Московский государственный университет имени М.В.Ломоносова

M.V.Lomonosov Moscow State University (MSU)

Address: MSU, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, 1 Leninskiye Gory, Main Building
Phone: (495) 939-12-44
Fax: (495) 939-20-90
WWW: http://www.math.msu.su/
E-mail: mmmf@mech.math.msu.su

Faculty of Mechanics and Mathematics

GRIGORIY MICHAILOVICH SOBKO

LIMIT THEOREMS FOR RANDOM WALKS ON DIFFERENTIABLE MANIFOLDS

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Science advisor: Professor V.N. Tutubalin, Ph, D.

Official opponents: V.M Maximov, Doctor of Science

V.V. Sazonov, Doctor of Science

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Classical results of Probability Theory related to summation of random variables provide description of the family of limit laws for distribution of sums of uniformly distributed infinitesimal independent terms and establish conditions for convergence of the sums to the laws of the family. The circle of problems concerned with the probability distributions for sums of “small “ random variables was expanding in the two major directions: first, it had been posed a more general problem to move from discrete random walks to Markov processes with continuous time (A.N. Kolmogorov [1], A. Y. Khintchin {2], A.V. Skorokhod [3]); second, the study shifted to the situation in which the random variables are taking their values on the sets of more general mathematical nature than *n*-dimensional Euclidean space, for example, on differentiable manifolds, Lie groups (A. N. Kolmogorov [4], K. Ito [5], G.A. Hunt [6], D. When [7].

In attempts to apply the theory for summation of asymptotically uniformly small independent random variables to Lie groups there appeared difficulties caused by a non-commutative (in general) group operation as well as by non-compactness of a group topological space. This forced to limit the study either by the case of commutative distributions ([6], [7](the more complete results were obtained for compact groups [8]), or assume (in the non-commutative case) additional restrictions that are not present in conditions of the corresponding classical limit theorems [7].

In this work we consider sequences of random walks (i.e. random processes with discrete time) of both Markov and non-Markov character on differentiable manifolds, and the limit transition from the walks to Markov random processes with continuous time so that the “steps” of walks are asymptotically uniformly small, and the number of “steps” goes to infinity. The mentioned Markov processes (diffusion and stochastically continuous without the second type of discontinuities) are introduces “constructively”, which provides the information about the properties of transition probabilities, necessary for proofs of the corresponding limit theorems. This way, we formulate in our dissertation conditions of convergence similar to the classical results.

Limit theorems for probability distributions of products of asymptotically uniformly small random elements on Lie groups are proved as a consequences of the corresponding statements for manifolds. We assume no conditions related to the commutative property of measure convolutions.

We can pose more general problem about the limit distributions of products of random elements on Lie groups, if we replace the condition of the asymptotic uniform smallness of multiplying elements by the condition of asymptotic uniform "compactness" of these elements. The last means that the distribution of each of multiplying elements is concentrating asymptotically on a certain compact subgroup $\mathcal{U}$ of the original group $\mathcal{G}$ .We solve this problem (about the "compact averaging") for rather general class of Lie groups.

In conclusion, we demonstrate how the obtained theoretical results can be used in some applications in Physics.

The dissertation consists of Introduction and four chapters.

In the Chapter I we consider a parabolic differential operator (in a canonical atlas $\mathcal{F}$ (see [9]):

$$L^{\partial} \equiv L^{\partial}_{s,x} = a^{ij}(s,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(s,x)\frac{\partial}{\partial x^i} + c(xs,x) + \frac{\partial}{\partial s}$$

on a differentiable manifold $\mathcal{M}$ of class $C^k \ (k \geq 2)$, under condition that matrices $\left[a^{ij}(s,x)\right]$ are strictly positively definite on the set $H_T = [0,T) \times \mathcal{M}$ $\left(0 < T < \infty\right)$. This allows define a Riemannian metric

$d_a r^2 = a_{ij}(x) dx^i dx^j$ and a measure $d_a x = \sqrt{a}\ dx^1 \cdots dx^m$,

where $a_{ij}(x) = a_{ij}(s_1, x), \left[a_{ij}(s,x)\right] = \left[a^{ij}(s,x)\right]^{-1}; \ a(x) = \det\left(\left[a_{ij}(s_1,x)\right]\right)$,

s_1 is some point from $[0,T]$.

Let $\mathcal{M}_\infty$ be a one-point compactification of the space $\mathcal{M}$, $\mathcal{C}$ a Banach space of continuous functions on $\mathcal{M}_\infty$ with the norm $\|f\| = \sup_{x\in\mathcal{M}} |f(x)|$ and

$\mathcal{B}$ a σ-algebra of Borel sets on $\mathcal{M}$.

A fundamental solution to the equation $L^\partial U = 0$ is called such a function

$u(s,x;t,y)$, where $0 \le s < t < T;\ x, y \in \mathcal{M}$, that the function

$U(s,x) = \int_{\mathcal{M}} u(s,x;t,y) f(y) d_a y$ satisfies the conditions

$$
\begin{aligned}
&L_{s,x}{}^{\partial} U(s,x) = 0 \quad \text{(uniformly on } \mathcal{M}\text{)},\\
&\lim_{s \nearrow t} U(s,x) = f(x),\\
&U(s,x) \in \mathcal{C},\ \lim_{x\to\infty} U(s,x) = f(\infty) \ \text{ for any } s \in [0,t)
\end{aligned}
$$

Under the assumptions for the coefficients of the equation of being Holder and locally bounded [9], we prove the existence and uniqueness theorems for the fundamental solution and establish some properties of smoothness for functions

$U(s,x) = \int_{\mathcal{M}} u(s,x;t,y) f(y) d_a y$.

In a similar way we define and construct the fundamental solution

$v(s,x;t,y)$ of the integro-differential equation $LV = 0$

where

$$
L \equiv A_{s,x}{}^{\partial} + A_{s,x}{}^{\Sigma} + \frac{\partial}{\partial s} \ .
$$

In a canonical atlas $\mathcal{F}_\delta = \{(S_x(\delta), h_x)\}$ $(0 < \delta \le 1)$,

$$
\begin{aligned}
&A_{s,x}{}^{\partial} = a^{ij}(s,x)\frac{\partial^2}{\partial x^i \partial x^j} + b^i(s,x)\frac{\partial}{\partial x^i}\ ,\\
&A_{s,x}{}^{\Sigma} f(x) = \int \left\{ f(y) - f(x) - \frac{\rho(x,y)}{1+\varphi(x,y)}(y^i - x^i)\frac{\partial f(x)}{\partial x^i} \right\} \frac{1+\varphi(x,y)}{\varphi(x,y)} G(s,x;dy)
\end{aligned}
$$

Here: $G(s,x;\Gamma)$ is a bounded measure on $\mathcal{B}$ for each $(s,x)\in H_T$ and is a Borel function for each $\Gamma\in\mathcal{B}$ on H_T;

$\rho(x,y)\in C^2$ in y for each $x\in\mathcal{M},\ 0\le\rho(x,y)\le 1$ and $\rho(x,y)=0$ as $y\in\mathcal{M}-S_x(1)$ and $\rho(x,y)=1$ as $y\in S_x\left(\frac{1}{2}\right)$;

$\varphi(x,y)\in C^2$ in y for each $x\in\mathcal{M},\ 0\le\varphi(x,y),\ \varphi(x,y)=\sum_{i=1}^{m}\left(y^i-x^i\right)^2$ as $y\in S_x(1)$ and $\varphi(x,y)\ge 1$ as $y\in\mathcal{M}-S_x(1)$; $\dfrac{\varphi(x,y)}{1+\varphi(x,y)}\in C$ in y for each $x\in\mathcal{M}$.

The assumed properties of measure $G(s,x;\Gamma)$ are similar to those of the function which defines jumps of a process in the book of A.V. Skorokhod [3].

The construction of fundamental solutions is done by applying the method of parametrix, and the proof of uniqueness is based on applying the maximum principle to the operators L^{∂} and L given the "boundary" conditions:

$$\lim_{s\nearrow t}V(s,x)=f(x) \text{ and } \lim_{x\to\infty}V(s,x)=f(\infty) \text{ for } f\in C.$$

The idea to use differential equations with partial derivatives to solve probability problems related to random walks ("the diffusion problems") was systematically developed in the monograph of A.J. Khintchin [2]. As shown in the Chapter III, the method of A.J. Khintchin can be generalized to the case of convergence of sequences of Markov chains to a non-homogeneous diffusion process on a manifold with some modification of the proofs.

The major difference from [2] is that in [2} the existence of diffusion processes with the properties that guarantee the proof of the corresponding limit theorems is postulated while in this work the proof of existence of the processes with the desirable properties follows directly from the results of Chapters I and Chapter II. By not using the method of "upper" and "lower" functions, we were able to prove the theorems under weaker assumptions imposed on convergence of the infinitesimal characteristics.

Some problems related to random walks on Lie groups (for example, about the compact averaging) lead to the consideration of non-Markov random walks.

In this regard, we expand the limit theorem for sums of dependent random variables to the case of arbitrary system of dependent random points

$\{\xi_\Delta(k);\ k=0,1,\dots,n\}$ on $\mathcal{M}$ which can be interpreted as a trajectory of a walking particle that belongs at the moments of "registration" t_k to the measurable space $(\mathcal{M}_k,\mathcal{B}_k)\equiv(\mathcal{M},\mathcal{B})$.

We associate the sequence of random walks $\{\xi_\Delta(k)\}$ with the sequence of specifically constructed non-homogeneous diffusion processes $\tilde{\xi}_\Delta(t)$ on $\mathcal{M}$ and derive the sufficient conditions in terms of convergence of the infinitesimal characteristics:

$$b_\Delta{}^i(t_{k+1};x_k,\dots,x_0)=\frac{1}{\Delta t_k}\int\limits_{S_k(\delta)}\left(y^i-x^i\right)P_k{}^\Delta(dy\mid x_k,\dots,x_0),$$

$$a_\Delta{}^{ij}(t_{k+1};x_k,\dots,x_0)=\frac{1}{2\Delta t_k}\int\limits_{S_k(\delta)}\left(y^i-x^i\right)\left(y^j-x^j\right)P_k{}^\Delta(dy\mid x_k,\dots,x_0)\ ,$$

$$\eta_\Delta(t_{k+1};x_k,\dots,x_0)=\int\limits_{\mathcal{M}-S_k(\delta)}P_k{}^\Delta(dy\mid x_k,\dots,x_0)$$

under which $\{\xi_\Delta(k)\}$ allows approximation by the sequence of indicated random processes $\tilde{\xi}_\Delta(t)$ in the sense of weak convergence (with respect to C) of the corresponding transition probabilities.

This theorem implies directly the statement about convergence of a sequence of Markov chains to the diffusion process on $\mathcal{M}$ with the generating operator $A^{\partial}{}_{s,x}$ that satisfies conditions of Chapter I.

Further, we consider a wider class of Markov processes on $\mathcal{M}$, which one can get by "imposing" infinitely many jumps on the continuous component. This is reflected in a specific form of the infinitesimal operators of these processes on a set of smooth finite function:

$$A\equiv A^{\partial}_{s,x}+A^{\Sigma}_{s,x}$$

We demonstrate here as an example the formulation of the corresponding limit theorem.

For a sequence of Markov chains on $\mathcal{M}$, we introduce the infinitesimal characteristics of the random walks in a canonical atlas $\mathcal{F}_\delta = \{(S_x(\delta), h_x)\}$ $(0 < \delta \le 1)$:

$$a_\Delta{}^{ij}(t_k, x) = \frac{1}{2\Delta t_k} \int_{S_k(\delta)} \left(y^i - x^i\right)\left(y^j - x^j\right) P_{k-1}{}^\Delta(dy \mid x),$$

$$b_\Delta{}^i(t_k, x) = \frac{1}{\Delta t_k} \int_{S_k(\delta)} \left(y^i - x^i\right) \frac{\rho(x, y)}{1 + \varphi(x, y)} P_{k-1}{}^\Delta(dy \mid x),$$

$$G_\Delta(t_k, x, \Gamma) = \frac{1}{\Delta t_k} \int_{\mathcal{M} - S_k(\delta)} \frac{\varphi(x, y)}{1 + \varphi(x, y)} P_{k-1}{}^\Delta(dy \mid x),$$

where $\rho(x, y), \varphi(x, y)$ are function defined for the operator $A_{s,x}^\Sigma$.

Let the measure $G(s, x; \Gamma)$ have the properties:

$$\lim_{x \to \infty} G(s, x; \Gamma) = 0 \quad \text{for all } s \in [0, T) \text{ and any compact } \Gamma \in \mathcal{B};$$

$$\lim_{k \to \infty} \left\{ \sup_{(s,x) \in H_T} G(s, x; \Gamma_k \right\} = 0 \text{ for any sequence of sets } \Gamma_k \in \mathcal{B} \text{ such that} \qquad (1)$$

$$\Gamma_k \supseteq \Gamma_{k+1} \text{ and } \bigcap_{k=1}^{\infty} \Gamma_k = \varnothing .$$

Assume the following conditions:

A. $\lim\limits_{|\Delta| \to 0} \sum\limits_{k=1}^{n} \max\limits_{i,j} \left\{ \left\| b_\Delta^i(t_k, x) - b^i(t_k, x) \right\| + \left\| a_\Delta^{ij}(t_k, x) - a^{ij}(t_k, x) \right\| \right\} \Delta t_k = 0.$

B. $\lim\limits_{|\Delta| \to 0} \sum\limits_{k=1}^{n} \left\| \int\limits_{\mathcal{M} - \{x\}} f(y) \left[G_\Delta(t_k, x, dy) - G(t_k, x, dy) \right] \right\| \Delta t_k = 0,$

where $\|F(\tau, \cdot)\| = \sup\limits_{x \in \mathcal{M}} |F(\tau, x)|$, $|\Delta| = \max\limits_{1 \le k \le n} \Delta t_k$.

Theorem. If the conditions of Chapters I and II and the condition (1) hold true for the operator $L \equiv A^{\partial}_{s,x} + A^{\Sigma}_{s,x} + \frac{\partial}{\partial s}$, then, under the assumptions A and B, for any function $f \in C$,

$$\lim_{|\Delta|\to 0} \left\| \int_{\mathcal{M}} f(y)\left[P_{\Delta}(t_k, x;, t, dy) - P(t_k, x;, t, dy)\right] \right\| = 0,$$

where $P_{\Delta}(t_k, x;, t, dy)$ is a transition probability of a random walk $\{\xi_{\Delta}(k)\}$
and
$P(t_k, x;, t, \Gamma) = \int_{\mathcal{M}} v(t_k, x;, t, y)\, d_a y$ is a transition probabiity of the Markov random process $\xi(t)$
with the generating operator $A \equiv A^{\partial}_{s,x} + A^{\Sigma}_{s,x}$,
$v(s, x;, t, y)$ is a fundamental solution of the equation $LV = 0$.

In Chapter IV, we consider an analog of the Central Limit Theorem on Lie groups.

Convolutions of probability measures correspond to probability distributions of random walks given by products of independent random group elements:

$$\xi_n(k+1) = \xi_n(k) \cdot g_{n,k+1}, \xi_n(0) = e$$

$P\left(\xi_n(k+1) \in \Gamma \mid \xi_n(k) = g\right) = \mu(g^{-1}\Gamma)$, where μ_{nk} belongs to some infinitesimal triangular system of Borel measures on $\mathcal{G} : \{\mu_{kn} \mid k = 1, \ldots, k_n; n = 1, 2, \ldots\}$.

Infinitesimal characteristics are written in canonical coordinates related to the Lie algebra basis $\{X_1, \ldots, X_m\}$:

$$b^i_n(t_{nk}) = \frac{1}{\Delta t_{nk}} \int_{\mathcal{N}_e(\delta)} x^i(g)\mu_{nk}(dg),$$

$$a^{ij}_n(t_{nk}) = \frac{1}{2\Delta t_{nk}} \int_{\mathcal{N}_e(\delta)} x^i(g)x^j(g)\mu_{nk}(dg),$$

$$\eta(t_{nk}) = \mu_{nk}(\mathcal{G} \setminus \mathcal{N}_e(\delta)), \text{ where } \mathcal{N}_e(\delta) = \exp(S_e(\delta)),\ S_e(\delta) = \left\{\lambda \in \mathbb{R}^m : |\lambda| < \delta\right\}$$

It follows from the results of Chapter III, that under certain conditions for

$b^i_n(t_{nk}),\ a^{ij}_n(t_{nk}), \eta_n(t_{nk})$, there exists a unique non-homogeneous semigroup of probabilistic operators

$\mathcal{T}(s,t)$, continuous for $0 \le s \le t < T$, such that $\mathcal{T}_n(s,t) \to \mathcal{T}(s,t)$ as $n \to \infty$,

in strong operator topology on C.

Here $\mathcal{T}_n(s,t)f(g) = \int_{\mathcal{G}} f(h)P_n(t_{nk}, g; t_{nl}, dh),$ if $t_{nk} \le s < t_{nk+1,} \, t_{nl} \le t < t_{nl+1};$

$P_n(t_{nk}, g; t_{nl}, \Gamma) = \mu_{n,k+1} * \cdots * \mu_{n,l}(g^{-1}\Gamma) \quad (0 \le k \le l \le n)$

Too burdening condition "5.4" of D. When [7] is a simple consequence of the Theorem

stated above.

A more general problem about limit behavior of products of a series of independent but, in general, not "small", random elements on Lie groups can be posed as follows.

Let $\left\{ g_p^{(n)} \mid p = 1, \ldots, p_n; n = 1, 2, \cdots \right\}$ be a triangular system of independent random elements of a group $\mathcal{G}$ such that probability distributions $\pi_p^{(n)}$ of elements $g_p^{(n)}$ satisfy the following conditions: for any neighborhood $\mathcal{N}_e$ of the identity element e of group $\mathcal{G}$,

$$\sup_{1 \le p \le p_n} \pi_p^{(n)}(\mathcal{G} \setminus \mathcal{U}\mathcal{N}_e) \to 0 \text{ as } n \to \infty, \tag{2}$$

where $\mathcal{U}$ is a fixed compact subgroup of the group $\mathcal{G}$.

When $\mathcal{U} = \{e\}$ where e is a group identity element, we have the property of infinitesimality mentioned above; condition (2) we call the property of "asymptotic

uniform compactness" of triangular systems $\left\{ g_p^{(n)} \right\}$ or $\left\{ \pi_p^{(n)} \right\}$.

We can define as above a sequence of random walks on group $\mathcal{G}$, if we set

$$x_0^{(n)} = e, \; x_{p+1}^{(n)} = x_p^{(n)} \cdot g_{p+1}^{(n)}$$

and consider the question of convergence of the sequence of random walks $\left\{ x_p^{(n)} \right\}$ to a random process on $\mathcal{G}$ with continuous time. We solve this problem in Chapter IV for a wide class of Lie groups which admit "polar decomposition":

$\mathcal{G} = \mathcal{R} \cdot \mathcal{U},$ where $\mathcal{R}$ is C^∞-submanifold in $\mathcal{G}$ such that an individual decomposition $g = \bar{r}(g) \cdot \bar{u}(g)$ with $\bar{r}(g) \in \mathcal{R}$ and $\bar{u}(g) \in \mathcal{U},$ is unique for each $g \in \mathcal{G}$.

Let the probability distribution $\lambda_p^{(n)}$ of an element $\bar{u}(g_p^{(n)})$ is such that

$$\sup_{1\le p\le p_n} \left\| \mathcal{T}_{\lambda_p^{(n)}}(\Lambda) \right\| = \tau(\Lambda) < 1 \text{ if } \Lambda \neq 0 \tag{3}$$

$$\mathcal{T}_{\lambda_p^{(n)}}(\Lambda) = \int_{\mathcal{G}} \mathcal{T}_u(\Lambda)\lambda_p^{(n)}(du),$$

where

$\{\mathcal{T}_u(\Lambda)\}$ is the collection of all irreducible unitary representations of the group $\mathcal{U}$ (Λ=0,1,2,…)

Then, the probability distribution $\nu_{p(n)}^{(n)}$ of the element $u_{p(n)}^{(n)}$ converges weakly, as $p(n) \to \infty$, to the Haar measure ν on $\mathcal{U}$ for any initial distribution $\nu_0^{(n)}$ of the element $u_0^{(n)}$.

This answers the question about asymptotic probability distribution of the "angular coordinate" $\bar{u}(x_{p_n}^{(n)})$ of the random walk $x_{p_n}^{(n)}$.

To obtain an asymptotic distribution of the "radial coordinate" $\bar{r}(x_{p_n}^{(n)})$ we apply a certain analog of the averaging principle suggested by N.N. Bogolyubov for a system of ordinary differential equations with a small parameter. By separating the coordinates which are not in the compact subgroup $\mathcal{U}$, we conclude, due to the assumption of the uniform asymptotic compactness, that there is a ''small shift" along these coordinates. For the "large" number of steps the coordinates on the compact subgroup will be "averaged". If meanwhile the trajectory would not move "very far" along the non-compact coordinates ("slow" diffusion and trend), then we can expect that in the limit we obtain some diffusion process on $\mathcal{G}$, "averaged" over the compact coordinates.

Let π_k^Δ be the probability distribution of the element $g_k^\Delta = g_{p_{k-1}(n)+1} \cdots g_{p_k(n)}$,
where $0 = p_0(n) < p_1(n) < \cdots < p_{k_n}(n) = p_n$ and $\Delta = \Delta(0 = t_0 < t_1 < \cdots < t_{k_n} = t)$,
$p_{k+1}(n) - p_k(n) \to \infty$ as $n \to \infty$.

The polar decomposition $g_k^\Delta = \bar{r}(g_k^\Delta)\cdot\bar{u}(g_k^\Delta)$ induces probability distributions μ_k^Δ and ν_k^Δ of elements $\bar{r}(g_k^\Delta)$ and $\bar{u}(g_k^\Delta)$, respectively.

By denoting $x_\Delta(k) = x^{(n)}_{p_k(n)}$, $x_\Delta(k) = r_\Delta(k) \cdot u_\Delta(k)$, where $r_\Delta(k) = \overline{r}(x_\Delta(k))$, $u_\Delta(k) = \overline{u}(x_\Delta(k))$,

We obtain the recurrent relationships: $r_\Delta(k+1) = r_\Delta(k) \cdot \overline{r}(u_\Delta(k) \cdot g^\Delta_{k+1})$, $r_\Delta(0) = e$,

which define the sequence of non-Markov random walks $\{r_\Delta(k)\}$ on $\mathcal{R}$
under the corresponding assumptions about measures $\pi^{(n)}_p$, $\lambda^{(n)}_p$, μ^Δ_k.

Further, it is shown that the random elements $u_\Delta(k)$ and $r_\Delta(k)$ are asymptotically independent as $|\Delta| \to 0$.

Limit theorem about convergence of compositions of infinitesimal and asymptotically compact measures on Lie groups lead to processes with the transition probabilities that have invariance with respect to left shifts by group elements (independence of increments) and, respectively, possess an additional property of invariance with respect to right shifts by elements of the fixed compact subgroup (compact invariance).

By expressing these properties in terms of invariance of the corresponding generating differential operators on $\mathcal{G}$, we establish the connection between the processes obtained as a result of compact averaging applied to sequences of asymptotically compact random walks, and the compactly invariant processes with independent increments on Lie groups.

Usefulness of the mathematical constructions described above is justified by the fact that one can use them to solve certain problems in Physics and technical applications. Such problems appear, in particular, in study of mathematical models of propagation of radio waves in waveguides with random irregularities, and as it is shown in Chapter IV, the principle of compact averaging can be used in calculations of waveguide parameters.

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