# The Chessboard Puzzle

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**Abstract**

We introduce compact subsets in the plane and in $\mathbb{R}^3$, which we call *Polyorthogon* and *Polycuboid*, respectively. We ask whether we can represent these sets by congruent bricks or mirrored bricks.

*Keywords and phrases:* rectangle, cuboid *MSC 2020 : 51P99*

## 1 Introduction

We ask whether a usual chessboard can be represented by congruent bricks formed by the squares of which it is made. Of course the number of the bricks have to be a devisor of 64. This question can be generalized.

**Definition 1.** We define a *Polyorthogon* as a subset of $\mathbb{R}^2$ such that it is homeomorphic to a circle area $\{x^2+y^2 \leq 1 \mid (x,y) \in \mathbb{R}^2\}$ and it is the union of a finite number of rectangles. We define a *Polycuboid* as a subset of $\mathbb{R}^3$ such that it is homeomorphic to a ball $\{x^2+y^2+z^2 \leq 1 \mid (x,y,z) \in \mathbb{R}^3\}$ and it is the union of a finite number of cuboids. The rectangles in a polyorthogon and the cuboids in a polycuboid intersect only on their boundaries.

**Remark 1.** In a polyorthogon there are exactly two sets of parallel rectangle sides. The second set is perpendicular to the first set. In a polycuboid there are exactly three sets of parallel cuboid sides. The second set is perpendicular to the first set. The third set is perpendicalar both to the second and the first set.

A *Polyomino* and a *Polycube* is well-known. See [1].

**Definition 2.** Let $X$ and $Y$ both be subsets of $\mathbb{R}^2$ or both be subsets of $\mathbb{R}^3$. We say that $X$ is a *mirror image* of $Y$ if and only if there is a symmetry axis or a symmetry plane which mirrors $X$ in $Y$.

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## 2 The Chessboard

**Definition 3.** We write $E \cong F$ if and only if $E$ and $F$ are either polyorthogons or both are polycuboids, and they have the same shape and size.

Please see the picture below. The polyorthogon $G$ is one big square, while $H$ consists of 64 small squares of sidelength 1, which form a polyorthogon of the same shape and size. We call $H$ a *chessboard*. It holds $G \cong H$.

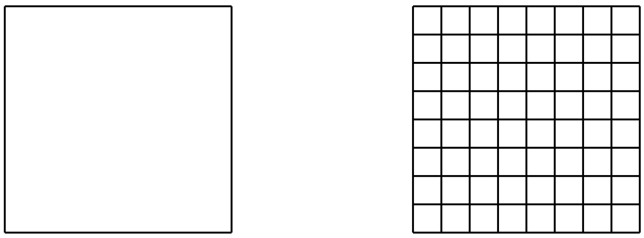


Figure 1:
On the left hand side we show two polyorthogons $G$ and $H$.
Both are also polyominoes.

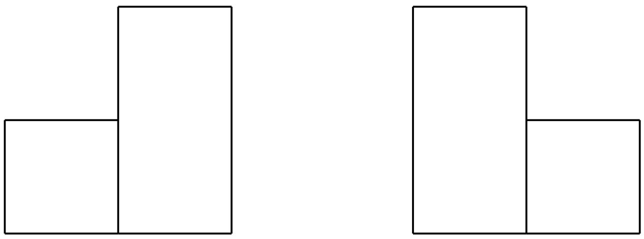


Figure 2:
On the left hand side we see a polyorthogon and its mirror image.

**Definition 4.** We say that '*B represents A*' or '$B$ is a *representation* of $A$' if and only if $A$ is a polyorthogon or a polycuboid or a polyomino or a polycube, and $B$ is the union

$$B = B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_{J-1} \cup B_J \tag{1}$$

and $B$ equals $A$ in shape and size. The sets $B_i$ are homeomorphic to a circle area $\{x^2 + y^2 \leq 1 \mid x, y \in \mathbb{R}\}$ or the ball $\{x^2 + y^2 + z^2 \leq 1 \mid x, y, z \in \mathbb{R}\}$. $B_i$ is called a *brick*. The bricks intersect only on their boundaries.

We pose an infinite set of questions. The answers can be 'yes' or 'no', where a negative answer generally would be difficult to prove.

Let $L, K$ be natural numbers.

**Question 1.** We presume that $P$ either is any polyorthogon or a polyomino. We ask whether there is a set $D(K) \subset \mathbb{R}^2$ such that $D(K)$ represents $P$, and

$$D(K) \;=\; W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{L-1} \cup W_L \tag{2}$$

where $W_i$ is a finite nonempty set of bricks for $i \;=\; 1,2,3,\ldots L-1,L$.
We need further conditions.
We demand for all $W_i$ that it has a cardinality larger than 1, i.e.

$$\text{cardinality}(W_i) > 1 \text{ for } i \in \{1,2,\ldots L-1,L\} \tag{3}$$

Further we demand that the sum of the cardinalities of the $W_i$'s is $K$, i.e.

$$K \;=\; \sum_{i=1}^{L} \text{cardinality}(W_i) \tag{4}$$

i.e. $P$ is represented by $K$ bricks. We demand if $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that $X$ is not congruent to $Y$, while if $i = j$ then $X$ is congruent to $Y$.
In the case that $D(K)$ exists, we say that 'the polyorthogon $P$ or the polyomino $P$ is $L$ *placeable* by $K$ bricks', or '$P$ is $L$ *placeable* by $K$ bricks' in brief. If $L = 1$ and $D(K)$ exists we say '$P$ is *placeable*'.

**Question 2.** We presume that $P$ either is a polyorthogon or a polyomino. We ask whether there is a set $E(K) \subset \mathbb{R}^2$ such that $E(K)$ represents $P$, and

$$E(K) \;=\; W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{L-1} \cup W_L \tag{5}$$

where $W_i$ is a finite nonempty set of bricks for $i \;=\; 1,2,3,\ldots L-1,L$. We demand if $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that neither $X$ is congruent to $Y$ nor $X$ is a mirror image of $Y$. If $i = j$ either $X$ is congruent to $Y$ or $X$ is a mirror image of $Y$. Also the rules (3) and (4) hold.
In the case that $E(K)$ exists, we say that 'the polyorthogon $P$ or the polyomino $P$ is $L$ *mirror-placeable* by $K$ bricks', or '$P$ is $L$ *mirror-placeable* by $K$ bricks' in brief. If $L = 1$ and $E(K)$ exists we say '$P$ is *mirror-placeable*'.

**Question 3.** We presume that $Q$ is a polycuboid or a polycube. We ask whether there is a set $F(K) \subset \mathbb{R}^3$ such that $F(K)$ represents $Q$, and

$$F(K) \;=\; W_1 \cup W_2 \cup W_3 \cup \ldots \cup W_{L-1} \cup W_L \tag{6}$$

where $W_i$ is a finite nonempty set of three dimensional bricks for $i \;=\; 1,2,3,\ldots L-1,L$. We demand if $X \in W_i$ and $Y \in W_j$ with $i \neq j$ that $X$ is not congruent to $Y$, while $X$ is congruent to $Y$ if $i = j$. Further we demand also that the rules (3) and (4) hold.
In the case that $F(K)$ exists, we say that 'the polycuboid $Q$ or the polycube $Q$ is $L$ *sectional* by $K$ bricks', or '$Q$ is $L$ *sectional* by $K$ bricks' in brief. If $L = 1$ and $F(K)$ exists we say '$Q$ is *sectional*'.

A similar question can be asked for a polycuboid $R$ or a polycube $R$, if we define the expression '$R$ is $L$ *mirror-sectional* by $K$ bricks'.

**Proposition 1.** *If a polyorthogon or a polyomino or a polycuboid or a polycube, respectively, is L placeable or L sectional, respectively, by K bricks, it is also L mirror-placeable or L mirror-sectional, respectively, by K bricks.*

**Proposition 2.** *If a polyorthogon or a polyomino or a polycuboid or a polycube is L mirror-placeable or L mirror-sectional, respectively, by K bricks, it is also $2 \cdot L$ placeable or $2 \cdot L$ sectional, respectively, by K bricks.*

**Proposition 3.** *Let P be a polyorthogon or a polycuboid or a polyomino or a polycube, respectively. Then it is placeable or sectional, respectively.*

*Proof.* Take a rectangle or a cuboid, respectively, from $P$. Take as bricks the two halves. If $P$ is a polyomino or a polycube, respectively, with consist of more than one square or one cube, respectively, we can also take one of the generating squares or cubes, respectively. $\square$

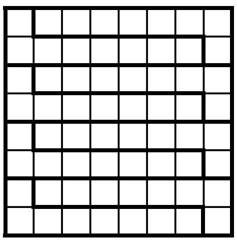
**Proposition 4.** *For every natural number K exists a polyorthogon and a polyomino and a polycuboid and a polycube, respectively, that is placeable or sectional, respectively, by $K+1$ bricks.*

*Proof.* We consider the most simple polyorthogon or polycuboid, respectively, i.e. a square or a cube, respectively. We take $K+1$ copies and put them together in a row. The constructed polyorthogon and polyomino or polycuboid and polycube, respectively, is placeable or sectional, respectively. For $K=1$ we also can take two halves of a square or a cube, respectively. $\square$

**Proposition 5.** *For every natural number K exists a representation of the chessboard by $K^2$ squares.*

*Proof.* Take the square with sidelength $\frac{8}{K}$. $\square$

We show once more that a chessboard is placeable. There are at least 9 trivial possibilities, if we use particular squares to represent the chessboard. They have sidelengths 1 or 2 or 4. We also can use 6 different rectangles to represent the chessboard. They have sidelengths $1 \times 2, 1 \times 4, 1 \times 8, 2 \times 4, 2 \times 8$ and $4 \times 8$. Further there are at least two representations of the chessboard, which we call ‘semi-trivial’. The bricks are L-shaped. Two examples are shown below. We add Malus square, which proves that the chessboard is mirror-placeable, which we know already by Proposition 1, and three non-trivial representations.



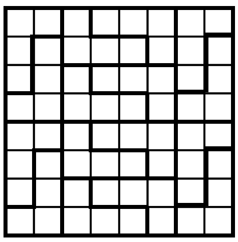
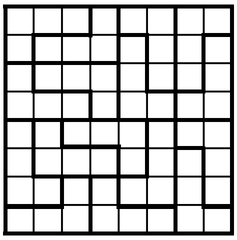


Figure 3:
On the left hand side we show two ‘semi-trivial’ representations of a chessboard.

A few years ago we have taught at a primary school. One day a clever child came to us and showed us a representation of a chessboard. Unfortunately we have forgotten the name. Her first name was ‘Malu’. Hence we dubbed it ‘Malus square’. It is shown now. Malu prompted this paper. Without her it would not have been written.



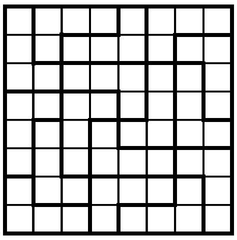
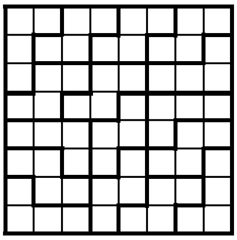


Figure 4:
We see two representations
of the chessboard.
The picture on the right
hand is Malus square.



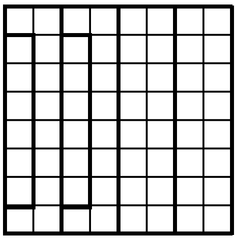


Figure 5:
The pictures on the left hand side prove
that the chessboard is 2 mirror-placeable
by 16 bricks and 3 placeable by 6 bricks.

# References

[1] Anthony J. Guttmann: *Polygons, Polyominoes and Polycubes*, Springer 2009

**Aknowledgements** We thank Rolf Baumgart for the word ‘polyorthogon’