Sums of p-Sequences

Asutosh Kumar

P. G. Department of Physics, Gaya College, Magadh University, Rampur, Gaya 823001, India

Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Prayagraj 211 019, India

Vaidic and Modern Physics Research Centre, Bhagal Bhim, Bhinmal, Jalore 343029, India

Abstract. In this article, we obtain closed expressions for odd and even sums, the sum of the first n numbers, and the sum of squares of the first n numbers of the *exponent* p-sequence whose *seeds* are $(0, 1, \cdots, p-1)$.

1 Introduction

The Fibonacci sequence and the golden ratio are two central concepts in mathematics [1, 2] (see also the references [3–28]). They appear almost everywhere in nature. The Fibonacci sequence is a 2-sequence, $f_{n+2} = f_{n+1} + f_n$, starting with the seeds 0 and 1. A natural extension of this sequence is the p-sequence [1]: $t_{n+p}(p) := t_{n+p-1}(p) + t_{n+p-2}(p) + \cdots + t_n(p)$, begining with p seeds $(s_0, s_1, \cdots, s_{p-1})$ such that $t_0 = s_0,\ t_1 = s_1, \cdots, t_{p-1} = s_{p-1}$. Thus, there can be an infinite number of p-sequences depending on the values of seeds; some of them being special.

2 Sums of p-sequences

In this section, we obtain closed expressions for odd and even sums, sum of the first n numbers, and the sum of squares of the first n numbers of the *exponent* p-sequences. Recall that the exponent p-sequence $S_X(p)$ whose seeds are $(0, 1, \cdots, p-1)$ is given as [1]

$$S_X(p) \equiv \{(s_k = k,\ 0 \leq k \leq p-1),\ t_n(p)\}, \tag{1}$$

where $t_n(p) := \sum_{k=n-p}^{n-1} t_k$. We state the results without giving any proof. We invite the enthusiastic readers to verify (using Tables 4, 5, 6 and 7 [1]) and prove them, and also obtain similar expressions for other p-sequences.

2.1 Sum of first n numbers

Firstly, we consider the general 1-sequence $S_G(1) = \{t_0 = s_0, t_1 = s_0 + a, t_n = t_{n-1} + a = s_0 + na\}$. The sum of first $n \geq 0$ terms of this sequence is

$$\Sigma_n[S_G(1)] = \sum_{k=0}^{n} t_k = \frac{(n+1)(t_n + t_0)}{2} = \frac{(n+1)(2s_0 + na)}{2}. \quad (2)$$

Next, we obtain the sum of first n numbers of the exponent p-sequence $S(Xp)$ for different values of p.

$$\sum_{k=1}^{n} t_k(X2) = t_{n+2} - t_2. \quad (3)$$

$$2\sum_{k=1}^{n} t_k(X3) = t_{n+2} + t_n - [t_3 - t_1]. \quad (4)$$

$$3\sum_{k=1}^{n} t_k(X4) = t_{n+2} + 2t_n + t_{n-1} - [t_3 - t_1]. \quad (5)$$

and

$$\begin{aligned}
&(p-1)\sum_{k=1}^{n} t_k(Xp) \\
= \ & \sum_{k=0}^{p-1}(p-k)t_{n-k} - \big(t_p - \sum_{k=1}^{p-2}(p-1-k)t_k\big), & (6)\\
= \ & t_{n+1} + \sum_{k=0}^{p-2}(p-1-k)t_{n-k} - \big(t_p - \sum_{k=1}^{p-2}(p-1-k)t_k\big), & (7)\\
= \ & t_{n+2} + \sum_{k=0}^{p-3}(p-2-k)t_{n-k} - \big(t_p - \sum_{k=1}^{p-2}(p-1-k)t_k\big), & (8)\\
= \ & t_{n+p} + \sum_{k=1}^{p-2}(p-1-k)t_{n+k} - \big(t_p - \sum_{k=1}^{p-2}(p-1-k)t_k\big). & (9)
\end{aligned}$$

2.2 Odd and even sums

- Sum of the first n odd numbers of
  (i) $S_X(2)$ is $\sum_{i=1}^{n} t_{2i-1} = t_{2n}$,

  (ii) $S_X(3)$ is $\sum_{i=1}^{n} t_{2i-1} = \frac{t_{2n}+t_{2n-1}-1}{2}$,

  (iii) $S_X(4)$ is $\sum_{i=1}^{n} t_{2i-1} = \frac{t_{2n+1}+t_{2n-2}-2}{3}$, and

  (iv) $S_X(5)$ is $\sum_{i=1}^{n} t_{2i-1} = \frac{t_{2n+1}+t_{2n-1}+t_{2n-2}+t_{2n-3}}{4}$.

- Sum of the first n even numbers of
  (i) $S_X(2)$ is $\sum_{i=1}^{n} t_{2i} = t_{2n+1} - 1$,

  (ii) $S_X(3)$ is $\sum_{i=1}^{n} t_{2i} = \frac{t_{2n+1}+t_{2n}-1}{2}$,

  (iii) $S_X(4)$ is $\sum_{i=1}^{n} t_{2i} = \frac{3t_{2n}+2t_{2n-1}}{3}$, and

  (iv) $S_X(5)$ is $\sum_{i=1}^{n} t_{2i} = \frac{4t_{2n}+2t_{2n-1}+t_{2n-2}}{4}$.

3 Sum of squares of first n numbers

Following, we obtain the sum of squares of first n terms of p-sequence. For $p = 1$ we consider the general sequence $S_G(1)$, and for $p = 2,\ 3,\ 4$ we consider the *exponent* sequence $S_X(p)$.

3.1 $p = 1$ (general sequence)

For the general 1-sequence $S_G(1) = \{t_0 = s_0, t_1 = s_0 + a, t_n = t_{n-1} + a = s_0 + na\}$, the sum of squares of first $n \geq 0$ terms is

$$\sum_{k=0}^{n} t_k^2(G1) = (n+1)\big[s_0^2 + ns_0a + \frac{n(2n+1)}{6}a^2\big]. \tag{10}$$

3.2 $p = 2$ (exponent sequence)

$$\sum_{k=1}^{n} t_k^2(X2) \quad = \quad t_n(X2)t_{n+1}(X2). \tag{11}$$

n	$\sum_{k=1}^{n} t_k^2(X2)$	$t_n(X2)t_{n+1}(X2)$
1	1^2	1×1
2	$1 + 1^2 = 2$	1×2
3	$2 + 2^2 = 6$	2×3
4	$6 + 3^2 = 15$	3×5
5	$15 + 5^2 = 40$	5×8
6	$40 + 8^2 = 104$	8×13
7	$104 + 13^2 = 273$	13×21
8	$273 + 21^2 = 714$	21×34
9	$714 + 34^2 = 1870$	34×55

Table 1: Illustration of sum of squares of numbers of 2-sequence $S_X(2)$.

n	$\sum_{k=1}^{n} t_k^2(X3)$	$t_n(X3)t_{n+1}(X3) - t_{n+1}^2[S_2(3)]$
1	1^2	$1 \times 2 - 1^2$
2	$1 + 2^2 = 5$	$2 \times 3 - 1^2$
3	$5 + 3^2 = 14$	$3 \times 6 - 2^2$
4	$14 + 6^2 = 50$	$6 \times 11 - 4^2$
5	$50 + 11^2 = 171$	$11 \times 20 - 7^2$
6	$171 + 20^2 = 571$	$20 \times 37 - 13^2$
7	$571 + 37^2 = 1940$	$37 \times 68 - 24^2$
8	$1940 + 68^2 = 6564$	$68 \times 125 - 44^2$
9	$6564 + 125^2 = 22189$	$125 \times 230 - 81^2$

Table 2: Illustration of sum of squares of numbers of 3-sequence $S_X(3)$.

3.3 $p = 3$ (exponent sequence)

$$\sum_{k=1}^{n} t_k^2(X3) = t_n(X3)t_{n+1}(X3) - t_{n+1}^2[S_2(3)]. \tag{12}$$

3.4 $p = 4$ (exponent sequence)

$$\sum_{k=1}^{n} t_k^2(X4) = t_n(X4)t_{n+1}(X4) - \left[(t_{n+2}[S_3(4)] + t_{n-1}[S_3(4)])^2 \pm \delta_n\right], \tag{13}$$

where $0 \leq \delta_n < t_{n-1}(X4)$. Note that here, unlike $p = 2$ & 3 cases, the formula is not exact.

n	$\sum_{k=1}^{n} t_k^2(X4)$	$t_n(X4)t_{n+1}(X4) - \left[(t_{n+2}[S_3(4)] + t_{n-1}[S_3(4)])^2 \pm \delta_n\right]$
1	1^2	$1 \times 2 - [(1+0)^2 + 0]$
2	$1 + 2^2 = 5$	$2 \times 3 - [(1+0)^2 + 0]$
3	$5 + 3^2 = 14$	$3 \times 6 - [(2+0)^2 + 0]$
4	$14 + 6^2 = 50$	$6 \times 12 - [(4+1)^2 - 3]$
5	$50 + 12^2 = 194$	$12 \times 23 - [(8+1)^2 + 1]$
6	$194 + 23^2 = 723$	$23 \times 44 - [(15+2)^2 + 0]$
7	$723 + 44^2 = 2659$	$44 \times 85 - [(29+4)^2 - 8]$
8	$2659 + 85^2 = 9884$	$85 \times 164 - [(56+8)^2 - 40]$
9	$9884 + 164^2 = 36780$	$164 \times 316 - [(108+15)^2 + 67]$

Table 3: Illustration of sum of squares of numbers of 4-sequence $S_X(4)$.

3.5 More sums

We further find that

- For 2-sequence $S_X(2)$,
  (i) $\sum_{i=1}^{n} t_i t_{i+1} = \frac{1}{2}(t_{n+2}^2 - t_n t_{n+1} - t_2^2)$, and

  (ii) $\sum_{i=1}^{n} t_i t_{i+1} = \sum_{j=1}^{n}(n+1-j)t_j^2$.

- For 3-sequence $S_X(3)$,
  $\sum_{i=1}^{n-1}(t_i t_{i+1} + t_{i+1}t_{i+2} + t_{i+2}t_i)$

  $= \frac{1}{2}\big[t_{n+2}(t_{n+2} - 1) + t_n(t_n - 1) - (t_3 - t_1)(t_3 + t_1 - 1)\big]$.

n	$S_1(2)$	$S_0(2)$	$S_C(2)$	$S_S(2)$	$S_G(2)$
0	0	1	1	1	2
1	1	0	1	2	21
2	1	1	2	3	23
3	2	1	3	5	44
4	3	2	5	8	67
5	5	3	8	13	111
6	8	5	13	21	178
7	13	8	21	34	289
8	21	13	34	55	467
9	34	21	55	89	756
10	55	34	89	144	1223
11	89	55	144	233	1979
12	144	89	233	377	3202
13	233	144	377	610	5181
14	377	233	610	987	8383
15	610	377	987	1597	13564
16	987	610	1597	2584	21947
17	1597	987	2584	4181	35511
18	2584	1597	4181	6765	57458
19	4181	2584	6765	10946	92969
20	6765	4181	10946	17711	150427
21	10946	6765	17711	28657	243396
22	17711	10946	28657	46368	393823
23	28657	17711	46368	75025	637219
24	46368	28657	75025	121393	1031042
25	75025	46368	121393	196418	1668261

Table 4: 2-sequences. (i) $S_C \equiv S_1 + S_0$. (ii) $S_X = S_1$. (iii) $S_1 \sim S_0 \sim S_C \sim S_S$. (iv) S_G is a general 2-sequence with seeds $s_1 = 2,\ s_2 = 21$. (v) For each of these 2-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.61803$.

n	$S_2(3)$	$S_1(3)$	$S_0(3)$	$S_C(3)$	$S_X(3)$	$S_S(3)$
0	0	0	0	1	0	1
1	0	1	0	1	1	2
2	1	0	1	1	2	4
3	1	1	1	3	3	7
4	2	2	2	5	6	13
5	4	3	4	9	11	24
6	7	6	7	17	20	44
7	13	11	13	31	37	81
8	24	20	24	57	68	149
9	44	37	44	105	125	274
10	81	68	81	193	230	504
11	149	125	149	355	423	927
12	274	230	274	653	778	1705
13	504	423	504	1201	1431	3136
14	927	778	927	2209	2632	5768
15	1705	1431	1705	4063	4841	10609
16	3136	2632	3136	7473	8904	19513
17	5768	4841	5768	13745	16377	35890
18	10609	8904	10609	25281	30122	66012
19	19513	16377	19513	46499	55403	121415
20	35890	30122	35890	85525	101902	223317
21	66012	55403	66012	157305	187427	410744
22	121415	101902	121415	289329	344732	755476
23	223317	187427	223317	532159	634061	1389537
24	410744	344732	410744	978793	1166220	2555757
25	755476	634061	755476	1800281	2145013	4700770

Table 5: 3-sequences. (i) $S_C \equiv S_2 + S_1 + S_0$. (ii) $S_2 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 3-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.83929$.

n	$S_3(4)$	$S_2(4)$	$S_1(4)$	$S_0(4)$	$S_C(4)$	$S_X(4)$	$S_S(4)$
0	0	0	0	1	1	0	1
1	0	0	1	0	1	1	2
2	0	1	0	0	1	2	4
3	1	0	0	0	1	3	8
4	1	1	1	1	4	6	15
5	2	2	2	1	7	12	29
6	4	4	3	2	13	23	56
7	8	7	6	4	25	44	108
8	15	14	12	8	49	85	208
9	29	27	23	15	94	164	401
10	56	52	44	29	181	316	773
11	108	100	85	56	349	609	1490
12	208	193	164	108	673	1174	2872
13	401	372	316	208	1297	2263	5536
14	773	717	609	401	2500	4362	10671
15	1490	1382	1174	773	4819	8408	20569
16	2872	2664	2263	1490	9289	16207	39648
17	5536	5135	4362	2872	17905	31240	76424
18	10671	9898	8408	5536	34513	60217	147312
19	20569	19079	16207	10671	66526	116072	283953
20	39648	36776	31240	20569	128233	223736	547337
21	76424	70888	60217	39648	247177	431265	1055026
22	147312	136641	116072	76424	476449	831290	2033628
23	283953	263384	223736	147312	918385	1592363	3919944
24	547337	507689	431265	283953	1770244	3068654	7555935
25	1055026	978602	831290	547337	3412255	5623572	14564533

Table 6: 4-sequences. (i) $S_C \equiv S_3 + S_2 + S_1 + S_0$. (ii) $S_3 \sim S_0 \sim S_S$. (iii) $S_1 \sim S_X$. (iv) For each of these 4-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.92756$.

n	$S_4(5)$	$S_3(5)$	$S_2(5)$	$S_1(5)$	$S_0(5)$	$S_C(5)$	$S_X(5)$	$S_S(5)$
0	0	0	0	0	1	1	0	1
1	0	0	0	1	0	1	1	2
2	0	0	1	0	0	1	2	4
3	0	1	0	0	0	1	3	8
4	1	0	0	0	0	1	4	16
5	1	1	1	1	1	5	10	31
6	2	2	2	2	1	9	20	61
7	4	4	4	3	2	17	39	120
8	8	8	7	6	4	33	76	236
9	16	15	14	12	8	65	149	464
10	31	30	28	24	16	129	294	912
11	61	59	55	47	31	253	578	1793
12	120	116	108	92	61	497	1136	3525
13	236	228	212	181	120	977	2233	6930
14	464	448	417	356	236	1921	4390	13624
15	912	881	820	700	464	3777	8631	26784
16	1793	1732	1612	1376	912	7425	16968	52656
17	3525	3405	3169	2705	1793	14597	33358	103519
18	6930	6694	6230	5318	3525	28697	65580	203513
19	13624	13160	12248	10455	6930	56417	128927	400096
20	26784	25872	24079	20554	13624	110913	253464	786568
21	52656	50863	47338	40408	26784	218049	498297	1546352
22	103519	99994	93064	79440	52656	428673	979626	3040048
23	203513	196583	182959	156175	103519	842749	1925894	5976577
24	400096	386472	359688	307032	203513	1656801	3786208	11749641
25	786568	754784	707128	603609	400096	3257185	7443489	23099186

Table 7: 5-sequences. (i) $S_C \equiv S_4 + S_3 + S_2 + S_1 + S_0$. (ii) $S_4 \sim S_0 \sim S_S$. (iii) For each of these 5-sequences, $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1.96595$.

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