}_{||} + \sin\frac{\theta}{2}\mathbf{i}\cdot\vec{x} = -\cos\frac{\theta}{2}\vec{x}_{||} + \sin\frac{\theta}{2}\mathbf{i}\vec{x}_{||}$. For the fifth equality we use for the last three terms in square brackets that $R^{-1} + 3\sin\frac{\theta}{2}\mathbf{i} + \cos\frac{\theta}{2} =$

$\cos\frac{\theta}{2} - \mathbf{i}\sin\frac{\theta}{2} + 3\sin\frac{\theta}{2}\mathbf{i} + \cos\frac{\theta}{2} = 2[\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{i}] = 2R$. For the last equality we use $R^{-1}\vec{x}_{||} + 2R\vec{x}_{\perp} = \vec{x}_{||}R + R\vec{x}_{\perp} + R\vec{x}_{\perp} = \vec{x}_{||}R + \vec{x}_{\perp}R + R\vec{x}_{\perp} = \vec{x}R + R\vec{x}_{\perp}$.

Finally, we obtain

**Proposition 7.** *The bivector rotation angle derivative of a rotor transformation $R^{-1}\vec{x}R$, with $R = e^{A/2}$, $A = A_1e_{23} + A_2e_{31} + A_3e_{12}$, $\theta = |A|$, rotation plane unit bivector $\mathbf{i} = A/\theta$, $\vec{x}_{||} = (\vec{x}\cdot\mathbf{i})\mathbf{i}^{-1}$, $\vec{x}_{\perp} = (\vec{x}\wedge\mathbf{i})\mathbf{i}^{-1}$, is*

$$\begin{aligned}\partial_A R^{-1}\vec{x}R &= -R^{-1}\vec{x}_{||}R - \frac{2}{\theta}\sin\frac{\theta}{2}[R^{-1}\vec{x}_{||} + 2R\vec{x}_{\perp}] \\ &= -R^{-1}\vec{x}_{||}R - \frac{2}{\theta}\sin\frac{\theta}{2}[\vec{x}R + R\vec{x}_{\perp}]. \qquad (80)\end{aligned}$$

**Remark 3.** *An alternative proof of Proposition 7 can be obtained by observing that*

$$\begin{aligned}R^{-1}\vec{x}R &= \cos\theta\,\vec{x}_{||} + \sin\theta\,\vec{x}_{||}\mathbf{i} + \vec{x}_{\perp} \\ &= \frac{1}{2}\cos\theta(\vec{x} + \mathbf{i}\vec{x}\mathbf{i}) + \frac{1}{2}\sin\theta(\vec{x} + \mathbf{i}\vec{x}\mathbf{i})\mathbf{i} + \frac{1}{2}(\vec{x} - \mathbf{i}\vec{x}\mathbf{i}) \\ &= \frac{1}{2}\cos\theta(\vec{x} + \mathbf{i}\vec{x}\mathbf{i}) + \frac{1}{2}\sin\theta(\vec{x}\mathbf{i} - \mathbf{i}\vec{x}) + \frac{1}{2}(\vec{x} - \mathbf{i}\vec{x}\mathbf{i}), \qquad (81)\end{aligned}$$

*and using Proposition 6 for computing the bivector argument derivative $\partial_A R^{-1}$ $\vec{x}R$.*

**Remark 4.** *We observe that in case of invariant rotation plane $\mathbf{i}$, the last term (originating from differentiating $\mathbf{i}$, indicated in black in Proposition 7) vanishes and only $\partial_A R^{-1}\vec{x}R = -R^{-1}\vec{x}_{||}R$ remains, which is natural in view of $R^{-1}\vec{x}R = R^{-1}(\vec{x}_{||} + \vec{x}_{\perp})R = R^{-2}\vec{x}_{||} + \vec{x}_{\perp} = e^{-A}\vec{x}_{||} + \vec{x}_{\perp}$, and both $\vec{x}_{||}$, $\vec{x}_{\perp}$ independent of $A$, if $\mathbf{i}$ is kept invariant.*

We therefore have the corollary

**Corollary 1.** *The bivector rotation angle derivative of a rotor transformation $R^{-1}\vec{x}R$, with $R = e^{A/2}$, $A = A_1e_{23} + A_2e_{31} + A_3e_{12}$, $\theta = |A|$, and* **fixed** *rotation plane unit bivector $\mathbf{i} = A/\theta$, $\vec{x}_{||} = (\vec{x}\cdot\mathbf{i})\mathbf{i}^{-1}$, is*

$$\partial_A R^{-1}\vec{x}R = -R^{-1}\vec{x}_{||}R. \qquad (82)$$

### 6.3. Directional differentiation wrt. bivector angle argument of rotor rotation

We now want to compute the directional derivative with respect to the bivector angle argument of the rotor rotation

$$\begin{aligned}&(B\cdot\partial_A)R^{-1}(A)\vec{x}R(A) = B\cdot\partial_A R^{-1}(A)\vec{x}R(A) \\ &= (B_1e_{23} + B_2e_{31} + B_3e_{12})\cdot\partial_A R^{-1}(A)\vec{x}R(A) \\ &= (B_1\partial_{A_1} + B_2\partial_{A_2} + B_3\partial_{A_3})R^{-1}(A)\vec{x}R(A) \\ &= [B\cdot\partial_A R^{-1}(A)]\vec{x}R(A) + B\cdot\dot{\partial}_A R^{-1}(A)\vec{x}\,\dot{R}(A), \qquad (83)\end{aligned}$$

where we assume that inner and outer products have priority over the geometric product. We first obtain from (77)

$$
\begin{aligned}
B\cdot\partial_A R^{-1}(A) = {}& \frac{1}{2}\frac{(-B_1A_1-B_2A_2-B_3A_3)}{\theta}R^{-1}\mathbf{i} \\
& -\frac{1}{\theta}\sin\frac{\theta}{2}[B_1e_{23}+B_2e_{31}+B_3e_{12}-\frac{(-B_1A_1-B_2A_2-B_3A_3)}{\theta}\mathbf{i}] \\
= {}& \frac{1}{2}\frac{B\cdot A}{\theta}R^{-1}\mathbf{i}-\frac{1}{\theta}\sin\frac{\theta}{2}(B+\frac{B\cdot A}{\theta}R^{-1}\mathbf{i}). \qquad (84)
\end{aligned}
$$

Then we get with (76)

$$
\begin{aligned}
(B\cdot\partial_A)R^{-1}\vec{x}R = {}& [B\cdot\partial_A R^{-1}]\vec{x}R + B\cdot\dot{\partial}_A R^{-1}\vec{x}\,\dot{R} \\
\overset{(84)}{=} {}& \frac{1}{2}\frac{B\cdot A}{\theta}R^{-1}\mathbf{i}\vec{x}R-\frac{1}{\theta}\sin\frac{\theta}{2}(B\vec{x}R+\frac{B\cdot A}{\theta}R^{-1}\mathbf{i}\vec{x}R) \\
& + B_1R^{-1}\vec{x}\partial_{A_1}R + B_2R^{-1}\vec{x}\partial_{A_2}R + B_3R^{-1}\vec{x}\partial_{A_3}R \\
\overset{(76)}{=} {}& \frac{1}{2}\frac{B\cdot A}{\theta}R^{-1}\mathbf{i}\vec{x}R-\frac{1}{\theta}\sin\frac{\theta}{2}(B\vec{x}R+\frac{B\cdot A}{\theta}R^{-1}\mathbf{i}\vec{x}R) \\
& -\frac{1}{2}\frac{B\cdot A}{\theta}R^{-1}\vec{x}\mathbf{i}R+\frac{1}{\theta}\sin\frac{\theta}{2}R^{-1}\vec{x}(B+\frac{B\cdot A}{\theta}\mathbf{i}) \\
= {}& \frac{B\cdot A}{\theta}R^{-1}\frac{1}{2}[\mathbf{i}\vec{x}-\vec{x}\mathbf{i}]R \\
& +\frac{1}{\theta}\sin\frac{\theta}{2}(-B\vec{x}R+R^{-1}\vec{x}B-\frac{B\cdot A}{\theta}\mathbf{i}\vec{x}R+\frac{B\cdot A}{\theta}R^{-1}\vec{x}\mathbf{i}) \\
= {}& \frac{B\cdot A}{\theta}\mathbf{i}R^{-1}\vec{x}_{||}R \\
& +\frac{1}{\theta}\sin\frac{\theta}{2}\{-B\vec{x}R+R^{-1}\vec{x}B+\frac{B\cdot A}{\theta}(-\mathbf{i}\vec{x}R+R^{-1}\vec{x}\mathbf{i})\}, \qquad (85)
\end{aligned}
$$

because

$$
\frac{1}{2}(\mathbf{i}\vec{x}-\vec{x}\mathbf{i}) = \mathbf{i}\frac{1}{2}(\vec{x}+\mathbf{i}\vec{x}\mathbf{i}) = \mathbf{i}\vec{x}_{||}. \qquad (86)
$$

We summarize in

**Proposition 8.** *The directional derivative in the direction of any bivector $B \in Cl^2(3,0)$ of the rotor rotation transformation $R^{-1}(A)\vec{x}R(A)$ with bivector angle argument $A$, $R = e^{A/2}$, $A = A_1e_{23}+A_2e_{31}+A_3e_{12}$, $\theta = |A|$, rotation plane unit bivector $\mathbf{i} = A/\theta$, $\vec{x}_{||} = (\vec{x}\cdot\mathbf{i})\mathbf{i}^{-1}$, $\vec{x}_{\perp} = (\vec{x}\wedge\mathbf{i})\mathbf{i}^{-1}$, is*

$$
\begin{aligned}
(B\cdot\partial_A)R^{-1}\vec{x}R = {}& (B\cdot\mathbf{i})\mathbf{i}R^{-1}\vec{x}_{||}R \\
& +\frac{1}{\theta}\sin\frac{\theta}{2}\{-B\vec{x}R+R^{-1}\vec{x}B+(B\cdot\mathbf{i})(-\mathbf{i}\vec{x}R+R^{-1}\vec{x}\mathbf{i})\}. \qquad (87)
\end{aligned}
$$

If we compute the directional derivative with respect to the the unit plane bivector $\mathbf{i}$ of the rotor rotation, i.e. we only vary the angle $\theta$ but keep the orientiation of the rotation plane fixed, we get from (87) that

$$
(\mathbf{i}\cdot\partial_A)R^{-1}\vec{x}R = -\mathbf{i}R^{-1}\vec{x}_{||}R, \qquad (88)
$$

because

$$
B\cdot\mathbf{i} \overset{B=\mathbf{i}}{=} \mathbf{i}^2 = -1. \qquad (89)
$$

**Remark 5.** *Equation* (88) *is exactly what we expect in view of Corollary* 1 *for fixed orientation of the rotation plane (i.e. fixed* $\mathbf{i}$*), because with* $\vec{x}_{||}$ *paralellel to* $\mathbf{i}$ *also* $R^{-1}\vec{x}_{||}R$ *is paralellel to* $\mathbf{i}$ *and therefore the inner product of* $\mathbf{i}$ *with* (82) *from the left is* $\mathbf{i}\cdot(-R^{-1}\vec{x}_{||}R) = -\mathbf{i}R^{-1}\vec{x}_{||}R$*, i.e. gives* (88)*, the outer product part* $\mathbf{i}\wedge(-R^{-1}\vec{x}_{||}R)$ *being zero.*

### 6.4. Vector differentiation of translation

We employ the conformal model of Euclidean space in CGA $Cl(4,1)$ and use vector differential calculus [12]. A point is given by

$$X = \vec{x} + \frac{1}{2}\vec{x}^2 e_\infty + e_0, \tag{90}$$

a translator for translation by Euclidean translation vector $\vec{a} \in \mathbb{R}^3$, $\vec{x} \in \mathbb{R}^3$ independent of $\vec{a}$, is given by

$$\begin{aligned} T(\vec{a}) &= 1 + \frac{1}{2}\vec{a}e_\infty = e^{\frac{1}{2}\vec{a}e_\infty}, \\ T^{-1}(\vec{a}) &= \widetilde{T}(\vec{a}) = 1 - \frac{1}{2}\vec{a}e_\infty = e^{-\frac{1}{2}\vec{a}e_\infty}, \end{aligned} \tag{91}$$

and the translated conformal point is

$$\begin{aligned} T^{-1}(\vec{a})XT(\vec{a}) &= \vec{x} + \vec{a} + \frac{1}{2}(\vec{x}+\vec{a})^2 e_\infty + e_0 \\ &= \vec{x} + \vec{a} + \frac{1}{2}\vec{x}^2 e_\infty + \vec{a}\cdot\vec{x}e_\infty + \frac{1}{2}\vec{a}^2 e_\infty + e_0. \end{aligned} \tag{92}$$

We could compute the vector derivative of $T^{-1}(\vec{a})XT(\vec{a})$ from

$$\partial_{\vec{a}}T^{-1}(\vec{a})XT(\vec{a}) = (\partial_{\vec{a}}T^{-1})XT + \dot{\partial}_{\vec{a}}T^{-1}X\dot{T}, \tag{93}$$

with

$$\begin{aligned} \partial_{\vec{a}} &= e^1 e_1\cdot\partial_{\vec{a}} + e^2 e_2\cdot\partial_{\vec{a}} + e^3 e_3\cdot\partial_{\vec{a}} \\ &= e_1\partial_{a_1} + e_2\partial_{a_2} + e_3\partial_{a_3}, \end{aligned} \tag{94}$$

and reciprocal vectors and partial derivatives

$$e^k = e_k, \qquad e_k\cdot\partial_{\vec{a}} = \partial_{a_k}, \qquad k = 1,2,3. \tag{95}$$

But obviously, the direct vector differentiation of the right hand side of (92) will be less cumbersome. So we compute

$$\begin{aligned} &\partial_{\vec{a}}T^{-1}(\vec{a})XT(\vec{a}) \\ &= \partial_{\vec{a}}\big(\vec{x} + \vec{a} + \frac{1}{2}\vec{x}^2 e_\infty + \vec{a}\cdot\vec{x}e_\infty + \frac{1}{2}\vec{a}^2 e_\infty + e_0\big) \\ &= \partial_{\vec{a}}\vec{a} + \partial_{\vec{a}}\vec{a}\cdot\vec{x}e_\infty + \frac{1}{2}\partial_{\vec{a}}\vec{a}^2 e_\infty = 3 + \vec{x}e_\infty + \frac{1}{2}2\vec{a}e_\infty \\ &= 3 + (\vec{x}+\vec{a})e_\infty. \end{aligned} \tag{96}$$

Therefore we arrive at

**Proposition 9.** *In CGA $Cl(4,1)$ the point $X = \vec{x} + \frac{1}{2}\vec{x}^2 e_\infty + e_0$ is translated by $\vec{a} \in \mathbb{R}^3$ with $T^{-1}(\vec{a})XT(\vec{a})$. Vector differentiation with respect to $\vec{a}$ gives*

$$\partial_{\vec{a}} T^{-1}(\vec{a})XT(\vec{a}) = 3 + (\vec{x} + \vec{a})\, e_\infty. \tag{97}$$

Finally, we compute coodinate free the *directional derivative* of $T^{-1}(\vec{a})XT(\vec{a})$ as a function of $\vec{a} \in \mathbb{R}^3$ in the direction $\vec{b} \in \mathbb{R}^3$ by

$$\begin{aligned} &\vec{b} \cdot \partial_{\vec{a}} T^{-1}(\vec{a})XT(\vec{a}) \\ &= \vec{b} \cdot \partial_{\vec{a}} \big(\vec{x} + \vec{a} + \frac{1}{2}\vec{x}^2 e_\infty + \vec{a} \cdot \vec{x} e_\infty + \frac{1}{2}\vec{a}^2 e_\infty + e_0\big) \\ &= \vec{b} \cdot \partial_{\vec{a}} \vec{a} + \vec{b} \cdot \partial_{\vec{a}}\, \vec{a} \cdot \vec{x} e_\infty + \vec{b} \cdot \partial_{\vec{a}} \frac{1}{2}\vec{a}^2 e_\infty \\ &= \vec{b} + \vec{b} \cdot \vec{x} e_\infty + \frac{1}{2} 2\vec{b} \cdot \vec{a} e_\infty = \vec{b} + \vec{b} \cdot (\vec{x} + \vec{a}) e_\infty. \end{aligned} \tag{98}$$

Therefore we have

**Proposition 10.** *In CGA $Cl(4,1)$ the point $X = \vec{x} + \frac{1}{2}\vec{x}^2 e_\infty + e_0$ is translated by $\vec{a} \in \mathbb{R}^3$ with $T^{-1}(\vec{a})XT(\vec{a})$. The directional derivative of $T^{-1}(\vec{a})XT(\vec{a})$ as a function of $\vec{a} \in \mathbb{R}^3$ in the direction $\vec{b} \in \mathbb{R}^3$ is*

$$\vec{b} \cdot \partial_{\vec{a}} T^{-1}(\vec{a})XT(\vec{a}) = \vec{b} + \vec{b} \cdot (\vec{x} + \vec{a}) e_\infty. \tag{99}$$

### 6.5. Bivector and vector differentiation of motor transformation

We now want to compute the directional derivatives of the motor transformation

$$\widetilde{M}(A, \vec{a})XM(A, \vec{a}), \qquad M(A, \vec{a}) = R(A)T(\vec{a}) \tag{100}$$

with $\vec{a} \in \mathbb{R}^3$ the vector of translation, and $A \in Cl^2(3,0)$ the bivector argument of the rotor $R(A)$.

Because the translator $T(\vec{a})$ is independent of the rotor bivector argument $A$ we can simply translate (87) to obtain

$$\begin{aligned} &B \cdot \partial_A \widetilde{M}(A, \vec{a})XM(A, \vec{a}) = (B \cdot \partial_A)\widetilde{T}\widetilde{R}\vec{x}RT \\ &= (B \cdot \mathbf{i})\widetilde{T}\widetilde{R}\,\mathbf{i}\vec{x}_{||}RT \\ &\quad + \frac{1}{\theta}\sin\frac{\theta}{2}\widetilde{T}\{-B\vec{x}R + \widetilde{R}\vec{x}B + (B \cdot \mathbf{i})(-\mathbf{i}xR + \widetilde{R}x\mathbf{i})\}T \\ &= (B \cdot \mathbf{i})\widetilde{M}\mathbf{i}\vec{x}_{||}M \\ &\quad + \frac{1}{\theta}\sin\frac{\theta}{2}\{-\widetilde{T}(B + (B \cdot \mathbf{i})\mathbf{i})\vec{x}M + \widetilde{M}\vec{x}(B + (B \cdot \mathbf{i})\mathbf{i})T\} \\ &= (B \cdot \mathbf{i})\widetilde{M}\mathbf{i}\vec{x}_{||}M \\ &\quad + \frac{1}{\theta}\sin\frac{\theta}{2}\{\widetilde{T}(B \times \mathbf{i})\mathbf{i}\vec{x}M + \widetilde{M}\vec{x}(\mathbf{i} \times B)\mathbf{i}T\}, \end{aligned} \tag{101}$$

because

$$B = -(B\mathbf{i})\mathbf{i} = -(B \cdot \mathbf{i} + B \times \mathbf{i})\mathbf{i}, \tag{102}$$

and therefore

$$B + (B \cdot \mathbf{i})\mathbf{i} = -(B \cdot \mathbf{i} + B \times \mathbf{i})\mathbf{i} + (B \cdot \mathbf{i})\mathbf{i} = -B \times \mathbf{i}\,\mathbf{i}, \tag{103}$$

with anti-symmetric commutator product $A \times B = \frac{1}{2}(AB - BA)$. Note that the commutator product has priority over the geometric product.

We furthermore compute the directional derivative

$$\begin{aligned}\vec{b} \cdot \partial_{\vec{a}} \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) &= \vec{b} \cdot \partial_{\vec{a}} \widetilde{T} X' T \\ &= \vec{b} + \vec{b} \cdot (\vec{x}' + \vec{a}) e_\infty = \vec{b} + \vec{b} \cdot (\widetilde{R} \vec{x} R + \vec{a}) e_\infty,\end{aligned} \tag{104}$$

where the conformal vector $X' = \widetilde{R} X R$ is independent of $\vec{a}$. We summarize in

**Proposition 11.** *The directional derivatives of the motor transformation $\widetilde{M}(A, \vec{a})\ XM(A, \vec{a})$, with $M(A, \vec{a}) = R(A)T(\vec{a})$, $A \in Cl^2(3,0)$ the rotation angle bivector, $\vec{a} \in \mathbb{R}^3$ the translation vector, are in the directions $B \in Cl^2(3,0)$ and $\vec{b} \in \mathbb{R}^3$ :*

$$\begin{aligned}&B \cdot \partial_A \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) \\ &= (B \cdot \mathbf{i}) \widetilde{M} \mathbf{i} \vec{x}_{||} M + \frac{1}{\theta} \sin \frac{\theta}{2} \{\widetilde{T}(B \times \mathbf{i}) \mathbf{i} \vec{x} M + \widetilde{M} \vec{x} (\mathbf{i} \times B) \mathbf{i} T\},\end{aligned} \tag{105}$$

*and respectively*

$$\vec{b} \cdot \partial_{\vec{a}}\, \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) = \vec{b} + \vec{b} \cdot (\widetilde{R} \vec{x} R + \vec{a}) e_\infty. \tag{106}$$

Because every bivector $B = B_1 e_{23} + B_2 e_{31} + B_3 e_{12}$ and every vector $\vec{b} = b_1 e_1 + b_2 e_2 + b_3 e_3$ we can express the directional derivatives of Proposition 11 componentwise.

**Corollary 2.**

$$\begin{aligned}B \cdot \partial_A \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) &= B_1\, e_{23} \cdot \partial_A \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) \\ &\quad + B_2\, e_{31} \cdot \partial_A \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) \\ &\quad + B_3\, e_{12} \cdot \partial_A \widetilde{M}(A, \vec{a}) X M(A, \vec{a}), \\ \vec{b} \cdot \partial_{\vec{a}}\, \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) &= b_1\, e_1 \cdot \partial_{\vec{a}}\, \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) \\ &\quad + b_2\, e_2 \cdot \partial_{\vec{a}}\, \widetilde{M}(A, \vec{a}) X M(A, \vec{a}) \\ &\quad + b_3\, e_3 \cdot \partial_{\vec{a}}\, \widetilde{M}(A, \vec{a}) X M(A, \vec{a}).\end{aligned} \tag{107}$$

## 7. Directional derivative of squared distance point to plane term in correspondence cost function $\Omega$

We have now assembled all algebraic and differential tools for computing the directional derivatives of a squared distance (51) term $d^2$ in the point to plane correspondence cost function $\Omega$ of (52).

To demonstrate the principle we assume the initial conformal plane (49) $Plane_0$ constructed from three initial conformal points $P_{10}, P_{20}, P_{30}$ in the plane in strip 1. The strip 1 adjustment motor $M(A, \vec{a})$ transforms the three points, and via the outermorphism rule of CGA likewise the whole plane and also the dual plane vector $Plane_0^*$, because dualization means multiplication with $I_5^{-1}$, and $I_5$ is invariant under motor transformations. So the motor $M(A,\vec{a})$ transforms

$$\begin{aligned} Plane^*(A,\vec{a}) &= \widetilde{M}((A,\vec{a}))Plane_0^*\, M(A,\vec{a}) \\ &= \widetilde{T}(\vec{a})\widetilde{R}(A)(\mathbf{n}_0+\delta_0\mathbf{e}_\infty)R(A)T(\vec{a}) \\ &= \widetilde{T}(\vec{a})\mathbf{n}_R T(\vec{a}) + \widetilde{T}(\vec{a})\delta_0\mathbf{e}_\infty T(\vec{a}) \\ &= (1-\frac{1}{2}\vec{a}\mathbf{e}_\infty)\mathbf{n}_R(1+\frac{1}{2}\vec{a}\mathbf{e}_\infty)+\delta_0\mathbf{e}_\infty \\ &= \mathbf{n}_R + \frac{1}{2}(\mathbf{n}_R\vec{a}+\vec{a}\mathbf{n}_R)\mathbf{e}_\infty + \delta_0\mathbf{e}_\infty \\ &= \mathbf{n}_R + (\mathbf{n}_R\cdot\vec{a}+\delta_0)\mathbf{e}_\infty \\ &= \widetilde{R}(A)\mathbf{n}_0 R(A) + [(\widetilde{R}(A)\mathbf{n}_0 R(A))\cdot\vec{a}+\delta_0]\mathbf{e}_\infty, \end{aligned} \tag{108}$$

with unit normal vector $\mathbf{n}_0$ of plane $Plane_0^*$, unit normal vector $\mathbf{n}_R = \widetilde{R}(A)\mathbf{n}_0 R(A)$ of the motor transformed plane $Plane^*(A,\vec{a})$ and distance $\delta_0$ to the origin of $Plane_0^*$, and distance $\mathbf{n}_R\cdot\vec{a}+\delta_0$ to the origin of $Plane^*(A,\vec{a})$.

In a second strip 2 we assume a corresponding conformal point $Q_0 = \vec{q}_0 + \frac{1}{2}\vec{q}_0^2\mathbf{e}_\infty + \mathbf{e}_0$, transformed by the strip 2 adjustment motor $N(Z,\vec{z}) = R(Z)T(\vec{z})$ into

$$Q(Z,\vec{z}) = \widetilde{N}(Z,\vec{z})Q_0 N(Z,\vec{z}). \tag{109}$$

The correspondence distance to be minimized between conformal plane $Plane$ in strip 1 and the corresponding point $Q$ in strip 2, supposed to belong to the plane is by (51) :

$$d = Plane^*\cdot Q = Plane^*(A,\vec{a})\cdot Q(Z,\vec{z}), \tag{110}$$

where $A$ is the bivector angle for the adjustment rotation of strip 1, $\vec{a}$ is the translation vector for the adjustment translation of strip 1, and $Z,\vec{z}$ are the corresponding entities for the adjustment motor transformation of strip 2. The correspondence cost function $\Omega$ of (52) contains the term $d^2$. We therefore need to compute the four directional derivatives $B\cdot\partial_A$, $\vec{b}\cdot\partial_{\vec{a}}$, $B\cdot\partial_Z$, $\vec{b}\cdot\partial_{\vec{z}}$ of $d^2$ in the following.

We find that

$$\begin{aligned} B\cdot\partial_A d^2 &= 2d\,B\cdot\partial_A d \\ &= 2d\,B\cdot\partial_A\Big(Plane^*(A,\vec{a})\cdot Q(Z,\vec{z})\Big) \\ &= 2d\Big([B\cdot\partial_A Plane^*(A,\vec{a})]\cdot Q(Z,\vec{z})\Big), \end{aligned} \tag{111}$$

for which we need to compute

$$\begin{aligned} & B\cdot\partial_A Plane^*(A,\vec{a}) \\ & \overset{(108)}{=} B\cdot\partial_A\Big(\widetilde{R}(A)\mathbf{n}_0 R(A) + [(\widetilde{R}(A)\mathbf{n}_0 R(A))\cdot\vec{a}+\delta_0]\mathbf{e}_\infty\Big) \\ & = B\cdot\partial_A\widetilde{R}(A)\mathbf{n}_0 R(A) + [B\cdot\partial_A\widetilde{R}(A)\mathbf{n}_0 R(A)]\cdot\vec{a}\,\mathbf{e}_\infty, \end{aligned} \tag{112}$$

for which in turn we need $B \cdot \partial_A \widetilde{R}(A)\mathbf{n}_0 R(A)$. This last computation can be done easily by replacing $\vec{x} \to \mathbf{n}_0$ in (87) of Proposition 8, then $B \cdot \partial_A d^2$ can be computed explicitly.

Next we compute the directional derivative for the $\vec{a}$ translation

$$\vec{b} \cdot \partial_{\vec{a}}\, d^2 = 2d\Big([\vec{b} \cdot \partial_{\vec{a}} Plane^*(A, \vec{a})] \cdot Q(Z, \vec{z})\Big), \tag{113}$$

for which we need to compute

$$\begin{aligned} &\vec{b} \cdot \partial_{\vec{a}} Plane^*(A, \vec{a}) \\ &\overset{(108)}{=} \vec{b} \cdot \partial_{\vec{a}} \Big(\widetilde{R}(A)\mathbf{n}_0 R(A) + [(\widetilde{R}(A)\mathbf{n}_0 R(A)) \cdot \vec{a} + \delta_0]\mathbf{e}_\infty\Big) \\ &= \vec{b} \cdot \partial_{\vec{a}}(\mathbf{n}_R \cdot \vec{a})\mathbf{e}_\infty = \mathbf{n}_R \cdot \vec{b}\,\mathbf{e}_\infty = [\widetilde{R}(A)\mathbf{n}_0 R(A)] \cdot \vec{b}\,\mathbf{e}_\infty, \end{aligned} \tag{114}$$

the result being a scalar factor times $\mathbf{e}_\infty$. This means that

$$\begin{aligned} [\vec{b} \cdot \partial_{\vec{a}} Plane^*(A, \vec{a})] \cdot Q(Z, \vec{z}) &= [\widetilde{R}(A)\mathbf{n}_0 R(A)] \cdot \vec{b}(\mathbf{e}_\infty \cdot \mathbf{e}_0) \\ &= -[\widetilde{R}(A)\mathbf{n}_0 R(A)] \cdot \vec{b}, \end{aligned} \tag{115}$$

because

$$\mathbf{e}_\infty \cdot Q(Z, \vec{z}) = \mathbf{e}_\infty \cdot (\vec{q} + \frac{1}{2}\vec{q}^2\mathbf{e}_\infty + \mathbf{e}_0) = \mathbf{e}_\infty \cdot \mathbf{e}_0 = -1. \tag{116}$$

Hence

$$\vec{b} \cdot \partial_{\vec{a}}\, d^2 = -2d\,\mathbf{n}_R \cdot \vec{b} = -2d\,[\widetilde{R}(A)\mathbf{n}_0 R(A)] \cdot \vec{b}. \tag{117}$$

Next we compute

$$\begin{aligned} B \cdot \partial_Z\, d^2 &= 2d\,B \cdot \partial_Z\, d = 2d\,B \cdot \partial_Z \Big(Plane^*(A, \vec{a}) \cdot Q(Z, \vec{z})\Big) \\ &= 2d\,\Big(Plane^*(A, \vec{a}) \cdot [B \cdot \partial_Z Q(Z, \vec{z})]\Big), \end{aligned} \tag{118}$$

and $B \cdot \partial_Z Q(Z, \vec{z})$ can be easily computed using (105) of Proposition 11.

Finally,

$$\begin{aligned} \vec{b} \cdot \partial_{\vec{z}}\, d^2 &= 2d\,\vec{b} \cdot \partial_{\vec{z}}\, d = 2d\,\vec{b} \cdot \partial_{\vec{z}} \Big(Plane^*(A, \vec{a}) \cdot Q(Z, \vec{z})\Big) \\ &= 2d\,\Big(Plane^*(A, \vec{a}) \cdot [\vec{b} \cdot \partial_{\vec{z}}\, Q(Z, \vec{z})]\Big), \end{aligned} \tag{119}$$

and $\vec{b} \cdot \partial_{\vec{z}}\, Q(Z, \vec{z})$ can be easily computed using (106) of Proposition 11 :

$$\begin{aligned} \vec{b} \cdot \partial_{\vec{z}}\, Q(Z, \vec{z}) &= \vec{b} \cdot \partial_{\vec{z}}\, \widetilde{T}(\vec{z})\widetilde{R}(Z)Q_0 R(Z)T(\vec{z}) \\ &= \vec{b} + \vec{b} \cdot [\widetilde{R}(Z)\vec{q}_0 R(Z) + \vec{z}]\, e_\infty. \end{aligned} \tag{120}$$

Therefore

$$\begin{aligned} &Plane^*(A, \vec{a}) \cdot [\vec{b} \cdot \partial_{\vec{z}}\, Q(Z, \vec{z})] \\ &= \Big(\mathbf{n}_R + (\mathbf{n}_R \cdot \vec{a} + \delta_0)\mathbf{e}_\infty\Big) \cdot \Big(\vec{b} + \vec{b} \cdot [\widetilde{R}(Z)\vec{q}_0 R(Z) + \vec{z}]\, e_\infty\Big) \\ &= \mathbf{n}_R \cdot \vec{b}, \end{aligned} \tag{121}$$

hence

$$\vec{b} \cdot \partial_{\vec{z}}\, d^2 = 2d\,\mathbf{n}_R \cdot \vec{b} = 2d\,\widetilde{R}(A)\mathbf{n}_0 R(A) \cdot \vec{b}. \tag{122}$$

# 8. Newton's Method in Geometric Algebra for ALS Adjustment

## 8.1. Taylor expansion in geometric algebra

We refer to Proposition 12 of [13].

**Proposition 12.** *For a multivector function $F = F(X)$ defined on the Clifford algebra $Cl(p,q)$*

$$F : X \in Cl(p,q) \to F(X) \in Cl(p,q), \tag{123}$$

*the Taylor expansion is given by*

$$\begin{aligned} F(X + P(A)) &= \sum_{k=0}^{\infty} \frac{1}{k!}(A * \partial_X)F = \exp(A * \partial_X)F \\ &= F(X) + (A * \partial_X)F(X) + \frac{1}{2}(A * \partial_X)^2 F(X) + \dots, \end{aligned} \tag{124}$$

*with $P(A)$ the projection of $A$ onto $Cl(p,q)$.*

## 8.2. Application of Newton's method in GA to adjustment solution by iteration

We assume two ALS strips with $n$ planes $Plane_{0i} = \mathbf{n}_{0i} + \delta_{0i}$, $1 \le i \le n$, in strip 1, with adjustment motor $M(A, \vec{a})$ that gives by (108)

$$\begin{aligned} Plane_i^*(A, \vec{a}) &= \widetilde{M}((A, \vec{a}))Plane_{0i}^*\, M(A, \vec{a}) \\ &= \widetilde{R}(A)\mathbf{n}_{0i}R(A) + [(\widetilde{R}(A)\mathbf{n}_{0i}R(A)) \cdot \vec{a} + \delta_{0i}]\mathbf{e}_\infty, \end{aligned} \tag{125}$$

Furthermore, we assume $n$ corresponding conformal points $Q_{0i} = \vec{q}_{0i} + \frac{1}{2}\vec{q}_{0i}^{\,2}\mathbf{e}_\infty + \mathbf{e}_0$ in strip 2, with adjustment motor $N(Z, \vec{z})$, such that

$$Q_i(Z, \vec{z}) = \widetilde{N}(Z, \vec{z})Q_{0i}N(Z, \vec{z}), \quad 1 \le i \le n. \tag{126}$$

The assumed $n$ correspondences result in $n$ distances

$$d_i = Plane_i^*(A, \vec{a}) \cdot Q_i(Z, \vec{z}), \quad 1 \le i \le n. \tag{127}$$

In this simplified model of two strips and $n$ correspondences we have the two bivector angle paramenters $\hat{A}, \hat{Z} \in Cl^2(3,0)$ and the two translation vector parameters $\hat{\vec{a}}, \hat{\vec{z}} \in \mathbb{R}^3$. The $n$-dimensional vector $f$ of distances $d_i$, $1 \le i \le n$ can therefore be linearized by

$$\begin{aligned} f(\hat{A}, \hat{\vec{a}}, \hat{Z}, \hat{\vec{z}}) &= f(A_0, \vec{a}_0, Z_0, \vec{z}_0) + \Big[(\Delta\hat{A} \cdot \partial_A) + (\Delta\hat{\vec{a}} \cdot \partial_{\vec{a}}) \\ &+ (\Delta\hat{Z} \cdot \partial_Z) + (\Delta\hat{\vec{z}} \cdot \partial_{\vec{z}})\Big] f\Big|_{(A,\vec{a},Z,\vec{z})=(A_0,\vec{a}_0,Z_0,\vec{z}_0)}. \end{aligned} \tag{128}$$

In view of Corollary 2 and Section 7, the functions $(\Delta\hat{A} \cdot \partial_A)f$, $(\Delta\hat{\vec{a}} \cdot \partial_{\vec{a}})f$, $(\Delta\hat{Z} \cdot \partial_Z)f$, and $(\Delta\hat{\vec{z}} \cdot \partial_{\vec{z}})f$ are linear in $\Delta\hat{A}$, $\Delta\hat{\vec{a}}$, $\Delta\hat{Z}$, and $\Delta\hat{\vec{z}}$, respectively. This means linearity in the six coordinates of the two bivectors $\Delta\hat{A}$ and $\Delta\hat{Z}$, and in the six coordinates of the two vectors $\Delta\hat{\vec{a}}$ and $\Delta\hat{\vec{z}}$, respectively. Using (112), (115), (118), and (122), respectively, we can express the linearization of

$f$ with the help of linear directional derivatives in terms of the 12-component coordinate vector of $\Delta\hat{A}$, $\Delta\hat{\vec{a}}$, $\Delta\hat{Z}$, and $\Delta\hat{\vec{z}}$.

$$\begin{aligned} f(\hat{A},\hat{\vec{a}},\hat{Z},\hat{\vec{z}}) &= f(A_0,\vec{a}_0,Z_0,\vec{z}_0) + D_{n\times 12}\Big[\Delta\hat{A}_1,\Delta\hat{A}_2,\Delta\hat{A}_3, \\ &\qquad \Delta\hat{\vec{a}}_1,\Delta\hat{\vec{a}}_2,\Delta\hat{\vec{a}}_3,\Delta\hat{Z}_1,\Delta\hat{Z}_2,\Delta\hat{Z}_3,\Delta\hat{\vec{z}}_1,\Delta\hat{\vec{z}}_2,\Delta\hat{\vec{z}}_3\Big]^T \\ &= f(A_0,\vec{a}_0,Z_0,\vec{z}_0) + D_{n\times 12}\,\Delta\hat{\mathbf{x}}^T, \end{aligned} \tag{129}$$

where $D_{n\times 12}$ is the $(n\times 12)$ *design* matrix resulting from the directional derivatives $(\Delta\hat{A}\cdot\partial_A)f$, $(\Delta\hat{\vec{a}}\cdot\partial_{\vec{a}})f$, $(\Delta\hat{Z}\cdot\partial_Z)f$, and $(\Delta\hat{\vec{z}}\cdot\partial_{\vec{z}})f$, when the 12-component vector $\Delta\hat{\mathbf{x}} = [\Delta\hat{A}_1,\Delta\hat{A}_2,\Delta\hat{A}_3,\Delta\hat{\vec{a}}_1,\Delta\hat{\vec{a}}_2,\Delta\hat{\vec{a}}_3,\Delta\hat{Z}_1,\Delta\hat{Z}_2,\Delta\hat{Z}_3,$ $\Delta\hat{\vec{z}}_1,\Delta\hat{\vec{z}}_2,$ $\Delta\hat{\vec{z}}_3]$ is factored out to the right.

Adjustment means that the point to plane correspondence distances should all be zero

$$d_i = 0 + v_i, \quad 1\le i\le n, \tag{130}$$

where $v_i$, $1\le i\le n$ are the residual distances. $v$ is the $n$-dimensional vector of residual distances. In vector notation this becomes

$$f(\hat{A},\hat{\vec{a}},\hat{Z},\hat{\vec{z}}) = v. \tag{131}$$

The determination of transformation parameter estimates and residuals occurs by

$$\begin{aligned} (\hat{A},\hat{\vec{a}},\hat{Z},\hat{\vec{z}}) &= (A_0,\vec{a}_0,Z_0,\vec{z}_0) + (\Delta\hat{A},\Delta\hat{\vec{a}},\Delta\hat{Z},\Delta\hat{\vec{z}}), \\ v &= D_{n\times 12}\,\Delta\hat{\mathbf{x}}^T. \end{aligned} \tag{132}$$

Including the diagonal weight matrix $P = \mathrm{diag}(w_1,\ldots,w_n)$ we determine

$$\Delta\hat{\mathbf{x}}^T = (D^TPD)^{-1}D^TP\Big(-f(A_0,\vec{a}_0,Z_0,\vec{z}_0)\Big). \tag{133}$$

Note that $D^TPD$ is an invertible $(12\times 12)$ matrix, $D^T$ a $(12\times n)$ matrix, $P$ a diagonal $(n\times n)$ matrix, and $\Big(-f(A_0,\vec{a}_0,Z_0,\vec{z}_0)\Big)$ a $(n\times 1)$ column vector.

Regarding initial estimates for the Newton method we refer to Sections 4 and 5 of [8]. Usually it can be assumed that the raw data strips are already aligned within a couple of meters. During the selection of matching point pairs, the value of local point cloud co-planarity, and plane pair normal direction orientation matching of within e.g. five degrees, etc., are considered, and outliers are excluded.

## 9. Computation of Design Matrix Entries

For brevity, we suppress the time argument in this section $X = X(t)$.

### 9.1. Directional Derivatives of Conformal Terrain Points wrt. Calibration Variables

**9.1.1. Directional Derivatives wrt. Scanner Parameters.** Every conformal terrain point $X^e$ (44), depends via a motor transformation on a conformal scanner point (37) $X^s$. A conformal point in vertical distance (range) $\rho$ from the scanner is

$$X_\rho = \rho\,\mathbf{e}_3^s + \frac{1}{2}\rho^2\mathbf{e}_\infty + \mathbf{e}_0. \tag{134}$$

Its partial derivative wrt. $\rho$ (directional derivative wrt. $\mathbf{e}_3^s$) is

$$\frac{\partial}{\partial\rho}X_\rho = \mathbf{e}_3^s + \rho\,\mathbf{e}_\infty. \tag{135}$$

Therefore, the partial derivative of the conformal scanner point is

$$\begin{aligned}\frac{\partial}{\partial\rho}X^s &= \frac{\partial}{\partial\rho}\widetilde{R}_\beta\widetilde{R}_\alpha X_\rho R_\alpha R_\beta = \widetilde{R}_\beta\widetilde{R}_\alpha(\frac{\partial}{\partial\rho}X_\rho)R_\alpha R_\beta \\ &= \widetilde{R}_\beta\widetilde{R}_\alpha(\mathbf{e}_3^s + \rho\,\mathbf{e}_\infty)R_\alpha R_\beta = \widetilde{R}_\beta\widetilde{R}_\alpha\mathbf{e}_3^s R_\alpha R_\beta + \rho\,\mathbf{e}_\infty,\end{aligned} \tag{136}$$

where in the last step we used that the point at infinity $\mathbf{e}_\infty$ is rotation invariant, and $\widehat{x}_0^s = \widetilde{R}_\beta\widetilde{R}_\alpha\mathbf{e}_3^s R_\alpha R_\beta$ represents the three dimensional unit vector in the direction of the original uncorrected scanner point. Because the range $\rho = \Delta\rho + \rho_0(1+\varepsilon_\rho)$ has both an offset $\Delta\rho$ and a scale parameter $\varepsilon_\rho$ for which we have

$$\frac{\partial}{\partial\Delta\rho}\rho = 1, \qquad \frac{\partial}{\partial\varepsilon_\rho}\rho = \rho_0, \tag{137}$$

the partial derivative relevant for the scale paramenter is via the chain rule

$$\frac{\partial}{\partial\varepsilon_\rho}X^s = \rho_0\frac{\partial}{\partial\rho}X^s = \rho_0\widetilde{R}_\beta\widetilde{R}_\alpha\mathbf{e}_3^s R_\alpha R_\beta + \rho_0\rho\,\mathbf{e}_\infty, \tag{138}$$

where $\vec{x}_0^s = \rho_0\widetilde{R}_\beta\widetilde{R}_\alpha\mathbf{e}_3^s R_\alpha R_\beta$ is the uncorrected original scanner point in three dimensions.

For the angle parameter $\alpha$ we obtain the partial derivative (directional derivative wrt. $e_{23}^s$)

$$\begin{aligned}\frac{\partial}{\partial\alpha}X^s &= \frac{\partial}{\partial\alpha}\widetilde{R}_\beta\widetilde{R}_\alpha X_\rho R_\alpha R_\beta = \widetilde{R}_\beta(\frac{\partial}{\partial\alpha}\widetilde{R}_\alpha)X_\rho R_\alpha R_\beta + \widetilde{R}_\beta\widetilde{R}_\alpha X_\rho(\frac{\partial}{\partial\alpha}R_\alpha)R_\beta \\ &= \widetilde{R}_\beta(-\frac{1}{2}e_{23}^s\widetilde{R}_\alpha)X^s R_\alpha R_\beta + \widetilde{R}_\beta\widetilde{R}_\alpha X_\rho(\frac{1}{2}R_\alpha e_{23}^s)R_\beta \\ &= \widetilde{R}_\beta[(\widetilde{R}_\alpha X_\rho R_\alpha) \times e_{23}^s]R_\beta,\end{aligned} \tag{139}$$

using the commutator product $A \times B = \frac{1}{2}(AB - BA)$ in the last line. For the angle $\alpha$ scale parameter $\varepsilon_\alpha$ we have via the chain rule

$$\frac{\partial}{\partial\varepsilon_\alpha}X^s = \alpha_0\widetilde{R}_\beta[(\widetilde{R}_\alpha X_\rho R_\alpha) \times e_{23}^s]R_\beta. \tag{140}$$

Similarly, for the angle parameter $\beta$ we obtain the partial derivative (directional derivative wrt. $e_{31}^s$), and the scaling parameter derivative

$$\frac{\partial}{\partial\beta}X^s = X^s \times e_{31}^s, \qquad \frac{\partial}{\partial\varepsilon_\beta}X^s = \beta_0 X^s \times e_{31}^s. \tag{141}$$

Sandwiching $X^s$ between with the motor of (44) we obtain the parameter derivatives of a terrain point $X^e$ in the following proposition.

**Proposition 13.** *A general conformal terrain point $X^e$ of* (44) *has the following six scan parameter derivatives, specific for each point $X^e$ :*

$$\begin{aligned}
&\frac{\partial}{\partial\rho}X^e = \widetilde{M}_s^e\Big(\widetilde{R}_\beta\widetilde{R}_\alpha\mathbf{e}_3^s R_\alpha R_\beta + \rho\,\mathbf{e}_\infty\Big)M_s^e, \qquad \frac{\partial}{\partial\varepsilon_\rho}X^e = \rho_0\frac{\partial}{\partial\rho}X^e,\\
&\frac{\partial}{\partial\alpha}X^e = \widetilde{M}_s^e\widetilde{R}_\beta\Big[(\widetilde{R}_\alpha X_\rho R_\alpha)\times e_{23}^s\Big]R_\beta M_s^e, \qquad \frac{\partial}{\partial\varepsilon_\alpha}X^e = \alpha_0\frac{\partial}{\partial\alpha}X^e,\\
&\frac{\partial}{\partial\beta}X^e = \widetilde{M}_s^e(X^s\times e_{31}^s)M_s^e, \qquad \frac{\partial}{\partial\varepsilon_\beta}X^e = \beta_0\frac{\partial}{\partial\beta}X^e. \qquad (142)
\end{aligned}$$

**9.1.2. Directional Derivatives wrt. Mounting Calibration Parameters.** The six directional derivatives that we compute now, consistently affect all points in all strips, due to the same motor transformation $M_s^i$ contained in the computation of every terrain point $X^e$. We can rewrite a general conformal terrain point $X^e$ (44) as

$$\begin{aligned}
&X^e = \widetilde{M}_i^e X^i M_i^e, \quad X^i = \widetilde{M}_s^i X^s M_s^i, \quad M_s^i = M(V,\vec{a}^i) = R_s^i(V)T(\vec{a}^i),\\
&V = \varphi_{si}e_{si} = V_1e_{23}^s + V_2e_{31}^s + V_3e_{12}^s. \qquad (143)
\end{aligned}$$

We have by Prop. 11 the (INS to scanner) misalignment directional derivatives

$$\begin{aligned}
B\cdot\partial_V X^e &= \widetilde{M}_i^e(B\cdot\partial_V X^i)M_i^e\\
&\overset{(105)}{=} \widetilde{M}_i^e\Big[(B\cdot e_{si})\widetilde{M}_s^i e_{si}\vec{x}^i_{||}M_s^i \qquad (144)\\
&\quad + \frac{1}{\varphi_{si}}\sin\frac{\varphi_{si}}{2}\{\widetilde{T}(B\times e_{si})e_{si}\vec{x}^i M_s^i + \widetilde{M}_s^i\vec{x}^i(e_{si}\times B)e_{si}T\}\Big]M_i^e,
\end{aligned}$$

with $\vec{x}^i\in\mathbb{R}^3$ the Euclidean part of $X^i$, $T = T(\vec{a}^i)$, and $\vec{x}^i_{||} = (\vec{x}^i\cdot e_{si})e_{si}{}^{-1}$ is computed wrt. the unit rotation plane bivector $e_{si}$. Note that by replacing $B$ above with $e_{23}^s$, $e_{31}^s$, or $e_{12}^s$, respectively, we obtain the partial derivatives wrt. $V_1$, $V_2$, or $V_3$, respectively, i.e. wrt. the three components of the bivector angle $V$.

The directional derivatives wrt. the translation vector (scanner to INS lever arm) $\vec{a}^i$ can be obtained applying (106) to $X^i$

$$\begin{aligned}
\vec{b}\cdot\partial_{\vec{a}^i}X^e &= \widetilde{M}_i^e\vec{b}\cdot\partial_{\vec{a}^i}X^iM_i^e = \widetilde{M}_i^e\vec{b}\cdot\partial_{\vec{a}^i}\widetilde{M}(V,\vec{a}^i)X^sM(V,\vec{a}^i)M_i^e\\
&= \widetilde{M}_i^e\Big[\vec{b}+\vec{b}\cdot\big(\widetilde{R}_s^i(V)\vec{x}^sR_s^i(V)+\vec{a}^i\big)e_\infty\Big]M_i^e. \qquad (145)
\end{aligned}$$

Note that replacing above $\vec{b}$ with $\mathbf{e}_1^i$, $\mathbf{e}_2^i$, or $\mathbf{e}_3^i$, respectively, yields the three partial derivatives with respect to the three coordinates of $\vec{a}^i$.

**9.1.3. Directional Derivatives for GNSS position calibration.** According to (44), a general terrain point can be represented as

$$X^e = \widetilde{T}(\vec{g}^e)\widetilde{R}^e_n X^n R^e_n T(\vec{g}^e), \quad X^n = \widetilde{R}^n_i(F) X^i R^n_i(F),$$
$$X^i = \widetilde{M}^i_s X^s M^i_s, \quad F = \phi_{in} e_{in} = F_1 e^i_{23} + F_2 e^i_{31} + F_3 e^i_{12}. \tag{146}$$

We want to compute the directional derivatives

$$B \cdot \partial_F X^e = B \cdot \partial_F \widetilde{T}(\vec{g}^e)\widetilde{R}^e_n X^n R^e_n T(\vec{g}^e) = \widetilde{T}(\vec{g}^e)\widetilde{R}^e_n B \cdot \partial_F X^n R^e_n T(\vec{g}^e), \tag{147}$$

hence we first compute by Prop. 11, setting $T(\vec{a}) = 1$,

$$\begin{aligned} B \cdot \partial_F X^n &= B \cdot \partial_F \widetilde{R}^n_i(F) X^i R^n_i(F) \\ &\overset{(105)}{=} (B \cdot e_{in})\widetilde{R}^n_i e_{in} \vec{x}^i_{||} R^n_i \\ &\quad + \frac{1}{\phi_{in}} \sin\frac{\phi_{in}}{2}\{(B \times e_{in}) e_{in} \vec{x}^i R^n_i + \widetilde{R}^n_i \vec{x}^i (e_{in} \times B) e_{in}\}, \end{aligned} \tag{148}$$

and finally obtain

$$\begin{aligned} B \cdot \partial_F X^e = \widetilde{T}(\vec{g}^e)\widetilde{R}^e_n \Big[ & (B \cdot e_{in})\widetilde{R}^n_i e_{in} \vec{x}^i_{||} R^n_i \\ & + \frac{1}{\phi_{in}} \sin\frac{\phi_{in}}{2}\{(B \times e_{in}) e_{in} \vec{x}^i R^n_i + \widetilde{R}^n_i \vec{x}^i (e_{in} \times B) e_{in}\}\Big] R^e_n T(\vec{g}^e), \end{aligned} \tag{149}$$

where $\vec{x}^i \in \mathbb{R}^3$ is the Eulidean part of $X^i$, and $\vec{x}^i_{||} = (\vec{x}^i \cdot e_{in}) e_{in}{}^{-1}$ the part parallel to the rotation plane bivector $e_{in}$. Note that by replacing $B$ above with $e^i_{23}$, $e^i_{31}$, or $e^i_{12}$, respectively, we obtain the partial derivatives wrt. $F_1$, $F_2$, or $F_3$, respectively, i.e. wrt. the three components of the bivector angle $F$.

Next we compute the directional derivatives

$$\vec{b} \cdot \partial_{\vec{g}^e} X^e = \vec{b} \cdot \partial_{\vec{g}^e} \widetilde{T}(\vec{g}^e) Y^e T(\vec{g}^e), \qquad Y^e = \widetilde{R}^e_n X^n R^e_n. \tag{150}$$

We obtain with (106), setting $R = 1$,

$$\vec{b} \cdot \partial_{\vec{g}^e} X^e = \vec{b} \cdot \partial_{\vec{g}^e} \widetilde{T}(\vec{g}^e) Y^e T(\vec{g}^e) = \vec{b} + \vec{b} \cdot (\vec{y}^e + \vec{g}^e) e_\infty, \tag{151}$$

where $\vec{y}^e$ is the Euclidean part of $Y^e = \widetilde{R}^e_n X^n R^e_n$. Note that replacing above $\vec{b}$ with $\mathbf{e}^e_1$, $\mathbf{e}^e_2$, or $\mathbf{e}^e_3$, respectively, yields the three partial derivatives with respect to the three coordinates of $\vec{g}^e$.

### 9.2. Directional Derivatives of Corresponding Point to Plane Distances

Differentiating the cost function (52), and for the computation of the design matrix (in Section 8) the entries of the vector of correspondence distances (128), requires differentiation of distances $d_{jk}$ (see (51)) between a plane $Plane^*_j$ (see (49), dual form (50)) identified in a strip $j$ and a point $Q_k$ (see (20)) identified as corresponding to the plane in an overlapping strip $k$. In photogrammetric evaluation of Lidar data, the planes will usually be determined via local covariance matrix computation [1]. As in Section 8, for demonstration purposes, we simply assume the initial conformal plane in strip $j$ to be determined by the outer product of three neighboring points $P_{j10}, P_{j20}, P_{j30}$, of strip $j$ in the plane, linear in each of the three points

$$Plane_j^* = P_{j1} \wedge P_{j2} \wedge P_{j3} \wedge \mathbf{e}_\infty I_5^{-1}. \tag{152}$$

Because of this linearity of $Plane_j^*$ in its constituent points, the application of directional (and partial) derivatives wrt. the correction parameters apply point wise, e.g., partial derivatives wrt. scanner ranges for each constituent point are simply computed as

$$\begin{aligned}
\frac{\partial}{\partial \rho_1} Plane_j^* &= (\frac{\partial}{\partial \rho_1} P_{j1}) \wedge P_{j2} \wedge P_{j3} \wedge \mathbf{e}_\infty I_5^{-1}, \\
\frac{\partial}{\partial \rho_2} Plane_j^* &= P_{j1} \wedge (\frac{\partial}{\partial \rho_2} P_{j2}) \wedge P_{j3} \wedge \mathbf{e}_\infty I_5^{-1}, \\
\frac{\partial}{\partial \rho_3} Plane_j^* &= P_{j1} \wedge P_{j2} \wedge (\frac{\partial}{\partial \rho_3} P_{j3}) \wedge \mathbf{e}_\infty I_5^{-1}.
\end{aligned} \tag{153}$$

Because by (51) the plane $Plane_j^*$ to point $Q_k$ correspondence distance $d_{jk}$ is the inner product of the two, linear in each factor, we can compute the directional (and partial) derivatives of $d_{jk}$ using the product rule, from the directional (and partial) derivatives of conformal terrain points established in Section 9.1. For example, the partial derivatives wrt. the scanner ranges $\rho_1$ of $P_{j1}$ and $\rho_k$ of $Q_k$ are

$$\begin{aligned}
\frac{\partial}{\partial \rho_1} d_{jk} &= \frac{\partial}{\partial \rho_1} Plane_j^* \cdot Q_k = (\frac{\partial}{\partial \rho_1} Plane_j^*) \cdot Q_k, \\
\frac{\partial}{\partial \rho_k} d_{jk} &= \frac{\partial}{\partial \rho_k} Plane_j^* \cdot Q_k = Plane_j^* \cdot (\frac{\partial}{\partial \rho_k} Q_k).
\end{aligned} \tag{154}$$

We can therefore compute all directional (and partial) derivatives of the cost function (52) and of the distance vector (128) wrt. the $6+6+12n$ correction paramenters, six for the scanner calibration, six for the mounting calibration, and 12 each for the GNSS trajectory correction of each of the $n$ ALS strips in a data set. In this way we obtain the design matrix of Section 8.

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