Proof of the Strong Goldbach Conjecture

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Abstract: The article provides a proof of the Strong (binary) Goldbach conjecture based on the regularities of the interval between successive primes.
Key words: strong, binary, Goldbach's conjecture, prime numbers, interval.

1 Introduction

Goldbach's conjecture is the statement that any even number, starting from 4, can be represented as the sum of two prime numbers.

Christian Goldbach, in 1742, sent a letter to Leonhard Euler [1], in which he made the following assumption: every odd number greater than 5 can be represented as the sum of three prime numbers. Euler became interested in the problem and put forward a stronger conjecture: every even number greater than two can be represented as the sum of two prime numbers. The first statement is called the Weak (sometimes ternary) Goldbach conjecture, the second is called the Strong (sometimes binary) Goldbach conjecture.

The Goldbach conjecture is one of the most famous open mathematical problems, included in the legendary list of Hilbert's problems [2] and is one of the few Hilbert problems still unsolved. This conjecture is also included in the list of four important mathematical problems of Landau [3].

2 Matrix of even numbers

First, we will investigate the sums of two odd numbers, for this we will compose a matrix of even numbers obtained by summing two numbers of the form $k = 6\text{n} \mp 1$, $\text{n} \in \mathbb{N}$, which is shown in Table 1 (see Appendix 1). Note that all prime numbers greater than 3 have the form $k = 6\text{n} \mp 1$, so the above matrix contains all even numbers that can be represented as the sum of two prime numbers, except for even numbers, in which one of the terms is 3. The even number is also not used in the matrix The prime number is 2, because if you add 2 to a prime number greater than 2, then you get an odd number.

Note that the set of numbers, each element of which is the sum of two numbers of the form $k = 6\text{n} \mp 1$, includes the complete set of even numbers starting from the number 10. It can also be noted that if for each member of the sequence of odd numbers of the form $k = 6\text{n} \mp 1$, $\text{n} \in \mathbb{N}$, add any pair of odd numbers $p = 6n - 1$ and $p = 6n + 1$, then the resulting set of numbers certainly includes the complete set of even numbers starting from a certain number.

For example, if for each member of a sequence of odd numbers of the form $k = 6\text{n} \mp 1$, $\text{n} \in \mathbb{N}$, we add a pair of odd numbers 11 and 13, which are numbers of the form $k = 6\text{n} \mp 1$, then the resulting set of numbers will include the complete set of even numbers starting from the number 16. Therefore, the above matrix of even numbers, obtained by summing two numbers of the form $k = 6\text{n} \mp 1$, contains many copies of each even number, and the larger the even number, the more copies it will have in the matrix.

As can be seen from the matrix, all identical even numbers obtained as a result of summing two odd numbers of the form $k = 6\text{n} \mp 1$ are located on the cells of the matrix, which form secondary diagonals.

The main diagonal of the matrix divides all even numbers into two identical parts, so we will consider even numbers located on the cells of the secondary diagonals, starting from the cell of the first column of the matrix (corresponding to the number 5) to the main diagonal, including the cell of the main diagonal. In other words, we will be interested in even numbers located on the main diagonal and below it. Even numbers located on the main diagonal of the matrix are expressed by two arithmetic progressions $10 + 12t$ and $14 + 12t$, $t = 0,1,2, \dots$. Table 1 shows that some secondary diagonals contain even numbers of only one type, while others contain paired even numbers, and these diagonals certainly alternate.

In the matrix (Table 1), rows and columns corresponding to composite odd numbers are highlighted in gray, so the matrix represents alternations of light and dark bands. If several dark bands are located side by side, which corresponds to a large interval between prime numbers, then dark zones are formed in the matrix. Even numbers located in light zones have a representation as the sum of two prime numbers, while other even numbers located in dark zones do not have such a representation.

3 Interval between two consecutive primes

It is not difficult to see that if the interval between two consecutive prime numbers is very large, then the probability that there will be an even number that cannot be represented as the sum of two prime numbers will be high. Obviously, if half of the diagonal with even numbers is completely in the dark strip, then the even numbers located on this diagonal will not be represented as the sum of two prime numbers. It is impossible to find a dark zone that completely covers half of at least one secondary diagonal with even numbers, since even anomalously long intervals exceeding the average intervals between two consecutive prime numbers by several tens of times can cover only a tiny part of the secondary diagonal. For example, for 2022, the most anomalous interval between prime numbers of length 8350, which differs from the average interval by almost 42 times, is behind the 87-digit prime number [4], i.e., the prime number is larger than the interval by about 10^{83} times. This means that any even number greater than 4 can be represented as the sum of two prime numbers.

Further, to confirm what has been said, we will prove that any interval between consecutive prime numbers cannot completely cover half of the secondary diagonal of the matrix of even numbers, on which copies of any even number are located.

4 Bertrand interval

Since there is no proven size of the interval between primes, we will first rely on Bertrand's postulate. According to Bertrand's postulate, for any natural number $n \geq 2$ there is a prime number p in the interval $n < p < 2n$. This postulate, formulated by Bertrand in 1845 [5], was proved in 1852 by Chebyshev [6]. For convenience, the interval between natural numbers of size $[n, 2n]$ is called the Bertrand interval.

We consider numbers of the form $k = 6\text{n} \mp 1$, so taking into account Bertrand's postulate, we can study the intervals $[k, 2k]$. Note that according to Bertrand's postulate, the boundary numbers k and $2k$ are not included in the length of the interval. Note that if the numerical segment is assigned in the form $[k, 2k]$, then the second boundary number will not have the form $k = 6\text{n} \mp 1$, so the number equal to $2k$ is not visible in Table 1.

Find intervals between two natural numbers where there is only one prime number, i.e., corresponding to Bertrand's postulate is impossible, since there are a lot of prime numbers between the natural numbers n and $2n$, which has been proved by many scientists, for example, one can refer to the works [7, 8, 9, 10]. Therefore, as an example, we take the Bertrand interval equal to [47, 94], which corresponds to the fragment of the matrix shown in Figure 1. Note that the number 94 is not a number of the form $k = 6\text{n} \mp 1$, therefore it is not visible in Table 1, therefore instead, the boundary number in the fragment of Table 1 shows the number 95, which is closest to 94.

First, we will assume that there is not a single prime number between the numbers k and $2k$ and call it the “dark zone”, i.e., for our example, we will assume that between the numbers 47 and 94 there is not a single prime number.

As can be seen from Figure 1, if we assume that there are no prime numbers between the boundary numbers 47 and 94, then only 3 even numbers (92, 96 and 98) will not be represented as the sum of two prime numbers. This is because all copies of these even numbers are in the dark zone or above the main diagonal when other even numbers in the dark zone have at least one instance in the light zone. The above 3 even numbers, all copies of which are completely in the dark zone, will be called disputed even numbers.

14	**43**	48	50	54	56	60	62	66	68	72	74	78	80	84	86	90	92
15	**47**	52	54	58	60	64	66	70	72	76	78	82	84	88	90	94	96
16	**49**	54	56	60	62	66	68	72	74	78	80	84	86	90	92	96	98
17	**53**	58	60	64	66	70	72	76	78	82	84	88	90	94	96	100	102
18	**55**	60	62	66	68	72	74	78	80	84	86	90	92	96	98	102	104
19	**59**	64	66	70	72	76	78	82	84	88	90	94	96	100	102	106	108
20	**61**	66	68	72	74	78	80	84	86	90	92	96	98	102	104	108	110
21	**65**	70	72	76	78	82	84	88	90	94	96	100	102	106	108	112	114
22	**67**	72	74	78	80	84	86	90	92	96	98	102	104	108	110	114	116
23	**71**	76	78	82	84	88	90	94	96	100	102	106	108	112	114	118	120
24	**73**	78	80	84	86	90	92	96	98	102	104	108	110	114	116	120	122
25	**77**	82	84	88	90	94	96	100	102	106	108	112	114	118	120	124	126
26	**79**	84	86	90	92	96	98	102	104	108	110	114	116	120	122	126	128
27	**83**	88	90	94	96	100	102	106	108	112	114	118	120	124	126	130	132
28	**85**	90	92	96	98	102	104	108	110	114	116	120	122	126	128	132	134
29	**89**	94	96	100	102	106	108	112	114	118	120	124	126	130	132	136	138
30	**91**	96	98	102	104	108	110	114	116	120	122	126	128	132	134	138	140
31	**95**	100	102	106	108	112	114	118	120	124	126	130	132	136	138	142	144
32	**97**	102	104	108	110	114	116	120	122	126	128	132	134	138	140	144	146

Figure 1. Fragment of Table 1

Note that the number of disputed even numbers does not depend on the size of the Bertrand interval. For example, if we consider the interval between the numbers 55 and 110 ($55 \cdot 2=110$), then the disputed even numbers will be 112, 114, 116, and for the intervals 65 and 130 ($65 \cdot 2=130$) and 79 and 158 ($79 \cdot 2=158$) disputed even numbers will be 128, 132, 134 and 160, 162, 164 respectively.

Since, according to Bertrand's Postulate, there must be at least one prime number between the numbers 47 and 94, then we will further consider the cases when one prime number appears in the dark zone. Of course, between the numbers 47 and 94 there are actually 9 prime numbers (53, 59, 61, 67, 71, 73, 79, 83, 89), but we will consider them conditionally composite numbers.

As can be seen from Table 1 and Figure 1, if in the middle part of the Bertrand interval there are two prime numbers of different types ($p = 6n - 1$ and $p = 6n + 1$), then all even numbers included in the Bertrand interval will be represented as the sum of two prime numbers. If these two prime numbers correspond to the extreme numbers of the Bertrand interval, or both prime numbers are of the same type, then in some cases one number out of three may remain in the dark zone.

Thus, even if based on Bertrand's postulate, it can be argued that all even numbers can be represented as the sum of two prime numbers, except in special cases. We repeat that the dark zone corresponding to the Bertrand interval cannot exist, since there will be not one or two, but a lot of prime numbers on the Bertrand interval, and the larger the interval, the more prime numbers there will be on the Bertrand interval, this has been proven by many scientists.

5 Proof of the strong Goldbach conjecture

It is known that in 1952 Jitsuro Nagura proved that for $n \geq 25$ there is always a prime number between natural numbers n and $(1+\frac{1}{5})n$ [7]. It follows from the works of Jitsuro Nagura that the Bertrand interval contains at least 4 primes, since the Bertrand interval is equal to almost four Nagura intervals. Note that we called n and $(1+\frac{1}{5})n$ the Nagura interval. In subsequent years, the result of Jitsuro Nagura was improved by various scientists [8, 9, 10], in particular, Pierre Dusart in 2016 proved that for $x \geq 468\,991\,632$, there is at least one prime in the interval $x < p \leq (1+\frac{1}{5000\ln^2 x})x$ [9]. This equation for $x = 468\,991\,632$ is approximately $468\,991\,632 < p \leq (1+\frac{1}{100\,000})468\,991\,632$. This means that if $n \geq 468\,991\,632$, then, according to the Pierre Dusart formula, the Bertrand interval contains approximately 100,000 primes. Note that if we count the number of primes in the interval between the numbers 468 991 632 and 937 983 264 using the formula $\pi(x) = \frac{x}{\ln(x)}$, then there are approximately 23 779 818 primes in this interval.

It should be noted, as mentioned earlier, that the prime numbers present on the Bertrand interval must be of different types $p = 6n - 1$ and $p = 6n + 1$, otherwise, there is a possibility that at least one even number will not have a representation as the sum of two prime numbers. Therefore, the question may arise: can prime numbers of only one type ($p = 6n - 1$ or $p = 6n + 1$) exist on the Bertrand interval?

It is known that the number of primes of the form $p = 4n + 3$ is greater than the number of primes of the form $p = 4n + 1$, i.e., there are more prime numbers with remainder 3 modulo 4 than there are prime numbers with remainder 1, this phenomenon is called the Chebyshev deviation [11]. Information about the deviation of prime numbers of the types $p = 6n - 1$ and и $p = 6n + 1$ was not found, which means that there will be an equal number of prime numbers of the types $p = 6n - 1$ and и $p = 6n + 1$ on the Bertrand interval. Note that the Chebyshev deviation is not significant, so it cannot affect the distribution of primes of the form $p = 6n - 1$ and и $p = 6n + 1$ on the Bertrand interval, moreover, each of these two types of primes $p = 4n + 3$ and $p = 4n + 1$, depending on the value of n, they will correspond to $p = 6n - 1$ and и $p = 6n + 1$. Thus, it can be argued that the number of prime numbers of two types $6n - 1$ and и $p = 6n + 1$, on the Bertrand interval will be approximately equal.

From the above it follows that on the Bertrand interval there is a very large number of primes of both types $p = 6n - 1$ and и $p = 6n + 1$compared compared to the minimum required number of prime numbers equal to approximately 2, and the larger the interval, the greater the number of primes. This means that the Strong Goldbach Conjecture is true and it proven.

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Annex 1.

Table 1. Representation of even numbers as two odd numbers of the form $k = 6n \mp 1$

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
		5	**7**	**11**	**13**	**17**	**19**	**23**	**25**	**29**	**31**	**35**	**37**	**41**	**43**	**47**	**49**	**53**	**55**	**59**	**61**	**65**	**67**	**71**
1	**5**	10	12	16	18	22	24	28	30	34	36	40	42	46	48	52	54	58	60	64	66	70	72	76
2	**7**	12	14	18	20	24	26	30	32	36	38	42	44	48	50	54	56	60	62	66	68	72	74	78
3	**11**	16	18	22	24	28	30	34	36	40	42	46	48	52	54	58	60	64	66	70	72	76	78	82
4	**13**	18	20	24	26	30	32	36	38	42	44	48	50	54	56	60	62	66	68	72	74	78	80	84
5	**17**	22	24	28	30	34	36	40	42	46	48	52	54	58	60	64	66	70	72	76	78	82	84	88
6	**19**	24	26	30	32	36	38	42	44	48	50	54	56	60	62	66	68	72	74	78	80	84	86	90
7	**23**	28	30	34	36	40	42	46	48	52	54	58	60	64	66	70	72	76	78	82	84	88	90	94
8	**25**	30	32	36	38	42	44	48	50	54	56	60	62	66	68	72	74	78	80	84	86	90	92	96
9	**29**	34	36	40	42	46	48	52	54	58	60	64	66	70	72	76	78	82	84	88	90	94	96	100
10	**31**	36	38	42	44	48	50	54	56	60	62	66	68	72	74	78	80	84	86	90	92	96	98	102
11	**35**	40	42	46	48	52	54	58	60	64	66	70	72	76	78	82	84	88	90	94	96	100	102	106
12	**37**	42	44	48	50	54	56	60	62	66	68	72	74	78	80	84	86	90	92	96	98	102	104	108
13	**41**	46	48	52	54	58	60	64	66	70	72	76	78	82	84	88	90	94	96	100	102	106	108	112
14	**43**	48	50	54	56	60	62	66	68	72	74	78	80	84	86	90	92	96	98	102	104	108	110	114
15	**47**	52	54	58	60	64	66	70	72	76	78	82	84	88	90	94	96	100	102	106	108	112	114	118
16	**49**	54	56	60	62	66	68	72	74	78	80	84	86	90	92	96	98	102	104	108	110	114	116	120
17	**53**	58	60	64	66	70	72	76	78	82	84	88	90	94	96	100	102	106	108	112	114	118	120	124
18	**55**	60	62	66	68	72	74	78	80	84	86	90	92	96	98	102	104	108	110	114	116	120	122	126
19	**59**	64	66	70	72	76	78	82	84	88	90	94	96	100	102	106	108	112	114	118	120	124	126	130
20	**61**	66	68	72	74	78	80	84	86	90	92	96	98	102	104	108	110	114	116	120	122	126	128	132
21	**65**	70	72	76	78	82	84	88	90	94	96	100	102	106	108	112	114	118	120	124	126	130	132	136
22	**67**	72	74	78	80	84	86	90	92	96	98	102	104	108	110	114	116	120	122	126	128	132	134	138
23	**71**	76	78	82	84	88	90	94	96	100	102	106	108	112	114	118	120	124	126	130	132	136	138	142

Continuation of Table 1

24	**73**	78	80	84	86	90	92	96	98	102	104	108	110	114	116	120	122	126	128	132	134	138	140	144
25	**77**	82	84	88	90	94	96	100	102	106	108	112	114	118	120	124	126	130	132	136	138	142	144	148
26	**79**	84	86	90	92	96	98	102	104	108	110	114	116	120	122	126	128	132	134	138	140	144	146	150
27	**83**	88	90	94	96	100	102	106	108	112	114	118	120	124	126	130	132	136	138	142	144	148	150	154
28	**85**	90	92	96	98	102	104	108	110	114	116	120	122	126	128	132	134	138	140	144	146	150	152	156
29	**89**	94	96	100	102	106	108	112	114	118	120	124	126	130	132	136	138	142	144	148	150	154	156	160
30	**91**	96	98	102	104	108	110	114	116	120	122	126	128	132	134	138	140	144	146	150	152	156	158	162
31	**95**	100	102	106	108	112	114	118	120	124	126	130	132	136	138	142	144	148	150	154	156	160	162	166
32	**97**	102	104	108	110	114	116	120	122	126	128	132	134	138	140	144	146	150	152	156	158	162	164	168
33	**101**	106	108	112	114	118	120	124	126	130	132	136	138	142	144	148	150	154	156	160	162	166	168	172
34	**103**	108	110	114	116	120	122	126	128	132	134	138	140	144	146	150	152	156	158	162	164	168	170	174
35	**107**	112	114	118	120	124	126	130	132	136	138	142	144	148	150	154	156	160	162	166	168	172	174	178
36	**109**	114	116	120	122	126	128	132	134	138	140	144	146	150	152	156	158	162	164	168	170	174	176	180
37	**113**	118	120	124	126	130	132	136	138	142	144	148	150	154	156	160	162	166	168	172	174	178	180	184
38	**115**	120	122	126	128	132	134	138	140	144	146	150	152	156	158	162	164	168	170	174	176	180	182	186
39	**119**	124	126	130	132	136	138	142	144	148	150	154	156	160	162	166	168	172	174	178	180	184	186	190
40	**121**	126	128	132	134	138	140	144	146	150	152	156	158	162	164	168	170	174	176	180	182	186	188	192
41	**125**	130	132	136	138	142	144	148	150	154	156	160	162	166	168	172	174	178	180	184	186	190	192	196
42	**127**	132	134	138	140	144	146	150	152	156	158	162	164	168	170	174	176	180	182	186	188	192	194	198
43	**131**	136	138	142	144	148	150	154	156	160	162	166	168	172	174	178	180	184	186	190	192	196	198	202