# On Fermat's Last Theorem (III)


Richard Wayte

29 Audley Way, Ascot, Berkshire SL5 8EE, England, UK

e-mail: rwayte@googlemail.com



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**Abstract.** Fermat's Last Theorem is proved using elementary arithmetic.


## 1. Introduction

Fermat's Last Theorem was formulated in 1637 and not proved until Andrew Wiles [1] did so in 1995. Over the years, enthusiasts have been encouraged by the simplicity of the theorem to prove it using elementary arithmetic [2].

**Theorem**

No three positive integers a, b, c, can satisfy the equation:

$$c^p = a^p + b^p \tag{1}$$

if (p) is an integer greater than two.

## 2. Proof for (p = 3)

Given the equation

$$c^3 = a^3 + b^3, \tag{2.0}$$

let (e) be a positive integer, and set up two expressions

$$[F(a,e) = c^3 - a^3 - e^3] \quad = \quad [b^3 - e^3 = F(b,e)]. \tag{2.1}$$

For the F(a,e) term, substitute

$$c = (a + e), \text{ and } h = (3e), \tag{2.1a}$$

then reduce to

$$F(a,e) = a(ha + 3e^2)\ . \tag{2.1b}$$

For the F(b,e) term, let (q) and (m) be real numbers such that

$$b = (q + e)\ , \tag{2.2}$$

$$m = (q + 3e) = (b + 2e), \tag{2.3a}$$

then reduce to

$$F(b,e) = q(mq + 3e^2). \tag{2.3b}$$

Now, equate 4xF(a,e) to 4xF(b,e), and expand

$$(1/h) \times \{(2ha)(2ha + 6e^2)\} = (1/m) \times \{(2mq)(2mq + 6e^2)\}\,. \tag{2.4a}$$

Make this expression more symmetrical by substituting

$$X = (2ha + 3e^2), \tag{2.4b}$$

$$Y = (2mq + 3e^2), \tag{2.4c}$$

then substitute ($E = 3e^2$), and reduce to

$$(1/h) \times \{(X - E)(X + E)\} = (1/m) \times \{(Y - E)(Y + E)\}. \tag{2.4d}$$

This equation is equivalent to Eq.(2.1), so for an integer (a) the left side will evaluate to an integer; and for an integer (b) the right side will evaluate to a different integer. However, *a balanced all-integer equation* can be invented by changing the definitions in Eq.(2.4b,c). Auspiciously, a worked example of this trick will reveal why the proper definitions always produce non-integers (Y,b) when (X,a) are made integers. For example, start with an arbitrary expression comprising integers such as

$$\{12 \times 108\} = \{18 \times 72\}\,, \tag{2.5a}$$

then calculate arithmetic means and expand thus

$$\{(60 - 48) \times (60 + 48)\} = \{(45 - 27) \times (45 + 27)\}\,. \tag{2.5b}$$

Express this in the same format as Eq.(2.4d)

$$(1/27^2) \times \{(60 - 48)(60 + 48)\} = (1/48^2) \times \{(80 - 48)(80 + 48)\}\,. \tag{2.5c}$$

Here, every factor is an integer traceable back to Eq.(2.5a), and the ones analogous to Eq.(2.4d) are ($\tilde{X} = 60$), ($\tilde{E} = 48$), ($\tilde{Y} = 80$). However, [$\tilde{h}$] employs [27 = (72 - 18)/2] from the right side of Eq.(2.5b) and [$\tilde{m}$] employs [48 = (108 - 12)/2] from the left side, in stark contrast to Eq.(2.4d) employing Eq.(2.1a) and Eq.(2.3a). Therefore, these ($\tilde{h}$, $\tilde{m}$) cannot be related to ($\tilde{X}$, $\tilde{Y}$) in the way given by Eq.(2.4b,c). Accordingly, Eq.(2.5a) is the *only way* to invent an all-integer equation like (2.5c) which contains an integer ($\tilde{Y}$) with an integer ($\tilde{X}$), but it is totally incompatible with the derivation of Eq.(2.4d). That is, Eq.(2.4d) is not able to revert to the form Eq.(2.5a) containing only integers.

Consequently, Eq.(2.4d) cannot contain an integer (Y) with an integer (X), so *(b) is not an integer if (a) is an integer; which means that Eq.(1) is proved for (p = 3).*

## 3. Proof for (p = 4)

Given the equation

$$c^4 = a^4 + b^4, \qquad (3.0)$$

let (e) be a positive integer, then set up two equal expressions

$$[F(a,e) = c^4 - a^4 - e^4] = [b^4 - e^4 = F(b,e)]. \qquad (3.1)$$

For F(a,e), substitute

$$c = (a + e), \text{ and } \quad H = (4ae + 6e^2), \qquad (3.1a)$$

then reduce to

$$F(a,e) = a\{Ha + 4e^3\}. \qquad (3.1b)$$

For F(b,e), substitute

$$b = (q + e), \qquad (3.2)$$

$$M = (q^2 + 4eq + 6e^2), \qquad (3.3a)$$

then reduce to

$$F(b,e) = q\{Mq + 4e^3\}. \qquad (3.3b)$$

Now, equate 4xF(a,e) to 4xF(b,e) and expand

$$(1/H) \times \{\, 2Ha(2Ha + 8e^3) \,\} = (1/M) \times \{\, 2Mq(2Mq + 8e^3) \,\}. \qquad (3.4a)$$

Make this more symmetrical by substituting

$$X = (2Ha + 4e^3), \qquad (3.4b)$$

$$Y = (2Mq + 4e^3), \qquad (3.4c)$$

then substitute ($E = 4e^3$) and reduce to

$$(1/H) \times \{(X - E)(X + E)\} = (1/M) \times \{(Y - E)(Y + E)\}. \qquad (3.4d)$$

This equation is identical in format to Eq.(2.4d) although factors are defined differently. All the logical argument which followed Eq.(2.4d) will lead to the same conclusion. That is, genuine values of (Y) calculated from Eq.(3.4c) do not occur in expressions of the form Eq.(2.5c) which is the *unique all-integer* format required for getting an integer (Y) with integer (X).

Thus, Eq.(3.4d) cannot contain an integer (Y) with an integer (X), so *(b) cannot be an integer if (a) is an integer; which means that Eq.(1) is proved for (p = 4).*

## 4. Proof for (p = 5)

Given the equation:

$$c^5 = a^5 + b^5, \qquad (4.0)$$

let (e) be a positive integer then set up two equal expressions

$$[F(a,e) = c^5 - a^5 - e^5\,] = \;[\,b^5 - e^5 = F(b,e)]\,. \qquad (4.1)$$

For F(a,e) substitute

$$c = (a + e),\ \text{and}\quad H = (5a^2e + 10ae^2 + 10e^3)\,, \qquad (4.1a)$$

then reduce to

$$F(a,e) = a\{Ha + 5e^4\}. \qquad (4.1b)$$

For F(b,e), substitute

$$b = (q + e), \qquad (4.2)$$

$$M = (q^3 + 5eq^2 + 10e^2q + 10e^3), \qquad (4.3a)$$

then reduce to

$$F(b,e) = q\{Mq + 5e^4\}\,. \qquad (4.3b)$$

Now, equate 4xF(a,e) to 4xF(b,e), and expand

$$(1/H) \times \{\,2Ha(2Ha + 10e^4)\,\} = \;(1/M) \times \{\,2Mq(2Mq + 10e^4)\,\}\,. \qquad (4.4a)$$

Make this more symmetrical by substituting

$$X = (2Ha + 5e^4), \qquad (4.4b)$$

$$Y = (2Mq + 5e^4), \qquad (4.4c)$$

then substitute ($E = 5e^4$) and reduce Eq.(4.4a) to

$$(1/H) \times \{(X - E)(X + E)\} \;=\; (1/M) \times \{(Y - E)(Y + E)\}. \qquad (4.4d)$$

This equation is identical in form to Eq.(2.4d) although factors are defined differently. All the logical argument following Eq.(2.4d) will lead to the same conclusion.

Thus, Eq.(4.4d) cannot contain an integer (Y) with an integer (X), so *(b) cannot be an integer if (a) is an integer; which means that Eq.(1) is proved for (p = 5).*

## 5. Proof for ( p > 2 )

Proofs for (p = 7, 11, 13) have been performed successfully, so a general proof for (p > 2) will be proposed as follows. Given the equation

$$c^p = a^p + b^p, \tag{5.0}$$

let (e) be a positive integer, then set up two equal expressions

$$[\ F(a, e) = c^p - a^p - e^p\ ] = \ [\ b^p - e^p = F(b, e)]. \tag{5.1}$$

For F(a,e), substitute

$$c = (a + e), \text{ and } \ H = [\{(a + e)^p - a^p - e^p\} - ape^{p-1}\ ]/a^2, \tag{5.1a}$$

then reduce to

$$F(a, e) = a\{Ha + pe^{p-1}\}. \tag{5.1b}$$

For F(b,e), substitute

$$b = (q + e), \tag{5.2}$$

$$M = [\{(q + e)^p - e^p\} - qpe^{p-1}\ ]/q^2\ , \tag{5.3a}$$

then reduce to

$$F(b, e) = q\{Mq + pe^{p-1}\}. \tag{5.3b}$$

Now, equate 4xF(a,e) to 4xF(b,e), and expand

$$(1/H) \times \{2Ha(2Ha + 2pe^{p-1})\} \ = \ (1/M) \times \{2Mq(2Mq + 2pe^{p-1})\}. \tag{5.4a}$$

Make this more symmetrical by substituting

$$X = (2Ha + pe^{p-1}), \tag{5.4b}$$

$$Y = (2Mq + pe^{p-1}), \tag{5.4c}$$

then substitute ($E = pe^{p-1}$) and reduce to

$$(1/H) \times \{(X - E)(X + E)\} \ = \ (1/M) \times \{(Y - E)(Y + E)\}. \tag{5.4d}$$

This equation is identical in form to Eq.(2.4d) although factors are defined differently. All the logical argument following Eq.(2.4d) will lead to the same conclusion.

Thus, Eq.(5.4d) cannot contain an integer (Y) with an integer (X), so *(b) cannot be an integer if (a) is an integer; which means that Eq.(1) is proved for (p >2).*

## 6. Conclusion

The simplicity of Fermat's Last Theorem stimulated a search for a simple proof. First, the cubic equation was transformed into an equation with new variables (X,a), (Y,b) in a balanced symmetrical format. As expected, (X) and (Y) could not both be integers, so to explain why, an independent but similar *all-integer equation* was invented. By comparing analogous factors of the two equations, it could be seen that the cubic equation would never be compatible with the unique *all-integer equation*. That is, the Theorem was proved for ($p = 3$).

The quartic and quintic equations were also transformed into symmetrical expressions which could not satisfy the analogous all-integer expression. Finally, the analysis was performed for the general ($p > 2$) case, with the same result, thereby completing the proof of Fermat's Last Theorem.

## References

[1] Wiles, A.J. (1995) Annals of Mathematics 141, No.3, pp 443-551

[2] Wikipedia. https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem