The Chessboard

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Abstract

We introduce compact subsets in the plane and in $\mathbb{R}^3$, which we call *Polyorthogon* and *Polycuboid*, respectively. We ask whether we can represent these sets by equal bricks or mirrored bricks.

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1 Introduction

We investigate whether a usual chessboard can be represented by equal bricks formed by the squares of which it is compounded. Of course, the number of the bricks have to be a devisor of 64. This question can be generalized.

Definition 1. We define a *Polyorthogon* as a subset of $\mathbb{R}^2$ such that it is homeomorphic to a circle area $\{x^2 + y^2 \leq 1 \mid (x,y) \in \mathbb{R}^2\}$ and it is the union of a finite number of rectangles. We define a *Polycuboid* as a subset of $\mathbb{R}^3$ such that it is homeomorphic to a ball $\{x^2 + y^2 + z^2 \leq 1 \mid (x,y,z) \in \mathbb{R}^3\}$ and it is the union of a finite number of cuboids. Two rectangles in a polyorthogon and two cuboids in a polycuboid intersect exclusively on their boundaries or they are disjunct.

Remark 1. In a polyorthogon there are exactly two sets of parallel rectangle sides. The second set is perpendicular to the first set. In a polycuboid there are exactly three sets of parallel cuboid sides. The second set is perpendicular to the first set. The third set is perpendicular both to the second and the first set.

A *Polyomino* and a *Polycube* are well-known. See [1].

Definition 2. Let X and Y both be subsets of $\mathbb{R}^2$ or both be subsets of $\mathbb{R}^3$. We say that X is a *mirror image* of Y if and only if there is a symmetry axis or a symmetry plane which mirrors X in Y.

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2 The Chessboard

Definition 3. We write $E \cong F$ if and only if E and F both are either polyorthogons or polyominoes or polycuboids or polycubes. and they have the same form and size.

Please see the picture 'Figure 1' below. The polyorthogon G is one big square, while H consists of 64 small squares, which is a polyorthogon of the same form and size. We call H a *chessboard*. We determine that the sidelength of the small squares is 1. It holds $G \cong H$.

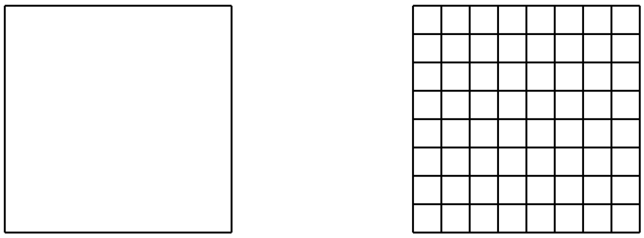

Figure 1:
On the left hand side, we show two polyorthogons G and H.
Both are also polyominoes.

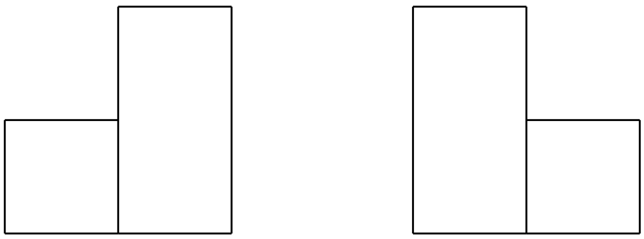

Figure 2:
On the left hand side, we see a polyorthogon and its mirror image.

Definition 4. We say that '*B represents A*' or 'B is a *representation* of A' if and only if A is a polyorthogon or a polycuboid or a polyomino or a polycube, J is a natural number, and B is the union

$$B = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_{J-1} \cup B_J \tag{1}$$

and B equals A in form and size. The sets B_i either all are homeomorphic to a circle area $\{x^2+y^2 \leq 1 \mid x,y \in \mathbb{R}\}$ or all are homeomorphic to a ball $\{x^2+y^2+z^2 \leq 1 \mid x,y,z \in \mathbb{R}\}$. B_i is called a *brick*. Two bricks intersect only on their boundaries or they are disjunct.

Remark 2. Note that if A is a polyomino which is not homeomorphic to a circle area, or a polycube which is not homeomorphic to a ball, the definition is not trivial.

Let X and Y both are polyorthogons or polycuboids or polyominoes or polycubes. We determine that the expression 'X is equal to Y' means that we can move and revolve Y such that Y is identical to X.

Let P and Q both are polyorthogons or polycuboids or polyominoes or polycubes. We say that 'P and Q have the same *shape*' if and only if they are equal.

Proposition 1. We presume that P either is any polyorthogon or a polyomino. There is a set $D_{Set} \subset \mathbb{R}^2$ such that D_{Set} represents P, and

$$D_{Set} \;=\; W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{S-1} \cup W_S \tag{2}$$

where W_i is a finite nonempty set of bricks for $i \;=\; 1,2,3,\ldots S-1,S$.
The sum of the cardinalities of the W_i's is called K, i.e.

$$K \;=\; \sum_{i=1}^{S} \text{cardinality}(W_i) \tag{3}$$

i.e. P is represented by K bricks. If $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that X is not equal to Y, while if $i = j$ then X is equal to Y.
We say that 'the polyorthogon P or the polyomino P is *placeable* by K bricks of S shapes', or 'P is *placeable* by K bricks of S shapes' in brief. If $S = 1$, we say 'P is *placeable* by K bricks'.

Proposition 2. We presume that P either is a polyorthogon or a polyomino. There is a set $E_{Set} \subset \mathbb{R}^2$ such that E_{Set} represents P, and

$$E_{Set} \;=\; W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{S-1} \cup W_S \tag{4}$$

where W_i is a finite nonempty set of bricks for $i \;=\; 1,2,3,\ldots S-1,S$. If $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that neither X is equal to Y nor X is a mirror image of Y. If $i = j$ either X is equal to Y or X is a mirror image of Y. Also, equation (3) holds.
We say that 'the polyorthogon P or the polyomino P is *mirror-placeable* by K bricks of S shapes', or 'P is *mirror-placeable* by K bricks of S shapes' in brief. If $S = 1$, we say 'P is *mirror-placeable* by K bricks'.

Proposition 3. We presume that Q is a polycuboid or a polycube. There is a set $F_{Set} \subset \mathbb{R}^3$ such that F_{Set} represents Q, and

$$F_{Set} \;=\; W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{S-1} \cup W_S \tag{5}$$

where W_i is a finite nonempty set of three dimensional bricks for $i \;=\; 1,2,3,\ldots S-1,S$. If $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that X is not equal to Y, while X is equal to Y if $i = j$. Further, equation (3) holds.
We say that 'the polycuboid Q or the polycube Q is *sectional* by K bricks of S shapes', or 'Q is *sectional* by K bricks of S shapes' in brief. If $S = 1$, we say 'Q is *sectional* by K bricks'.

A similar proposition can be stated for a polycuboid R or a polycube R, if we define the expression 'R is *mirror-sectional* by K bricks of S shapes'.

Proof. The existence of the sets D_{Set}, E_{Set}, and F_{Set} follows from the definitions of a polyomino and a polyorthogon and a polycube and a polycuboid, respectively. □

Remark 3. The number $S = 1$ is always possible.

Proposition 4. If a polyorthogon or a polyomino or a polycuboid or a polycube, respectively, is placeable or sectional by K bricks of S shapes, it is also mirror-placeable or mirror-sectional, respectively, by K bricks of S shapes.

Proposition 5. If a polyorthogon or a polyomino or a polycuboid or a polycube is mirror-placeable or mirror-sectional, respectively, by K bricks of S shapes, it is also placeable or sectional, respectively, by K bricks of $2 \cdot S$ shapes.

Proof. Both popositions are trivial. □

Proposition 6. Let P be a polyorthogon or a polycuboid or a polyomino or a polycube, respectively. Then it is placeable or sectional, respectively, by some bricks of some shapes.

Proof. This follows directly from the definitions. □

Proposition 7. For every natural number K exists a polyorthogon and a polyomino and a polycuboid and a polycube, respectively, that is placeable or sectional, respectively, by K bricks of a single shape.

Proof. We consider the most simple polyorthogon or polycuboid, respectively, i.e. a square or a cube, respectively. We take K copies and put them together in a row. □

Proposition 8. For every natural number K exists a representation of the chessboard by K^2 squares.

Proof. Take the square with sidelength $\frac{8}{K}$. □

We show once more that a chessboard is placeable by some bricks.
There are at least 10 trivial possibilities, if we use particular squares and other rectangles to represent the chessboard. The squares have sidelengths 1 or 2 or 4 or 8. We also can use 6 different rectangles to represent the chessboard. They have sidelengths $1 \times 2, 1 \times 4, 1 \times 8, 2 \times 4, 2 \times 8$ and 4×8. Further there are at least two representations of the chessboard, which we call 'semi-trivial'. The bricks have the form of an L.Two examples are shown below. We add Malu's square, which proves that the chessboard is mirror-placeable by 16 bricks, which we know already by Proposition 4 (we use the representation with rectangles of measures 1×4), and three non-trivial representations.

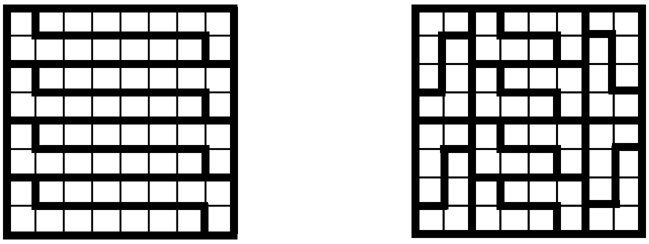

Figure 3:
On the left hand side, we show two 'semi-trivial' representations of a chessboard.

A few years ago we have taught at a primary school. One day a clever child came to us and showed a representation of a chessboard. Unfortunately we have forgotten the name. Her first name was 'Malu'. Hence we dubbed it 'Malu's square'. It is shown now. Malu prompted this paper. Without her it would not have been written.

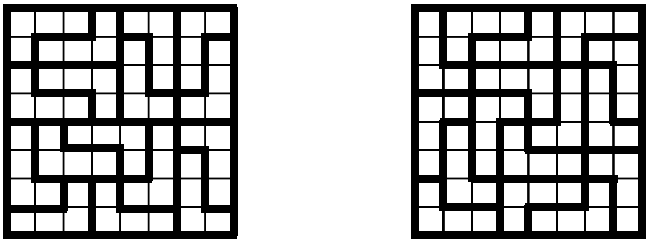

Figure 4:
We see two representations of the chessboard.
The picture on the right hand is Malu's square.

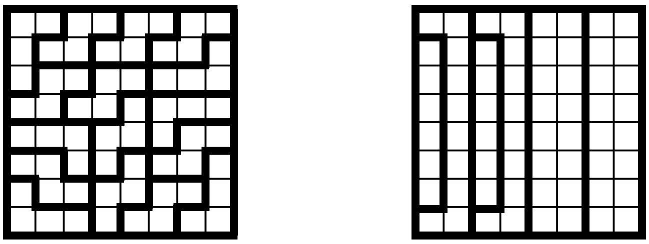

Figure 5:
The pictures on the left hand side prove that the chessboard is mirror-placeable by16 bricks of 2 shapes, and placeable by 6 bricks of 3 shapes.

References

[1] Anthony J. Guttmann: *Polygons, Polyominoes and Polycubes*, Springer 2009

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