Family of Determinantal Identities

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Abstract. In this article, we establish a family of determinantal identities of which the Cassini's identity is a particular case.

1 Introduction

The Fibonacci sequence: 0,1,1, 2, 3, 5, 8, 13, . . ., is a sequence of positive integers whose terms are defined by the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$ with the initial conditions $f_0 = 0$ and $f_1 = 1$. This sequence appears in the book *Liber Abaci* published by Fibonacci (initially, Leonardo of Pisa) in 1202. And, the *golden ratio* is a famous number that is related to the Fibonacci sequence as a limit to the ratio of consecutive Fibonacci numbers. For the theory of Fibonacci numbers and the golden ratio with their generalizations see [1–15], and the references given therein. We have reviewd the Fibonacci sequence, the golden ratio and introduced generalized additive sequences [1]. We also investigated golden ratios and golden angles of p-sequences [2], and obtained the sums of numbers of the *exponent* p-sequences [3]. In this article, we consider the generating Q-matrix and the *determinantal identities* of p-sequences.

2 Q-matrix and determinantal identity

We have seen that the Fibonacci sequence can be generalized in many ways such as generalizations of the Euclid's theorem, the recurrence relations, and the characteristic equations. There is yet another way to study and generalize the Fibonacci sequence and derive many interesting properties of these numbers using a matrix representation [16–27]. By matrix methods, while Silvester [18] derived many interesting properties of the Fibonacci numbers, Kalman [19] generalized Fibonacci numbers.

For the Fibonacci sequence recurrence relation $f_n = f_{n-1} + f_{n-2}$ given $f_1 = 1$ and $f_0 = 0$, using the generating Q-matrix

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_2 & f_1 \\ f_1 & f_0 \end{pmatrix}, \tag{1}$$

we have, for $n \geq 1$,

$$\begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} f_1 \\ f_0 \end{pmatrix}, \tag{2}$$

and

$$Q^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}. \tag{3}$$

From Eq. (3) follows the Cassini's identity

$$(\det Q)^n = \begin{vmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{vmatrix} = (-1)^n. \tag{4}$$

n	$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$
1	$1 \times 0 - 1^2 = -1$
2	$2 \times 1 - 1^2 = 1$
3	$3 \times 1 - 2^2 = -1$
4	$5 \times 2 - 3^2 = 1$
5	$8 \times 3 - 5^2 = -1$
6	$13 \times 5 - 8^2 = 1$
7	$21 \times 8 - 13^2 = -1$
8	$34 \times 13 - 21^2 = 1$
9	$55 \times 21 - 34^2 = -1$

Table 1: Illustration of the Cassini's identity for the Fibonacci sequence $S_1(2)$ (see Table 2 [1]).

In the following, we confine ourselves to establishing a family of *determinantal identities* of which the Cassini's identity is a particular case. We begin with introducing a square matrix Q_p of order p

$$Q_p = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{5}$$

where elements of the first row are seeds of the *coefficient* p-sequence $S_C(p)$, those of the second row are seeds of p-sequence $S_0(p)$, the third row are seeds of p-sequence $S_1(p)$, and so on, until the last row whose elements are seeds of p-sequence $S_{p-2}(p)$ [1]. The determinant of this matrix Q_p is

$$\det Q_p = \begin{cases} -1 & (p \text{ even}), \\ 1 & (p \text{ odd}). \end{cases} \tag{6}$$

The matrix Q_p can be seen as a special case of the generating matrix $\tilde{Q}_p$ [19],

$$\tilde{Q}_p = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_p \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \tag{7}$$

for the generalized recurrence relation $t_n(p) = \sum_{k=1}^{p} c_k t_{n-k}(p)$, where $\{c_k\}$s are constants, such that

$$\begin{pmatrix} t_{n+p-1} \\ \vdots \\ t_{n+1} \\ t_n \end{pmatrix} = \tilde{Q}_p^n \begin{pmatrix} t_{p-1} \\ \vdots \\ t_1 \\ t_0 \end{pmatrix} \quad (n \geq 1), \tag{8}$$

and

$$\det \tilde{Q}_p = (-1)^{p+1} c_p. \tag{9}$$

We can, further, identify the matrix Q_p as

$$Q_p = \begin{pmatrix} t_p & t_{p-1} & \cdots & t_{p-1} & t_{p-1} \\ t_{p-1} & 0 & \cdots & 0 & t_{p-2} \\ t_{p-2} & t_{p-1} & \cdots & 0 & t_{p-3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ t_1 & 0 & \cdots & t_{p-1} & t_0 \end{pmatrix}, \tag{10}$$

where t_k's in the first row, the first and the last columns, and the diagonal below the main diagonal are terms of the p-sequence $S_{p-1}(p)$, and other elements of the matrix are zero. Now, making use of Eqs. (5) and (10), we can construct the determinantal identities, like the Cassini's identity, for $p \geq 3$. Following, we illustrate the case for $p = 3,\ 4$.

2.1 Illustration for $p = 3$

Starting with

$$Q_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} t_3 & t_2 & t_2 \\ t_2 & 0 & t_1 \\ t_1 & t_2 & t_0 \end{pmatrix}, \tag{11}$$

we have

$$\begin{aligned} Q_3^2 &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t_3 & t_2 & t_2 \\ t_2 & 0 & t_1 \\ t_1 & t_2 & t_0 \end{pmatrix} \\ &= \begin{pmatrix} t_4 & 2t_2 & t_3 \\ t_3 & t_2 & t_2 \\ t_2 & 0 & t_1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{12}$$

and likewise

$$Q_3^n = \begin{pmatrix} t_{n+2}[S_2(3)] & t_n[S_X(3)]\, t_2[S_2(3)] & t_{n+1}[S_2(3)] \\ t_{n+1}[S_2(3)] & t_{n-1}[S_X(3)]\, t_2[S_2(3)] & t_n[S_2(3)] \\ t_n[S_2(3)] & t_{n-2}[S_X(3)]\, t_2[S_2(3)] & t_{n-1}[S_2(3)] \end{pmatrix} \quad (n \geq 2), \tag{13}$$

where the notations have their usual meanings (see Table 3 [1]).

Hence, the determinantal identity is

$$(\det Q_3)^n = \begin{vmatrix} t_{n+2}[S_2(3)] & t_n[S_X(3)]\, t_2[S_2(3)] & t_{n+1}[S_2(3)] \\ t_{n+1}[S_2(3)] & t_{n-1}[S_X(3)]\, t_2[S_2(3)] & t_n[S_2(3)] \\ t_n[S_2(3)] & t_{n-2}[S_X(3)]\, t_2[S_2(3)] & t_{n-1}[S_2(3)] \end{vmatrix} = 1. \tag{14}$$

2.2 Illustration for $p = 4$

For $p = 4$, we have

$$Q_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} t_4 & t_3 & t_3 & t_3 \\ t_3 & 0 & 0 & t_2 \\ t_2 & t_3 & 0 & t_1 \\ t_1 & 0 & t_3 & t_0 \end{pmatrix}, \tag{15}$$

$$Q_4^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} t_4 & t_3 & t_3 & t_3 \\ t_3 & 0 & 0 & t_2 \\ t_2 & t_3 & 0 & t_1 \\ t_1 & 0 & t_3 & t_0 \end{pmatrix} = \begin{pmatrix} t_5 & 2t_3 & 2t_3 & t_4 \\ t_4 & t_3 & t_3 & t_3 \\ t_3 & 0 & 0 & t_2 \\ t_2 & t_3 & 0 & t_1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{16}$$

and, for $n \geq 3$,

$$Q_4^n = \begin{pmatrix} t_{n+3}[S_3(4)] & t_{n+3}[S_2(4)]\ t_3[S_3(4)] & t_n[S_X(4)]\ t_3[S_3(4)] & t_{n+2}[S_3(4)] \\ t_{n+2}[S_3(4)] & t_{n+2}[S_2(4)]\ t_3[S_3(4)] & t_{n-1}[S_X(4)]\ t_3[S_3(4)] & t_{n+1}[S_3(4)] \\ t_{n+1}[S_3(4)] & t_{n+1}[S_2(4)]\ t_3[S_3(4)] & t_{n-2}[S_X(4)]\ t_3[S_3(4)] & t_n[S_3(4)] \\ t_n[S_3(4)] & t_n[S_2(4)]\ t_3[S_3(4)] & t_{n-3}[S_X(4)]\ t_3[S_3(4)] & t_{n_1}[S_3(4)] \end{pmatrix}, \tag{17}$$

where the notations have their usual meanings (see Table 4 [1]).

Hence, the determinantal identity for 4-sequence $S_3(4)$ is

$$(\det Q_4)^n = \begin{vmatrix} t_{n+3}[S_3(4)] & t_{n+3}[S_2(4)]\ t_3[S_3(4)] & t_n[S_X(4)]\ t_3[S_3(4)] & t_{n+2}[S_3(4)] \\ t_{n+2}[S_3(4)] & t_{n+2}[S_2(4)]\ t_3[S_3(4)] & t_{n-1}[S_X(4)]\ t_3[S_3(4)] & t_{n+1}[S_3(4)] \\ t_{n+1}[S_3(4)] & t_{n+1}[S_2(4)]\ t_3[S_3(4)] & t_{n-2}[S_X(4)]\ t_3[S_3(4)] & t_n[S_3(4)] \\ t_n[S_3(4)] & t_n[S_2(4)]\ t_3[S_3(4)] & t_{n-3}[S_X(4)]\ t_3[S_3(4)] & t_{n_1}[S_3(4)] \end{vmatrix} = (-1)^n. \tag{18}$$

n	$S_1(2)$	$S_0(2)$	$S_C(2)$	$S_S(2)$	$S_G(2)$
0	0	1	1	1	2
1	1	0	1	2	21
2	1	1	2	3	23
3	2	1	3	5	44
4	3	2	5	8	67
5	5	3	8	13	111
6	8	5	13	21	178
7	13	8	21	34	289
8	21	13	34	55	467
9	34	21	55	89	756
10	55	34	89	144	1223
11	89	55	144	233	1979
12	144	89	233	377	3202
13	233	144	377	610	5181
14	377	233	610	987	8383
15	610	377	987	1597	13564
16	987	610	1597	2584	21947
17	1597	987	2584	4181	35511
18	2584	1597	4181	6765	57458
19	4181	2584	6765	10946	92969
20	6765	4181	10946	17711	150427
21	10946	6765	17711	28657	243396
22	17711	10946	28657	46368	393823
23	28657	17711	46368	75025	637219
24	46368	28657	75025	121393	1031042
25	75025	46368	121393	196418	1668261

Table 2: 2-sequences. (i) $S_C \equiv S_1 + S_0$. (ii) $S_X = S_1$. (iii) $S_1 \sim S_0 \sim S_C \sim S_S$. (iv) S_G is a general 2-sequence with seeds $s_0 = 2,\ s_1 = 21$. (v) For each of these 2-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.61803$.

n	$S_2(3)$	$S_1(3)$	$S_0(3)$	$S_C(3)$	$S_X(3)$	$S_S(3)$
0	0	0	0	1	0	1
1	0	1	0	1	1	2
2	1	0	1	1	2	4
3	1	1	1	3	3	7
4	2	2	2	5	6	13
5	4	3	4	9	11	24
6	7	6	7	17	20	44
7	13	11	13	31	37	81
8	24	20	24	57	68	149
9	44	37	44	105	125	274
10	81	68	81	193	230	504
11	149	125	149	355	423	927
12	274	230	274	653	778	1705
13	504	423	504	1201	1431	3136
14	927	778	927	2209	2632	5768
15	1705	1431	1705	4063	4841	10609
16	3136	2632	3136	7473	8904	19513
17	5768	4841	5768	13745	16377	35890
18	10609	8904	10609	25281	30122	66012
19	19513	16377	19513	46499	55403	121415
20	35890	30122	35890	85525	101902	223317
21	66012	55403	66012	157305	187427	410744
22	121415	101902	121415	289329	344732	755476
23	223317	187427	223317	532159	634061	1389537
24	410744	344732	410744	978793	1166220	2555757
25	755476	634061	755476	1800281	2145013	4700770

Table 3: 3-sequences. (i) $S_C \equiv S_2 + S_1 + S_0$. (ii) $S_2 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 3-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.83929$.

n	$S_3(4)$	$S_2(4)$	$S_1(4)$	$S_0(4)$	$S_C(4)$	$S_X(4)$	$S_S(4)$
0	0	0	0	1	1	0	1
1	0	0	1	0	1	1	2
2	0	1	0	0	1	2	4
3	1	0	0	0	1	3	8
4	1	1	1	1	4	6	15
5	2	2	2	1	7	12	29
6	4	4	3	2	13	23	56
7	8	7	6	4	25	44	108
8	15	14	12	8	49	85	208
9	29	27	23	15	94	164	401
10	56	52	44	29	181	316	773
11	108	100	85	56	349	609	1490
12	208	193	164	108	673	1174	2872
13	401	372	316	208	1297	2263	5536
14	773	717	609	401	2500	4362	10671
15	1490	1382	1174	773	4819	8408	20569
16	2872	2664	2263	1490	9289	16207	39648
17	5536	5135	4362	2872	17905	31240	76424
18	10671	9898	8408	5536	34513	60217	147312
19	20569	19079	16207	10671	66526	116072	283953
20	39648	36776	31240	20569	128233	223736	547337
21	76424	70888	60217	39648	247177	431265	1055026
22	147312	136641	116072	76424	476449	831290	2033628
23	283953	263384	223736	147312	918385	1592363	3919944
24	547337	507689	431265	283953	1770244	3068654	7555935
25	1055026	978602	831290	547337	3412255	5623572	14564533

Table 4: 4-sequences. (i) $S_C \equiv S_3 + S_2 + S_1 + S_0$. (ii) $S_3 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 4-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.92756$.

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