An Inequality Approach of the Collatz Conjecture

Adarsha Chandra

Abstract

We define the Collatz Function F_col:N->N(n) as follows:

F_col(n):= n/2 if n is even, and

F_col(n):= 3n+1 if n is odd

We define the two branches f:N->N and g:N->N of the above function as follows:

f(n):= n/2 if n is even and g(n):= 3n+1 if n is odd

Also, we define the 'functional sequence' of a number n as the set of functions applied consecutively on n (obeying the obtained parities), and show that any two g's in a functional sequence must be separated by at least one f.

Next, we prove that all numbers n, under repetitive execution of the Collatz function, eventually yield a certain E < n. This is obvious for even n values, since-

F_col(n) = n/2 < n for even n

For odd n values, we prove that any odd number n which does not yield an E < n under repetitive execution of the Collatz function, must possess a functional sequence of the form-

S={gfgfgfgfgfgf...}

We then prove that the existence of a number possessing such a functional sequence is not possible, implying that our statement is true for odd numbers as well.

Hence, it follows that any natural number n, under repetitive execution of the Collatz function, must yield an E < n.

The truth of the Collatz Conjecture follows immediately from the above.

The Collatz Conjecture, also referred to as the Ulam Conjecture, the Kakutani Problem, the Thwaites Conjecture, Hasse's Algorithm or the Syracuse Problem, was proposed in 1937 by German Mathematician Lothar Collatz.

The conjecture states that if we define the function

$F_{col}: \mathbb{N} \to \mathbb{N}$ such that-

$$F_{col}(N) \coloneqq \frac{N}{2}, if \;\; N \; is \; even$$

$$F_{col}(N) \coloneqq 3N + 1, if \; N \; is \; odd$$

Then $\forall \, N \in \mathbb{N}, \exists \, k \in \, \mathbb{N}$ such that ${F_{col}}^k(N) = 1$, where

${F_{col}}^k(N) = F_{col}(F_{col}(F_{col}(\ldots(N))\ldots)$, where F_{col} is repeated k times.

This conjecture has been verified to be true for all natural numbers till an approximate value of 2^{68}

Since the posing of the problem, there have been many partial results on it, the most recent of which is the partial result established by mathematician Terence Tao, stating that 'almost' all numbers, under repetitive execution of the function $F_{col}: \mathbb{N} \to \mathbb{N}$, attain almost bounded values. Before that, partial results were established by mathematician Riho Terras in 1976 that 'almost' all numbers x yield an $\Omega < x$, under repetitive execution of the function $F_{col}: \mathbb{N} \to \mathbb{N}$. This upper bound was later improved to $x^{0.869}$ in 1979, and then it was further improved to $x^{0.7925}$ in 1994.

1. SOME NOTATIONS

Let us define $f: \mathbb{N} \to \mathbb{N}(N) := \frac{N}{2}, if\ \ N\ is\ even$

$$g: \mathbb{N} \to \mathbb{N}(N) := 3N + 1, if\ N\ is\ odd$$

Also, let us define the 'functional sequence' for any $N \in \mathbb{N}$ as the set of functions applied consecutively on a certain natural number until the number 1 is obtained, and let S_N denote the functional sequence of N.

For instance, if we have the natural number 5, we have the following continuous mapping obtained by repetitive execution of the functions $f: \mathbb{N} \to \mathbb{N}$ and $g: \mathbb{N} \to \mathbb{N}$, in obedience with the obtained parities:

$$5 \to 16 \to 8 \to 4 \to 2 \to 1$$

Note that the functions applied, in consecutive order, are $g, f, f, f\ and\ f$. Hence, the functional sequence of 5 is:

$$S_5 \equiv \{gffff\}$$

We can also shorten the sequence to:

$$S_5 \equiv \{gf^4\}$$

Since there are 4 consecutive repetitions of $f: \mathbb{N} \to \mathbb{N}$.

2. SOME IMPORTANT RESULTS

<u>Theorem 2.1</u>

$For\ any\ n \in \mathbb{N}, S_n\ does\ not\ contain\ two\ consecutive\ g's.$

$\underline{Proof}$

Let us assume contrarily that $\exists\, n \in \mathbb{N}$ such that S_n contains two consecutive $g's$. Hence, a certain number of executions of $F_{col}: \mathbb{N} \rightarrow \mathbb{N}$ yields a certain $\Omega \in \mathbb{N}$ such that the function $g: \mathbb{N} \rightarrow \mathbb{N}$ is applicable twice. This is possible if and only if both Ω and $g(\Omega) = 3\Omega + 1$ are odd. We know that the subtraction of two odd numbers is always even. Hence, $(3\Omega + 1) - \Omega = 2\Omega + 1$ is even, $\Omega \in \mathbb{N}$. Evidently, this is a contradiction. Thus, the assumption must be incorrect and hence,

$$For\ any\ n \in \mathbb{N}, S_n\ does\ not\ contain\ two\ consecutive\ g's.$$

This concludes the proof □

The above theorem implies that any two consecutive $g's$ are separated by at least one f. Thus, for any odd n, S_n is of the form:

$$S_n \equiv \{gf^{\alpha_1}gf^{\alpha_2}gf^{\alpha_3}gf^{\alpha_4}\ldots\}, where\ \alpha_i \in \mathbb{N}, \forall i \in \mathbb{N}$$

Theorem 2.2

$$\forall\, N \in \mathbb{N}_{\geq 2}, \exists\, k \in \mathbb{N}\ \underline{such\ that}\ {F_{col}}^k(N) < N$$

Proof

The result is obvious for even values of N. This is because if N is even, then $f(N) = \frac{N}{2} < N$. Hence, we can concern ourselves with the odd values of N only.

Let us now consider a certain odd natural number $n \in \mathbb{N}$. Hence, the functional sequence of n must be of the form:

$$S_n \equiv \{gf^{\alpha_1}gf^{\alpha_2}gf^{\alpha_3}gf^{\alpha_4}\ldots\}, where\ \alpha_i \in \mathbb{N}, \forall\ i \in \mathbb{N}$$

Let us define $\Psi_k(n)$ as the value obtained by the execution of the functional sequence:

$$S(k) \equiv \{gf^{\alpha_1} ... gf^{\alpha_k}\}$$

on n.

Now, let us assume contrarily that

$$\forall\, k \in \mathbb{N}, {F_{col}}^k(n) \geq n$$

Hence, it is evident that

$$\Psi_k(n) \geq n, \forall\, k \in \mathbb{N} \qquad (*)$$

<u>Claim 2.2.1</u>

$$\Psi_x(n)\ is\ recursive\ and\ follows\ the\ identity:$$

$$\Psi_{k+1}(n) = \frac{3\Psi_k(n)+1}{2^{\alpha_{k+1}}}, \forall\, k \in \mathbb{N}$$

$\underline{Proof}$

Note that, by definition, we can write

$$\Psi_{k+1}(n) = f^{\alpha_{k+1}}(g(f^{\alpha_k}(g(...(g\left(f^{\alpha_1}(n)\right)...)$$

and, $$\Psi_k(n) = (f^{\alpha_k}(g(...(g\left(f^{\alpha_1}(n)\right)...)$$

The substitution of the second equation into the first gives-

$$\Psi_{k+1}(n) = f^{\alpha_{k+1}}\left(g\left(\Psi_k(n)\right)\right) = f^{\alpha_{k+1}}(3\Psi_k(n) + 1)$$

Hence,

$$\Psi_{k+1}(n) = \frac{3\Psi_k(n)+1}{2^{\alpha_{k+1}}}$$

This concludes the proof □

Claim 2.2.2

$\Psi_k(n)$ *is given by* –

$$\Psi_k(n) = \frac{3^k n + 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_k}}$$

Proof

Notice that for $k = 1$, $\Psi_1(n)$ is the value obtained by the execution of the functional sequence

$$S \equiv \{g f^{\alpha_1}\}$$

on n.

Hence,

$$\Psi_1(n) = f^{\alpha_1}\big(g(n)\big) = f^{\alpha_1}(3n+1) = \frac{3n+1}{2^{\alpha_1}}$$

which is in accordance with our claim.

This serves as the base for an inductive process.

Let us now assume that

$$\Psi_x(n) = \frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}}$$

for a certain $x \in \mathbb{N}$.

Hence, using Claim 2.2.1, we can write $\Psi_{x+1}(n)$ as-

$$\Psi_{x+1}(n) = \frac{3\Psi_k(n)+1}{2^{\alpha_{k+1}}} = \frac{3\left(\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x-1} 3^{x-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_x}}\right)+1}{2^{\alpha_{x+1}}}$$

Hence, $\Psi_{x+1}(n) = \frac{\left(\frac{3^{x+1} n + 3^x + \sum_{i=2}^{x} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}} + 2^{\alpha_1+\ldots+\alpha_x}}{2^{\alpha_1+\ldots+\alpha_x}}\right)}{2^{\alpha_{x+1}}}$

Note that,

$$\sum_{i=2}^{x} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}} + 2^{\alpha_1+\ldots+\alpha_x} = \sum_{i=2}^{x+1} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}$$

Hence,

$$\Psi_{x+1}(n) = \frac{\left(\frac{3^{x+1} n + 3^x + \sum_{i=2}^{x+1} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_x}}\right)}{2^{\alpha_{x+1}}}$$

Which implies,

$$\Psi_{x+1}(n) = \frac{3^{x+1} n + 3^x + \sum_{i=2}^{x+1} 3^{(x+1)-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_{x+1}}}$$

Thus, our claim holds true for $(x+1)$.

Hence, by the principle of mathematical induction,

$$\forall\, k \in \mathbb{N},\ \Psi_k(n) = \frac{3^k n + 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_k}}$$

This concludes the proof □

<u>Claim 2.2.3</u>

Let $\Phi: \mathbb{N} \to \mathbb{N}$ *be any arbitrary function such that*

$$\Psi_x(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

Then,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n) - 1}{3}$$

<u>Proof</u>

We have,

$$\Psi_x(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

Hence,

$$\Psi_x(n) \geq \Phi(n) \text{ and } \Psi_{x+1}(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

This, along with <u>Claim 2.2.2</u> gives,

$$\Psi_x(n) = \frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} . 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}} \geq \Phi(n) \qquad (**)$$

$$\Psi_{x+1}(n) = \frac{3^{x+1} n + 3^x + \sum_{i=2}^{x+1} 3^{(x+1)-i} . 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_{x+1}}} \geq \Phi(n) \qquad (***)$$

Multiplying both sides of the inequality established in $(***)$ by $\frac{2^{\alpha_{x+1}}}{3}$;

$$\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x+1} 3^{x-i} \cdot 2^{\alpha_1 + \ldots + \alpha_{i-1}}}{2^{\alpha_1 + \ldots + \alpha_x}} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

Which implies,

$$\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}} + 3^{-1} 2^{\alpha_1+\ldots+\alpha_x}}{2^{\alpha_1+\ldots+\alpha_x}} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

Thus,

$$\frac{3^x n + 3^{x-1} + \sum_{i=2}^{x} 3^{x-i} \cdot 2^{\alpha_1+\ldots+\alpha_{i-1}}}{2^{\alpha_1+\ldots+\alpha_x}} + 3^{-1} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

Hence,

$$\Psi_x(n) + \frac{1}{3} \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n)}{3}$$

And thus,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n) - 1}{3}$$

Hence,

If $\Phi: \mathbb{N} \to \mathbb{N}$ *is an arbitrary function such that*

$$\Psi_x(n) \geq \Phi(n), \forall\, x \in \mathbb{N}$$

Then,

$$\Psi_x(n) \geq \frac{2^{\alpha_{x+1}} \cdot \Phi(n) - 1}{3}$$

This concludes the proof □

Let us choose the function $\Phi(n) = n$. We can make this selection because-

$$\Psi_k(n) \geq n, \forall\, k \in \mathbb{N}$$

Also, let us define the function

$$\varphi(x,\eta) = \frac{2^{\alpha_{x+1}} \cdot \eta - 1}{3}$$

Hence, Claim 2.2.3 implies-

$$\Psi_x(n) \geq \varphi(x,n)$$

We can apply this argument over and over again, to state that-

$$\Psi_x(n) \geq \varphi^k(x,n), \forall\, k \in \mathbb{N}$$

Claim 2.2.4

$$\alpha_i = 1, \forall\, i \in \mathbb{N}$$

Proof

It can easily be verified that $\alpha_1 = 1$, since-

$$\Psi_1(n) = \frac{3n+1}{2^{\alpha_1}} \geq n$$

can hold true if and only if $\alpha_1 = 1$

Let us now assume contrarily that for some $j \in \mathbb{N}_{\geq 2}$, we have-

$$\alpha_j \geq 2$$

Thus,

$$2^{\alpha_j} \geq 4$$

And hence,

$$2^{\alpha_j}\eta \geq 4\eta, \forall\, \eta \in \mathbb{N}$$

Thus,

$$2^{\alpha_j}\eta - 1 \geq 4\eta - 1, \forall\, \eta \in \mathbb{N}$$

And thus,

$$\frac{2^{\alpha_j}\eta - 1}{3} \geq \frac{4\eta - 1}{3}, \forall\, \eta \in \mathbb{N}$$

But, $\eta \in \mathbb{N}$. Hence,

$$\eta \geq 1$$

Thus,

$$4\eta - 3\eta \geq 1$$

And hence,

$$4\eta \geq 3\eta + 1$$

Hence,

$$4\eta - 1 \geq 3\eta$$

Which implies,

$$\frac{4\eta - 1}{3} \geq \eta$$

Hence, we have,

$$\frac{2^{\alpha_j}\eta - 1}{3} \geq \frac{4\eta - 1}{3} \; and \frac{4\eta - 1}{3} \geq \eta$$

Thus,

$$\frac{2^{\alpha_j}\eta - 1}{3} \geq \eta$$

But, by definition,

$$\varphi(j-1, \eta) = \frac{2^{\alpha_j}\eta - 1}{3}$$

Thus,

$$\varphi(j-1, \eta) \geq \eta, \forall \, \eta \in \mathbb{N}$$

We can use the above argument over and over again to state that-

$$\varphi^{k+1}(j-1,\eta) \geq \varphi^{k}(j-1,\eta), \forall\, k,\eta \in \mathbb{N}$$

Hence,

$$\lim_{k\to\infty} \varphi^{k}(j-1,\eta) \to \infty, \forall\, \eta \in \mathbb{N}$$

But,

$$\Psi_x(n) \geq \varphi^{k}(x,n), \forall\, k \in \mathbb{N}$$

Thus,

$$\Psi_{j-1}(n) \geq \lim_{k\to\infty} \varphi^{k}(j-1,n)$$

This implies that-

$$\Psi_{j-1}(n) \to \infty$$

But, Claim 2.2.1 suggests that-

$$If\ \Psi_{k+1}(n) \to \infty\ then\ \Psi_k(n) \to \infty$$

We can use the above argument over and over again to argue that-

$$\Psi_1(n) \to \infty$$

Which is possible if and only if

$$n \to \infty$$

Which is a contradiction. Hence, our assumption must be incorrect and hence,

$$\alpha_i = 1, \forall\, i \in \mathbb{N}$$

This concludes the proof □

Claim 2.2.5

$$\nexists\, n \in \mathbb{N}\ having\ the\ functional\ sequence\ -$$

$$S_n \equiv \{gfgfgfgfgfgf\ldots\}$$

Proof

Let us assume contrarily that $\exists\ n \in \mathbb{N}$ having the functional sequence-

$$S_n \equiv \{gfgfgfgfgfgf \dots\}$$

Thus,

$$\alpha_i = 1, \forall\ i \in \mathbb{N}$$

Thus, Claim 2.2.2 implies that $\forall\ k \in \mathbb{N}$ -

$$\Psi_k(n) = \frac{3^k n + 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{\alpha_1 + \dots + \alpha_{i-1}}}{2^{\alpha_1 + \dots + \alpha_k}} = \frac{3^k n + 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{i-1}}{2^k}$$

Now, note that the sum-

$$S(k) \coloneqq 3^{k-1} + \sum_{i=2}^{k} 3^{k-i} \cdot 2^{i-1} = \sum_{i=1}^{k} 3^{k-i} \cdot 2^{i-1}, \forall\ k \in \mathbb{N}$$

is the sum of the first k terms of a Geometric Progression, starting with 3^{k-1} and having a common ratio $\left(\frac{2}{3}\right)$. Hence, $\forall\ k \in \mathbb{N}$,

$$S(k) = 3^{k-1} \left(\frac{1 - \left(\frac{2}{3}\right)^k}{1 - \frac{2}{3}} \right) = 3^{k-1} \left(\frac{1 - \left(\frac{2}{3}\right)^k}{\frac{1}{3}} \right) = 3^k \left(1 - \left(\frac{2}{3}\right)^k \right)$$

Hence,

$$S(k) = 3^k - 2^k, \forall\ k \in \mathbb{N}$$

Thus,

$$\Psi_k(n) = \frac{3^k n + S}{2^k} = \frac{3^k n + 3^k - 2^k}{2^k} = \left(\frac{3^k n + 3^k}{2^k} \right) - 1, \forall\ k \in \mathbb{N}$$

This implies,

$$\Psi_k(n) = \left(\frac{3^k(n+\ 1)}{2^k}\right) - 1, \forall\ k \in \mathbb{N}$$

Note that

$$\Psi_k(n) \in \mathbb{N}, \forall\ k \in \mathbb{N}$$

Thus,

$$\left(\frac{3^k(n+\ 1)}{2^k}\right) \in \mathbb{N}, \forall\ k \in \mathbb{N}$$

Hence,

$$2^k | 3^k(n+\ 1), \forall\ k \in \mathbb{N}$$

But, $\forall\ k \in \mathbb{N}$, $2^k \nmid 3^k$. Hence,

$$2^k | (n+\ 1), \forall\ k \in \mathbb{N}$$

Thus,

$$\lim_{x\to\infty} 2^x\ | (n+\ 1)$$

Which is possible if and only if

$$n \to \infty$$

Which is a contradiction. Hence, our assumption must be incorrect and hence,

$$\nexists\ n \in \mathbb{N}\ having\ the\ functional\ sequence -$$

$$S_n \equiv \{gfgfgfgfgfgf\ldots\}$$

This concludes the proof □

Now, note that <u>Claim 2.2.4</u> is in contradiction with <u>Claim 2.2.5</u>. Thus, our assumption must be incorrect, and thus,

$$\forall N \in \mathbb{N}_{\geq \mathbf{2}}, \exists k \in \mathbb{N} \underline{\textit{ such that}} \; F_{col}{}^{k}(N) < N$$

This concludes the proof □

Remark

The Collatz Conjecture follows immediately from Theorem 2.2 due to the principle of mathematical induction. Theorem 2.2 suggests that every number $n \in \mathbb{N}$ yields an $\acute{\varepsilon} < n$. This argument can be used over and over again to argue that all natural numbers eventually yield 1.

3. PROOF OF THE COLLATZ CONJECTURE

Theorem 3.1

The Collatz Conjecture is true. In other words, if we define the function

$F_{col}: \mathbb{N} \rightarrow \mathbb{N}$ such that-

$$F_{col}(N) \coloneqq \frac{N}{2}, \textit{if } N \textit{ is even}$$

$$F_{col}(N) \coloneqq 3N + 1, \textit{if } N \textit{ is odd}$$

Then $\forall N \in \mathbb{N}, \exists k \in \mathbb{N}$ such that $F_{col}{}^{k}(N) = 1$

Proof

The Collatz Conjecture has been verified to be true for all natural numbers till an approximate value of 2^{68}. This serves as an appropriate base for an inductive process.

Let us now assume that the Collatz Conjecture is true $\forall N \in \mathbb{N}_{\leq \beta}$,

$\beta \in \mathbb{N}$.

Thus, $\forall m \in \mathbb{N}_{\leq \beta}, \exists \gamma_m \in \mathbb{N}$ such that

$${F_{col}}^{\gamma_m}(m) = 1$$

Let us now consider the case of $N = (\beta + 1)$. Thus, Theorem 2.2 implies that

$$\exists\ \delta \in \mathbb{N}\ \underline{such\ that}\ {F_{col}}^{\delta}(\beta + 1) < \beta + 1$$

This implies that-

$$\exists\ \delta \in \mathbb{N}\ \underline{such\ that}\ {F_{col}}^{\delta}(\beta + 1) = þ \leq \beta \qquad (****)$$

Also, note that $þ \leq \beta$.Thus, $þ \in \mathbb{N}_{\leq \beta}$ and hence, $\exists \gamma_{þ} \in \mathbb{N}$ such that-

$${F_{col}}^{\gamma_{þ}}(þ) = 1 \qquad (*****)$$

Now, from $(****)$,

$${F_{col}}^{\delta}(\beta + 1) = þ$$

Hence,

$${F_{col}}^{\delta+\gamma_{þ}}(\beta + 1) = {F_{col}}^{\gamma_{þ}}(þ)$$

But, from $(*****)$, ${F_{col}}^{\gamma_{þ}}(þ) = 1$. Hence,

$${F_{col}}^{\delta+\gamma_{þ}}(\beta + 1) = 1$$

Also, note that $\delta + \gamma_{þ} \in \mathbb{N}$. Thus, $\exists\ \gamma_{\beta+1} = \delta + \gamma_{þ} \in \mathbb{N}$ such that-

$${F_{col}}^{\gamma_{\beta+1}}(\beta + 1) = 1$$

Implying that the conjecture holds true for $(\beta + 1)$.

Hence, by the principle of mathematical induction, if we define the function

$F_{col}: \mathbb{N} \rightarrow \mathbb{N}$ such that-

$$F_{col}(N) \coloneqq \frac{N}{2}, \text{if } N \text{ is even}$$

$$F_{col}(N) \coloneqq 3N + 1, \text{if } N \text{ is odd}$$

Then $\forall\, N \in \mathbb{N}, \exists\, k \in \mathbb{N}$ such that ${F_{col}}^{k}(N) = 1$, implying that the Collatz Conjecture is true!