Decimal Bases and Irrationality Proofs

Timothy W. Jones

October 11, 2021

Abstract

We develop an aspect of decimal representation of rational numbers and use it to prove a family of series converges to an irrational number.

1 Introduction

A rule for forming base 10 decimals of rational numbers is just do the division given by the form a/b. So $1/2$, $1/3$, $1/4$, $1/5$, $1/6$, and $1/7$ yields $.5$, $.\overline{3}$, $.25$, $.2$, $.1\overline{6}$, and $.\overline{142857}$, respectively. One can reason out that some rational numbers require finite decimals ($1/2$, $1/4$, and $1/5$), mixed decimals ($1/6$), and pure repeating decimals ($1/3$ and $1/7$). Natural questions of interest are the relationship between base 10 and the denominators of the rational number to be represented in this base. A further natural question is how many digits are required in each form.

Hardy answers these questions, first for base 10 representations and then for general bases [4]. The answer can be suggested by the base 10 results just given. If the divisors of b in a/b are the same as those of the base, then the fraction can be represented by a finite decimal: $1/2$, $1/4$, and $1/5$. If b shares some, but not all prime divisors with the base, like 6 shares 2 with 10, but not 3, then the representation is mixed: a finite non-repeating part followed by a repeating part. Finally, if b and the base are relatively prime, share no prime factors, then the decimal is pure repeating: as in $1/3$ and $1/7$.

How many digits and which digits are needed are harder question. We observe that 4 is 2^2 and $1/4$ requires two digits: this fraction can be made into a fraction with denominator a power of ten by multiplying by an appropriate form of 1:

$$\frac{1}{2^2}\frac{5^2}{5^2} = \frac{25}{10^2} = .25.$$

This same idea applies to all finite decimals in base 10. But why 1/3 requires one repeating digit and 1/7 requires 6 is more difficult. For decimals with repeating patterns, the length of the repeating part is termed its period. So 1/3 has period 1 and 1/7 has period 6. The basis of the determination of the digits used is given by the remainders, upon successive divisions as given by the division algorithm – long division. So the digits are limited to the digits of 10: 1, 2, 3, 4, 5, 6, 7, 8, and 9; a remainder of 0 implies divisibility. Eventually repetition of the algorithm yields a repeat remainder and this restarts the pattern. This rule is misleading though.

Consider $.\overline{122}$. Using some arithmetic we determine this represents 122/999:

$$1000(.\overline{122}) - .\overline{122} = 122,$$

but also

$$1000(.\overline{122}) - .\overline{122} = 999(.\overline{122}),$$

giving

$$\frac{122}{999} = .\overline{122}. \tag{1}$$

As $122 = 2\cdot 61$ and $999 = 3^3\cdot 37$, this is a reduced fraction with a denominator having nothing in common with the base 10. But how can the remainder of 2 be repeated? The answer is the base is really not 10, but 10^3; the 122 is a unique digit in this base. The representation is $.\overline{(122)}_{10^3}$ where the parentheses indicates a single symbol and the subscript indicates the base.

One can use the above example to understand why pure repeating decimals have the periods they do. One can observe, using (1), that $10^3 - 1 \equiv 0 \mod 999$. That is for any reduced fraction like 122/999 with a denominator relatively prime to 10 there is a power of 10, in this case 3, such that 10 to that power is equivalent to 1 mod that denominator. Thus $10 \equiv 1 \mod 3$ and $10^6 \equiv 1 \mod 7$; this last can be verified with

$$\frac{142857}{999999} = \frac{1}{7}$$

and this means there exists the number 142857 such that $7 \cdot 142857 = 999999 = 10^6 - 1$. One more: $.\overline{01}$ is 1/11 with the same reasoning used for $.\overline{122}$. Using Maple or a computer algebra system like it we can predict

1/11 will need a period of 2 by filling in natural numbers in $\bmod(10^k - 1, 11)$ until one gets a return of 0. An application of Euler Fermat's theorem shows that such a power exists [4].

To get a further understanding of this phenomenon, consider the geometric series with the first $k = 0$ term skipped. We have the formula

$$.\overline{x_1 x_2 \ldots x_n} = x_1 x_2 \ldots x_n \sum_{k=1}^{\infty} \left(\frac{1}{10^n}\right)^k = \frac{x_1 x_2 \ldots x_n}{10^n - 1} \tag{2}$$

and numbers with all nine digits, i.e. these $10^n - 1$ denominators, don't have an even or 5 unit digit and so are relatively prime to 10; they are strings of 9s, hence end in 9. Reduction of the fraction can pull out (cancel) denominator factors but not add in those not there already; $10^n - 1$ has no factors in common with powers of 10.

This reasoning shows that pure repeating decimals in base b converge to fractions with denominators relatively prime to b. The proof of the existence of the numerator in (2) is dependent on the existence of an exponent n such that $10^n \equiv 1$ mod a where $(10, a) = 1$. As mentioned this is an application of the Euler Fermat Theorem:

$$a^{\phi(b)} \equiv 1 \text{ mod b}, \tag{3}$$

where $\phi(b)$ is the Euler phi function. This theorem can be proven by noting that the classes of integers relatively prime to b define a group. Each element of this group defines a cyclic subgroup. By Lagrange's theorem the order of these subgroups divides the order of the group; that order is given by $\phi(b)$. In turn, (3) implies the existence of an integers r and x such that $rb = a^x - 1$, this r is the numerator in (2).

Single Decimals

Developing this aspect of decimal representations, we have that all pure repeating decimals in base b are of the form $.\overline{(x)}_{b^n}$ in a suitable power of b giving the period of the decimal. This has the interesting implication that there is a finite number of fractions that can be represented given n.

Continuing with our $.\overline{122}$ example, in base 10^3, these are at most the 999 decimal combinations in $\{1, 2, 3, \ldots, 997, 998, 999\}$ that can be used to represent 999 fractions: $.\overline{100}$, $.\overline{200}$, $\ldots$, $.\overline{997}$, $\ldots$, $.\overline{999}$. It is at most this number because $.\overline{x} = .\overline{xx}_{10^2} = .\overline{xxx}_{10^3}$, etc.., so there is an overlap with shorter period decimals. None-the-less one can easily say the number of fractions represented by pure repeating decimals with periods 1 to 3 is finite; in the case of $b = 10$, it is 999. The terminating decimals of n digits are also just the digits used, n in number. The total number of terminating 1 to 3 digit base 10 numbers is less than 9+99+999; noting that .1 is different than .01, but .10 is the same as .1. We just need that it is a finite number.

It is interesting that per (2) all these pure repeating decimals distill to fractions with the same, at first, denominator: $10^n - 1$ for some n. These fractions are not generally reduced. This same pattern repeats for all bases. It's all rather suggestive; numbers relatively prime to 10 grow with an exponent; what is the ratio of numbers relatively prime to 10 and all numbers? It seems like the first insights leading to the prime number theorem. This is far afield of our purposes.

Family of series

Suppose a series has the following two properties. Property one: its terms have denominators that when used as number bases can represent all fractions between 0 and 1 as finite decimals; property two: its partial sums are fractions in reduced form not given by finite decimals when the denominators of its terms are used as number bases. Expressed succinctly (albeit loosely), a series with these two properties has terms that cover the rational numbers and partials that escape their terms. We claim that at least for one family of series, these properties imply irrational convergence points.

Consider the series

$$z_n = \sum_{j=2}^{\infty} \frac{1}{j^n}.$$

This series occurs in elementary calculus courses; it is a family of convergent series that are frequently used with the comparison test to establish the convergence of other series. In calculus books it is referred to as the p-series, where the exponent, our n, is p. In number theory, these series are natural number greater than one arguments of the zeta function without the first term:

$$\zeta(n) - 1 = \sum_{j=1}^{\infty} \frac{1}{j^n} - 1 = z_n.$$

This family of series has the first property. This follows as any $a/b \in \mathbb{Q}(0,1)$ can be given as a single decimal in base b^n; it is $a/b = .(a(b^{n-1}))_{b^n}$ and, as b^n occurs as a denominator of a term of z_n, all such fractions can be represented.

Now suppose we can show that the partial sums of such series

$$s_k^n = \sum_{j=2}^{k} \frac{1}{j^n}$$

are fractions with reduced forms having denominators exceeding k^n. This means the partials can't be represented as finite decimals using the denominators of its terms as bases; this series will have property two. We will show prove this in a later section.

If a partial can't be expressed as a finite decimal using the denominator of its terms as bases, then in those bases a mixed or repeating decimal is required. But as all rational numbers are represented as finite decimals using such bases, convergence to a rational implies there exist a base such that partials have fixed decimal of the form $.(a-1)\overline{(b-1)}_R$, where R indicates the number of repetitions of the decimal $b-1$: a contradiction – that's not a mixed or pure repeating decimal, that's a finite decimal expressed in infinite form.

One can get a visceral sense of this contradiction. Consider that a mixed or pure repeating decimal must converge to a number that is not given by a finite decimal in the basis used to form the mixed or pure repeating decimal. Using our earlier examples, $1/6$ and $1/3$ require mixed and pure repeating decimals; they can't be represented by finite decimals or the infinite form of these decimals. But both of these numbers can be represented as finite decimal in a base; $1/6 = .(1)_6 = .0\overline{5}_6$ and $1/3 = .(1)_3 = .0\overline{2}_3$; in any bases a power of 6 or power of 3 this remains true:

$$\frac{1}{6} \cdot \frac{6^{r-1}}{6^{r-1}} = \frac{6^{r-1}}{6^r} = .(6^{r-1})_{6^r},$$

more familiarly $.1 = 10/100 = .(10)_{100}$.

If all bases express a mixed or repeating decimal representation of partials from some point on, given that decimals become fixed, the fixed head part will be repeated in the unfixed tail part, but the total fixed plus unfixed parts combine for an approximation of the convergent point and can't be the convergent point. The fixed digits must be expanded into an indefinitely larger period, an irrational number.

The second property

In this section we show the second property. The central technique is suggested by a textbook problem and its solution [2, 5].

Lemma 1. *If $s_k^n = r/s$ with r/s a reduced fraction, then 2^n divides s.*

Proof. The set $\{2, 3, \ldots, k\}$ will have a greatest power of 2 in it, a; the set $\{2^n, 3^n, \ldots, k^n\}$ will have a greatest power of 2, na. Also $k!$ will have a powers of 2 divisor with exponent b; and $(k!)^n$ will have a greatest power of 2 exponent of nb. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + (k!)^n/3^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{4}$$

The term $(k!)^n/2^{na}$ will pull out the most 2 powers of any term, leaving a term with an exponent of $nb - na$ for 2. As all other terms but this term will have more than an exponent of 2^{nb-na} in their prime factorization, we have the numerator of (4) has the form

$$2^{nb-na}(2A + B),$$

where $2 \nmid B$ and A is some positive integer. This follows as all the terms in the factored numerator have powers of 2 in them except the factored term $(k!)^n/2^{na}$. The denominator, meanwhile, has the factored form

$$2^{nb}C,$$

where $2 \nmid C$. This leaves 2^{na} as a factor in the denominator with no powers of 2 in the numerator, as needed. $\square$

Lemma 2. *If $s_k^n = r/s$ with r/s a reduced fraction and p is a prime such that $k > p > k/2$, then p^n divides s.*

Proof. First note that $(k, p) = 1$. If $p|k$ then there would have to exist r such that $rp = k$, but by $k > p > k/2$, $2p > k$ making the existence of such a natural number $r > 1$ impossible.

The reasoning is much the same as in Lemma 1. Consider

$$\frac{(k!)^n}{(k!)^n} \sum_{j=2}^{k} \frac{1}{j^n} = \frac{(k!)^n/2^n + \cdots + (k!)^n/p^n + \cdots + (k!)^n/k^n}{(k!)^n}. \tag{5}$$

As $(k,p)=1$, only the term $(k!)^n/p^n$ will not have p in it. The sum of all such terms will not be divisible by p, otherwise p would divide $(k!)^n/p^n$. As $p<k$, p^n divides $(k!)^n$, the denominator of r/s, as needed. □

Lemma 3. *For any $k \geq 2$, there exists a prime p such that $k<p<2k$.*

Proof. This is Bertrand's postulate [4]. □

Theorem 1. *If $s_k^n = \frac{r}{s}$, with r/s reduced, then $s > k^n$.*

Proof. Using Lemma 3, for even k, we are assured that there exists a prime p such that $k>p>k/2$. If k is odd, $k-1$ is even and we are assured of the existence of prime p such that $k-1>p>(k-1)/2$. As $k-1$ is even, $p \neq k-1$ and $p>(k-1)/2$ assures us that $2p>k$, as $2p=k$ implies k is even, a contradiction.

For both odd and even k, using Lemma 3, we have assurance of the existence of a p that satisfies Lemma 2. Using Lemmas 1, 2, and 3 we have $2^n p^n$ divides the denominator of r/s and as $2^n p^n > k^n$, the proof is completed. □

Conclusion

For other treatments of the irrationality of z_n see [1, 3, 6, 7].

References

[1] Apéry, R. (1979). Irrationalité de $\zeta(2)$ et $\zeta(3)$. *Astérisque* 61: 11-13.

[2] Apostol, T. M. (1976). *Introduction to Analytic Number Theory.* New York: Springer.

[3] Beukers, F. (1979). A Note on the irrationality of $\zeta(2)$ and $\zeta(3)$, *Bull. London Math. Soc.* 11: 268–272.

[4] Hardy, G. H., Wright, E. M., Heath-Brown, R. , Silverman, J. , Wiles, A. (2008). *An Introduction to the Theory of Numbers*, 6th ed. London: Oxford Univ. Press.

[5] Hurst, G. (2014). Solutions to Introduction to Analytic Number Theory by Tom M. Apostol. https://greghurst.files.wordpress.com/2014/02/apostol_intro_to_ant.pdf

[6] Rivoal, T. (2000). La fonction zeta de Riemann prend une infinit de valeurs irrationnelles aux entiers impairs, *Comptes Rendus de l'Acadmie des Sciences, Srie I. Mathmatique* 331: 267-270.

[7] Zudilin, W. W. (2001). One of the numbers $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational. *Russian Mathematical Surveys* 56(4): 747–776.