The Egregema Explored

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Abstract

The article derives the Gauss Egregema by a novel technique and arrives at a surprising result that the Gauss curvature is equal to the normal curvature. This strange result leads to discrepancies like that the Gauss curvature of the sphere being zero. We arrive at a shocking result as that normal curvature should be zero.The difficulties with tensor transformations their corresponding Jacobian has been dealt with in detail.

Keywords: Egregema of Gauss, First Fundamental Form, Normal Curvature, Gauss Curvature, Christoffel Symbols, Taylor Series.

Introduction

The Egregema of Gauss is derived by a novel technique and analyzed. We arrive at a surprising result that the Gauss curvature is equal to the normal curvature of the surface at a point. This strange result leads to discrepancies like that the Gauss curvature of the sphere being zero. We have contradictions like an object being a vector and not a vector simultaneously. Finally we arrive at a shocking result as that normal curvature should be zero.

Gauss Egregema—A Novel Derivation

The first fundamental form[1]

$$|d\vec{r}(u,v)|^2 = Edu^2 + 2Fdudv + Gdv^2 \quad (1.1)$$

$$E = \frac{\partial\vec{r}}{\partial u}.\frac{\partial\vec{r}}{\partial u}, F = \frac{\partial\vec{r}}{\partial u}.\frac{\partial\vec{r}}{\partial v}, G = \frac{\partial\vec{r}}{\partial v}.\frac{\partial\vec{r}}{\partial v}$$

The first point to take cognizance of is that the derivatives $\frac{\partial\vec{r}}{\partial u}$ and $\frac{\partial\vec{r}}{\partial v}$ are not vectors themselves: they do not transform like vectors in that their dot products are not preserved. E, F and G are not invariants but are the components of a tensor. Consequently at every step we shall verify whether an object termed as a vector is truly a vector or not in terms of the requisite transformation properties.

Now,

$$|d\vec{r}(u,v)|^2 = ds^2$$

Thus we have the first fundamental form as

$$ds^2 = Edu^2 + 2Fdudv + Fdv^2 \ (1.2)$$

$$ds^2 = g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \ (1.3)$$

Even if we apply to the tangent plane $|g_{ii}| = 1, g_{ik} = 0$ only for the rectangular Cartesian system that is withy the x-y syatem.But the same is not valid for an arbitrary u-v system though we are not in the flat space context.

Again[|Expression for the normal curvature[2]

$$\kappa_N = L\left(\frac{du}{ds}\right)^2 + 2F\frac{du}{ds}\frac{dv}{ds} + F\left(\frac{dv}{ds}\right)^2 \ (2.1)$$

$$\Rightarrow \kappa_N ds^2 = Ldu^2 + 2Fdudv + Fdv^2$$

$$ds^2 = \frac{L}{\kappa_N}du^2 + 2\frac{M}{\kappa_N}dudv + \frac{N}{\kappa_N}dv^2 \ (2.2)$$

$$ds^2 = \frac{G_{11}}{\kappa_N}du^2 + 2\frac{G_{12}}{\kappa_N}dudv + \frac{G_{22}}{\kappa_N}dv^2 \ (2.3)$$

$$1 = \frac{G_{11}}{\kappa_N}\left(\frac{du}{ds}\right)^2 + 2\frac{G_{12}}{\kappa_N}\frac{du}{ds}\frac{dv}{ds} + \frac{G_{22}}{\kappa_N}\left(\frac{dv}{ds}\right)^2 \ (2.4)$$

In the above we have applied the notation,

$$G_{11} = L, G_{12} = M, G_{22} = N$$

Indeed by subtracting (2.2) from (1.2) we obtain

$$0 = \left(E - \frac{L}{\kappa_N}\right)du^2 + 2\left(F - \frac{M}{\kappa_N}\right)dudv + \left(G - \frac{N}{\kappa_N}\right)dv^2$$

From th arbitrariness of du and dv we claim

$$E - \frac{L}{\kappa_N} = 0, F - \frac{M}{\kappa_N} = 0, G - \frac{N}{\kappa_N} = 0$$

$$\Rightarrow E = \frac{L}{\kappa_N}, F = \frac{M}{\kappa_N}, G = \frac{N}{\kappa_N}$$

$$\Rightarrow g_{11} = \frac{G_{11}}{\kappa_N}, g_{12} = \frac{G_{12}}{\kappa_N}, g_{22} = \frac{G_{22}}{\kappa_N} \ (3)$$

Thus it follows that $\left(\frac{G_{11}}{\kappa_N}, \frac{G_{12}}{\kappa_N}, \frac{G_{11}}{\kappa_N}\right) = \left(\frac{L}{\kappa_N}, \frac{M}{\kappa_N}, \frac{N}{\kappa_N}\right) \equiv (g_{11}, g_{12}, g_{22})$ is a tensor

Now,

$$\frac{\bar{G}_{\mu\nu}}{\bar{\kappa}_N} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} \frac{G_{\alpha\beta}}{\kappa_N}$$

$$\kappa_N \bar{G}_{\mu\nu} - \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} G_{\alpha\beta} \bar{\kappa}_N = 0 \ (4)$$

In the above we have three linear homogeneous equations and two unknown quantities. That should make $\bar{\kappa}_N = \kappa_N = 0$unless two of the equations are identical and the determinant of the coefficient matrixof remaining two equations is zero .Then we might expect non trivial solutions for $\bar{\kappa}_N$ and κ_N

In the orthogonal system (4) reduces to two equations and two unknowns.

The issue of $\bar{\kappa}_N = \kappa_N = 0$ disappears if the $G \equiv \{G_{11}, G_{12}, G_{22}\}$ happens to be a tensor. We automatically do have $\bar{\kappa}_N = \kappa_N \neq 0$

Is $G \equiv \{G_{11}, G_{12}, G_{22}\}$ being a tensor a unique resolution to the issue? In order to have a clearer view of the situation we proceed as follows:

First we write from (4)

$$\bar{\kappa}_N = \frac{\kappa_N \bar{G}_{\mu\nu}}{\frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} G_{\alpha\beta}} \ (5)$$

It is assumed that $\bar{\kappa}_N \neq \kappa_N$,each non zero .Vow considering (2.4) in a transformed frame we have

$$1 = \frac{\bar{G}_{11}}{\bar{\kappa}_N} \left(\frac{dp}{ds}\right)^2 + 2\frac{\bar{G}_{12}}{\bar{\kappa}_N} \frac{dp}{ds} \frac{dq}{ds} + \frac{\bar{G}_{22}}{\bar{\kappa}_N} \left(\frac{dq}{ds}\right)^2 \ (6)$$

Applying (5) on (6) we obtain

$$1 = \frac{\bar{G}_{11} \frac{\partial x^\alpha}{\partial \bar{x}^1} \frac{\partial x^\beta}{\partial \bar{x}^1} G_{\alpha\beta}}{\kappa_N \bar{G}_{11}} \left(\frac{dp}{ds}\right)^2 + 2\frac{\bar{G}_{12} \frac{\partial x^\alpha}{\partial \bar{x}^1} \frac{\partial x^\beta}{\partial \bar{x}^2} G_{\alpha\beta}}{\kappa_N \bar{G}_{12}} \frac{dp}{ds} \frac{dq}{ds} + \frac{\bar{G}_{22} \frac{\partial x^\alpha}{\partial \bar{x}^2} \frac{\partial x^\beta}{\partial \bar{x}^2} G_{\alpha\beta}}{\kappa_N \bar{G}_{22}} \left(\frac{dq}{ds}\right)^2$$

$$1 = \frac{\bar{G}_{11} \kappa_N \frac{\partial x^\alpha}{\partial \bar{x}^1} \frac{\partial x^\beta}{\partial \bar{x}^1} \frac{G_{\alpha\beta}}{\kappa_N}}{\kappa_N \bar{G}_{11}} \left(\frac{dp}{ds}\right)^2 + 2\frac{\bar{G}_{12} \kappa_N \frac{\partial x^\alpha}{\partial \bar{x}^1} \frac{\partial x^\beta}{\partial \bar{x}^2} \frac{G_{\alpha\beta}}{\kappa_N}}{\kappa_N \bar{G}_{12}} \frac{dp}{ds} \frac{dq}{ds} + \frac{\bar{G}_{22} \kappa_N \frac{\partial x^\alpha}{\partial \bar{x}^2} \frac{\partial x^\beta}{\partial \bar{x}^2} \frac{G_{\alpha\beta}}{\kappa_N}}{\kappa_N \bar{G}_{22}} \left(\frac{dq}{ds}\right)^2$$

$$1 = \frac{\bar{G}_{11} \kappa_N \bar{G}_{11}}{\kappa_N \bar{G}_{11}} \left(\frac{dp}{ds}\right)^2 + 2\frac{\bar{G}_{12} \kappa_N \bar{G}_{12}}{\kappa_N \bar{G}_{12}} \frac{dp}{ds} \frac{dq}{ds} + \frac{\bar{G}_{22} \kappa_N \bar{G}_{22}}{\kappa_N \bar{G}_{22}} \left(\frac{dq}{ds}\right)^2$$

$$1 = \bar{G}_{11}\left(\frac{dp}{ds}\right)^2 + 2\bar{G}_{12}\frac{dp}{ds}\frac{dq}{ds} + \bar{G}_{22}\left(\frac{dq}{ds}\right)^2$$

$$ds^2 = \bar{G}_{11}dp^2 + 2\bar{G}_{12}dp\,dq + \bar{G}_{22}dq^2 \ (7.1)$$

Equation (6) is equivalent to

$$\bar{\kappa}_N ds^2 = \bar{G}_{11}dp^2 + 2\bar{G}_{12}dp\,dq + \bar{G}_{22}dq^2 \ (7.2)$$

From (6) and (7.2)

$$\bar{\kappa}_N ds^2 = ds^2 \Longrightarrow \bar{\kappa}_N = 1 \ (8.1)$$

Considering κ_N in terms of $\bar{\kappa}_N$ we may show from equation (2.4)

$$\kappa_N = 1 \ (8.2)$$

$\bar{\kappa}_N \neq \kappa_N$ has led to an inconsistency. Therefore we undo our initial assumption $\bar{\kappa}_N \neq \kappa_N$ and consider $\bar{\kappa}_N = \kappa_N$ instead.

We arrive at the same conclusion if an orthogonal system is considered.

Incidentally using various alternative replacements from (5) into (6) has the same effect. For example

$$1 = \frac{\bar{G}_{11}\kappa_N\bar{G}_{22}}{\kappa_N\bar{G}_{22}}\left(\frac{dp}{ds}\right)^2 + 2\frac{\bar{G}_{12}\kappa_N\bar{G}_{33}}{\kappa_N\bar{G}_{33}}\frac{dp}{ds}\frac{dq}{ds} + \frac{\bar{G}_{22}\kappa_N\bar{G}_{11}}{\kappa_N\bar{G}_{11}}\left(\frac{dq}{ds}\right)^2$$

implies (7.1),(7.2),(8.1) and (8.2).

Now

$$A_{\alpha\beta}dx^\alpha dx^\beta = \bar{A}_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu$$

$$\Longrightarrow A_{\alpha\beta}dx^\alpha dx^\beta = \bar{A}_{\mu\nu}\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\partial x^\beta}dx^\alpha dx^\beta$$

$$\Longrightarrow A_{\alpha\beta} = \bar{A}_{\mu\nu}\frac{\partial\bar{x}^\mu}{\partial x^\alpha}\frac{\partial\bar{x}^\nu}{\partial x^\beta}$$

Therefore $A_{\alpha\beta}$ is a rank two covariant tensor. But we require to vary $dx^\alpha$and dx^β. Therefore the invatriance has to hold over a continuous region no matter how small it is.We recall (7.2)

$$\kappa_N ds^2 = G_{11}du^2 + 2G_{12}dudv + G_{22}dv^2$$

Thus$\left(\frac{L}{\kappa_N},\frac{M}{\kappa_N},\frac{N}{\kappa_N}\right)$as a tensor implies (L,M,N) is a rank two covariant tensor in two dimensions[due to the invariance of κ_N.

$$\bar{K}=\frac{\bar{G}_{uu}\bar{G}_{vv}-\bar{G}_{uv}{}^2}{\bar{g}_{uu}\bar{g}_{vv}-\bar{g}_{uv}{}^2}$$

$$=\frac{\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^u}G_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}G_{\gamma\delta}-\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^v}G_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}G_{\gamma\delta}}{\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^u}g_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}g_{\gamma\delta}-\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^v}g_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}g_{\gamma\delta}}$$

$$=\frac{\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^u}G_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}G_{\gamma\delta}-\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^v}G_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}G_{\gamma\delta}}{\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^u}g_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}g_{\gamma\delta}-\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^v}g_{\alpha\beta}\frac{\partial x^\gamma}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}g_{\gamma\delta}};Each\ of\ \alpha,\beta,\gamma,\delta=u,v$$

$$\bar{K}=\frac{\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\gamma}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}\left[G_{\alpha\gamma}G_{\beta\delta}-G_{\alpha\beta}G_{\gamma\delta}\right]}{\frac{\partial x^\alpha}{\partial\bar{x}^u}\frac{\partial x^\gamma}{\partial\bar{x}^u}\frac{\partial x^\beta}{\partial\bar{x}^v}\frac{\partial x^\delta}{\partial\bar{x}^v}\left[g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\beta}g_{\gamma\delta}\right]}=\kappa_N{}^2=K;Each\ of\ \alpha,\beta,\gamma,\delta=u,v\ (6)(9)$$

Thus we obtain

$$\frac{G_{uu}G_{vv}-G_{uv}{}^2}{g_{uu}g_{vv}-g_{uv}{}^2}=\frac{\bar{G}_{uu}\bar{G}_{vv}-\bar{G}_{uv}{}^2}{\bar{g}_{uu}\bar{g}_{vv}-\bar{g}_{uv}{}^2}\ (10)$$

$$K=\bar{K}$$

$$K=\frac{LN-M^2}{EG-F^2}=\frac{\bar{L}\bar{N}-\bar{M}^2}{\bar{E}\bar{G}-\bar{F}^2}=\bar{K}=\kappa_N(11.1))$$

$$K=\kappa_N=\bar{K}\ (11.2)$$

Thus we arrive at the Theorema Egregium[3]:The Gaussian curvature of an embedded surface in $R^3$is invariant under local isometric. But we do have an extra bonus in tha we have discovered that the Gaussian curvature should be the same as normal curvature.

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[It is important to take note of the fact a zero of $\kappa_N=\bar{\kappa}_N$ should simplify everything; nevertheless we are considering $\kappa_N=\bar{\kappa}_N\neq 0$ cases in the following portion of this section]

For a given surface K is independent of the curves; it does not change as we pass from one curve to another. Therefore K (or equivalently $\bar{K}$) is independent of the curves

$$\kappa\hat{n}.\hat{N}$$
$$=\kappa_N\ independent\ of\ curves\ for\ a\ given\ surface\ \ and\ invariant\ in\ respect\ of\ transformations.$$

$$\kappa_1\hat{n}_1.\hat{N} = \kappa_2\hat{n}_2.\hat{N} = \kappa_3\hat{n}_3.\hat{N} =, \dots . = \kappa_i\hat{n}_i.\hat{N} \dots \dots . . = \kappa_N \quad (12)$$

If for one curve at a point P[for example the generator of a cylinder]$\kappa = 0$ we have $\kappa_N = 0$.Therefore

$$\kappa_i\hat{n}_i.\hat{N} = 0 \; (13)$$

$\kappa_N = 0$implies that all normals to the curves on the cylinder should lie on the tangent plane for all i.This is not true of the cylinder.We consider a circle on the surface of a cylinder with the axis passing through the center of the circle chosen. Normal to this circle at point P should lie on the tangent plane at P.We consider the solid of revolution for this circle along with the normal lying on the tangent plane. We end up with a sphere where the old normal [to the circle] by rotation has generated a tangent lane of the sphere. We have$\kappa_N = 0$ implying Gaussian curvature $K = 0$. This is not true. The same contradiction exists for an ellipsoid[by considering a curve with its normal lying on the tangent plane and then by rotating the curve; the tangent plane to this curve has to be considered where the major axis touches the surface for some ellipse]

Next we consider a point P on a curved surface. We assume that Gaussian curvature at P $\neq 0$. At P we merge an infinitesimally small straight line [by lifting off and replacing a small portion of some existing curve portion at P]with the final surface maintaining continuity and differentiability inclusive of the point P. We make sure that at least some part of the surface at P remains isometrically transformed. For other portions it is not necessary to preserve isometry while carrying out a replacement. But we have to preserve continuity and differentiability.

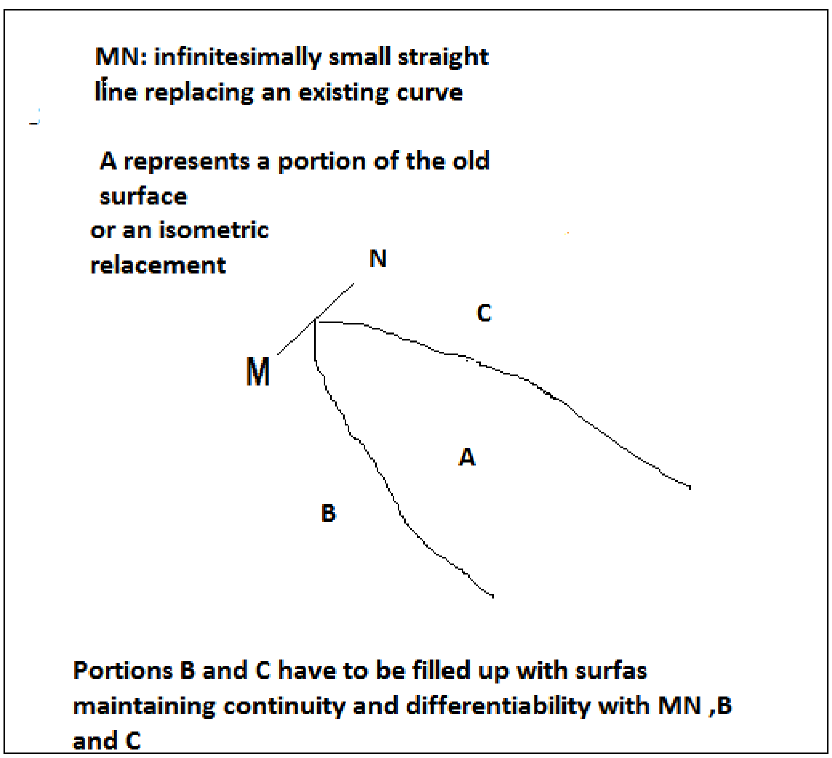

Figure I

Referring to figure I, MN is the straight line replacing an infinitesimally small portion of the curve through P.'A' represents the old surface or an isometric replacement of it. 'B' and 'C' are replacements in consideration of continuity and differentiability. It is not necessary to preserve isometry with 'B' and with 'C'

A similar operation is performed on the other side of the line MN keeping in the mind that the curves of A passing through P have to pass through its isometric counterpart A' on the other side without distortion.*In case we could do that in view of the failures with the sphere and the ellipsoid shown just now we have the following*

$\kappa_N = \kappa\hat{n}.\hat{N} = 0$ for the straight line. Therefore or all other curves it should be zero inclusive of curves on the isometric part. But this stands out to be a contradiction since for the isometric part by Egregema ,Gaussian curvature and hence normal curvature κ_N should not change. They should remain non zero.

If the surface A touched the straight line along it or a part of it then considering the fact that the common portion earlier was a curved line w are assuming a coincidence between a straight line and a curved line on an infinitesimal scale. This is impossible because we cannot alter the curvature of a line at a point by taking a very small part of it in the neighborhood of the point concerned. A practical way of visualizing this would be to consider the motion of a particle along a curved line at some point P.It would have a non zero centripetal acceleration. If we considered a straight line through P even its infinitesimal size would not allow any acceleration in a direction normal to it. Reducing the length of the curve round P will not reduce the acceleration at P to zero. We do not have this problem if the surface A touches the line MN at a point.

Christoffel Symbols

The second order derivative$\frac{d^2\vec{r}}{ds^2}$ is not a tensor unless the transformation is of a linear nature[see Appendix I, equation (29)].

But the expression $\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}$ represents a tensor.

Since$\frac{d\vec{r}}{ds}$ is a tensor th following inner product is an invariant

$$g_{\epsilon\alpha}\frac{dx^\epsilon}{ds}.\left[\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}\right] = INV$$

$$g_{\epsilon\alpha}\frac{dx^\epsilon}{ds}\frac{d^2x^\alpha}{ds^2} + g_{\epsilon\alpha}\frac{dx^\epsilon}{ds}\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = \text{INV}\ (14)$$

Conventional material:

$$\frac{d\vec{r}}{ds}.\frac{d^2\vec{r}}{ds^2} = \hat{T}.\kappa\hat{n} = 0 \ (15)$$

As fallout of this conventional stuff

$$g_{\epsilon\alpha}\frac{dx^\epsilon}{ds}\frac{d^2x^\alpha}{ds^2} = 0$$

From equations (14) and (15) we have,

$$g_{\epsilon\alpha}\frac{dx^\epsilon}{ds}\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = \mathrm{INV} \ (16)$$

By quotient law

$$\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}$$

is a rank one contravariant tensor

But the affine connection $\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}$,is not supposed to be a tensor.

Looking at

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds},$$

if $\Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}$ and $\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds}$ are both tensors their difference $\frac{d^2x^\alpha}{ds^2}$ will also be a tensor

But $\frac{d^2\vec{r}}{ds^2}$ as we know is not a tensor for non linear transformations.

NB: All objects have been considered in the ambient space[the space in which the surface has been considered as embedded].It is also important to take note of the fact that the Christoffel symbols pertaining to flat space time reduce to zero value only in the Cartesian[rectangular] system and not in other systems like the spherical coordinate system.

Shifting the Origin

Next[Shifting the origin]

$$\frac{d(\vec{r}.\hat{T})}{ds} = \frac{d\vec{r}}{ds}.\hat{T} + \vec{r}.\frac{d\vec{T}}{ds} \ (17)$$

$$\frac{dr_T}{ds} = \hat{T}.\hat{T} + \vec{r}.\frac{d^2\vec{r}}{ds^2}$$

$$\frac{dr_T}{ds} = 1 + \kappa\vec{r}.\hat{n}$$

$$\frac{dr_T}{ds} = 1 + \kappa r_n \ (18)$$

We keep changing the origin so that r_n changes enormously in comparison with $\frac{dr_T}{ds}$. That upsets the equation. κ being an intrinsic property does not depend on the origin.

NB:$d\vec{r}.\hat{n} = 0$ but $\vec{r}.\hat{n} \neq 0$.In a given frame of reference $\vec{r}$ is a vector in the ambient space like $\hat{n}$ and$\widehat{N}$ .The ambient space is Euclidean R^3 .But $\vec{r}$ is not a vector if the transformation is non linear.

A Contradiction Arising from the Invariance of the Normal Curvature

We consider the standard result[3]

$$\kappa\hat{n}.\widehat{N} = \kappa_N \ (25.1)$$

[$\hat{n}$:normal to the curve, $\widehat{N}$: normal to the surface,κ: $curvature$, κ_N: normal curvature

$$\kappa\hat{n} = \frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2$$

We have from Appendix II, equation (54)

$$\frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2 = \frac{d^2\vec{r}}{ds^2} - \frac{\partial\vec{r}}{\partial u}\left(\frac{du}{ds}\right)^2 - \frac{\partial\vec{r}}{\partial v}\left(\frac{dv}{ds}\right)^2$$

$$\kappa\hat{n} = \frac{d^2\vec{r}}{ds^2} - \frac{\partial\vec{r}}{\partial u}\left(\frac{du}{ds}\right)^2 - \frac{\partial\vec{r}}{\partial v}\left(\frac{dv}{ds}\right)^2$$

$$\kappa\hat{n}.\widehat{N} = \left[\frac{d^2\vec{r}}{ds^2} - \frac{\partial\vec{r}}{\partial u}\left(\frac{du}{ds}\right)^2 - \frac{\partial\vec{r}}{\partial v}\left(\frac{dv}{ds}\right)^2\right].\frac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}$$

$$\kappa\hat{n}.\widehat{N} = \frac{d^2\vec{r}}{ds^2}.\frac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}$$

$$\kappa_N = \frac{d^2\vec{r}}{ds^2}.\frac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|} \ (19)$$

Now, $\frac{\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}}{\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right|}$ is a vector and $\kappa_N$is an invariant. Therefore $\frac{d^2\vec{r}}{ds^2}$ is a vector. But this is not true from equation (22) unless the transformation is of a linear nature. Thus we have arrived at a contradiction.

Taylor series Considerations

$$\Delta\vec{r}(u,v) = \frac{\partial \vec{r}}{\partial u}\Delta u + \frac{\partial \vec{r}}{\partial v}\Delta v + \frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \Delta u^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u \partial v}\right]_o \Delta u\Delta v + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \Delta v^2\right] + \frac{1}{3!}\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \Delta u^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2 \partial v}\right]_o \Delta u^2\Delta v + 3\left[\frac{\partial^3\vec{r}}{\partial v^2 \partial v}\right]_o \Delta v^2\Delta u + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \Delta v^3\right] + H.O\ terms\ (20)$$

$$\Delta\vec{r}(u,v) - \left(\frac{\partial \vec{r}}{\partial u}\Delta u + \frac{\partial \vec{r}}{\partial v}\Delta v\right) = \frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \Delta u^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u \partial v}\right]_o \Delta u\Delta v + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \Delta v^2\right] + \frac{1}{3!}\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \Delta u^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2 \partial v}\right]_o \Delta u^2\Delta v + 3\left[\frac{\partial^3\vec{r}}{\partial v^2 \partial v}\right]_o \Delta v^2\Delta u + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \Delta v^3\right] + H.O\ terms$$

$$\Delta\vec{r}(u,v) - \Delta s\left(\frac{\partial \vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial \vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right) = \frac{1}{2}\Delta s^2\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u \partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right] + \frac{1}{3!}\Delta s^3\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2 \partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2\frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3\vec{r}}{\partial v^2 \partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\frac{\Delta u}{\Delta s} + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] + \Delta s^4(\ldots\ldots.\ldots)$$

[In the above $\Delta s^4(\ldots\ldots.\ldots)$ represent the higher order terms]

$$\frac{\Delta\vec{r}(u,v) - \Delta s\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)}{\Delta s^2}$$
$$= \frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]$$
$$+ \frac{1}{3!}ds\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2\partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3\vec{r}}{\partial v^2\partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s}\right.$$
$$\left. + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] + \Delta s^2(\dots\dots.\dots)$$

[$ds^2(\dots\dots.\dots)$ comprises higher order terms]

$$\frac{\left[\frac{\Delta\vec{r}}{\Delta s}ds - \Delta s\left(\frac{\partial\vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{dv}{ds}\right)\right]}{\Delta s^2}$$
$$= \frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]$$
$$+ \frac{1}{3!}ds\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2\partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3\vec{r}}{\partial v^2\partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s}\right.$$
$$\left. + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] + \Delta s^2(\dots\dots.\dots)$$

[$ds^2(\dots\dots.\dots)$ comprises higher order terms]

$$\frac{\left[\frac{\Delta\vec{r}}{\Delta s} - \left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s}$$
$$= \frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]$$
$$+ \frac{1}{3!}ds\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2\partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3\vec{r}}{\partial v^2\partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s}\right.$$
$$\left. + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] + \Delta s^2(\dots\dots.\dots)$$

[$ds^2(\dots\dots.\dots)$ comprises higher order terms]

$$lim_{\Delta s\to 0}\frac{\left[\frac{\Delta\vec{r}}{\Delta s} - \left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s}$$
$$= \frac{1}{2}lim_{\Delta s\to 0}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]$$

$$lim_{\Delta s\to 0}\frac{\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s}=\frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o\left(\frac{du}{ds}\right)^2+2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o\frac{du}{ds}\frac{dv}{ds}+\left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o\left(\frac{dv}{ds}\right)^2\right]\ (21)$$

We may evaluate

$$lim_{\Delta s\to 0}\frac{\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s}$$

by applying L'Hospital's rule since the limit is 0/0 form

$$lim_{\Delta s\to 0}\frac{\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s}$$

$$lim_{\Delta s\to 0}\frac{\frac{d}{d(\Delta s)}\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\frac{d}{d(\Delta s)}\Delta s}$$

$$lim_{\Delta s\to 0}\frac{\frac{d}{d(\Delta s)}\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{1}$$

Each term of the numerator with increasing smallness of Δs becomes point functions and cease to depend on Δs. As for example with$\Delta s\to 0\Longrightarrow\Delta u,\Delta v\longrightarrow 0$, $\frac{\Delta\vec{r}}{\Delta s},\frac{\Delta u}{\Delta s}$and $\frac{\Delta v}{\Delta s}$become point functions $\frac{d\vec{r}}{ds},\frac{du}{ds}$and $\frac{dv}{ds}$ which are independent of Δs[they depend on s].Derivatives do not depend on the differences like $dr,d^2r,\dots\dots$

Therefore $lim_{\Delta s\to 0}\frac{d}{d(\Delta s)}\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]=0$

Hence

$$lim_{\Delta s\to 0}\frac{\frac{d}{d(\Delta s)}\left[\frac{\Delta\vec{r}}{\Delta s}-\left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s}+\frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{1}=0\ (22)$$

$$\Longrightarrow\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o\left(\frac{du}{ds}\right)^2+2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o\frac{du}{ds}\frac{\Delta v}{\Delta s}+\left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o\left(\frac{dv}{ds}\right)^2=0\ (23)$$

We have,

$$\kappa\hat{n}=0\Longrightarrow\kappa_N=0\ (24.1)$$

But from equation (11.2) Gauss curvature $K=\kappa_N$

$$\Rightarrow K = 0 \ \ (24.2)$$

The Egregema loses significance.

All terms of the Taylor series like

$$\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial^2 u\partial v}\right]_o \left(\frac{du}{ds}\right)^2 \frac{dv}{ds} + 3\left[\frac{\partial^3\vec{r}}{\partial^2 v\partial u}\right]_o \left(\frac{dv}{ds}\right)^2 \frac{du}{ds} + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{dv}{ds}\right)^3 = 0$$

and the subsequent ones disappear

We recall

$$\frac{\left[\frac{\Delta\vec{r}}{\Delta s} - \left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s}$$
$$= \frac{1}{2}\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]$$
$$+ \frac{1}{3!}ds\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2\partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3\vec{r}}{\partial v^2\partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s}\right.$$
$$\left. + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] + \Delta s^2(\dots\dots.\dots)$$

[$ds^2(\dots\dots.\dots)$ comprises higher order terms]

$$\frac{\left[\frac{\Delta\vec{r}}{\Delta s} - \left(\frac{\partial\vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial\vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s^2}$$
$$= \frac{1}{2}\frac{\left[\left[\frac{\partial^2\vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2\vec{r}}{\partial u\partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2\vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]}{\Delta s}$$
$$+ \frac{1}{3!}\left[\left[\frac{\partial^3\vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3\vec{r}}{\partial u^2\partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3\vec{r}}{\partial v^2\partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s}\right.$$
$$\left. + \left[\frac{\partial^3\vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] + \Delta s(\dots\dots.\dots)$$

$$lim_{\Delta s \longrightarrow 0} \frac{\left[\frac{\Delta \vec{r}}{\Delta s} - \left(\frac{\partial \vec{r}}{\partial u}\frac{\Delta u}{\Delta s} + \frac{\partial \vec{r}}{\partial v}\frac{\Delta v}{\Delta s}\right)\right]}{\Delta s^2}$$

$$= \frac{1}{2} lim_{\Delta s \longrightarrow 0} \frac{\left[\left[\frac{\partial^2 \vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2 \vec{r}}{\partial u \partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2 \vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]}{\Delta s}$$
$$+ \frac{1}{3!}\left[\left[\frac{\partial^3 \vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3 \vec{r}}{\partial u^2 \partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3 \vec{r}}{\partial v^2 \partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s}\right.$$
$$\left. + \left[\frac{\partial^3 \vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right]$$

$$lim_{\Delta s \longrightarrow 0} \frac{\left[\left[\frac{\partial^2 \vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2 \vec{r}}{\partial u \partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2 \vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]}{\Delta s}$$

We apply Hospital's rule to

$$lim_{\Delta s \longrightarrow 0} \frac{\left[\left[\frac{\partial^2 \vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2 \vec{r}}{\partial u \partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2 \vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]}{\Delta s}$$

$$lim_{\Delta s \longrightarrow 0} \frac{\frac{d}{d(\Delta s)}\left[\left[\frac{\partial^2 \vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2 \vec{r}}{\partial u \partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2 \vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]}{1}$$

The numerator is zero since with $\Delta s \longrightarrow 0$ each term becomes a function of and is not a function of Δs

$$lim_{\Delta s \longrightarrow 0} \frac{\left[\left[\frac{\partial^2 \vec{r}}{\partial u^2}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 + 2\left[\frac{\partial^2 \vec{r}}{\partial u \partial v}\right]_o \frac{\Delta u}{\Delta s}\frac{\Delta v}{\Delta s} + \left[\frac{\partial^2 \vec{r}}{\partial v^2}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2\right]}{\Delta s} = 0$$

With the limit on the left side we apply Hospital twice to prove it is zero

Thus we conclude

$$\left[\left[\frac{\partial^3 \vec{r}}{\partial u^3}\right]_o \left(\frac{du}{ds}\right)^3 + 3\left[\frac{\partial^3 \vec{r}}{\partial u^2 \partial v}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^2 \frac{\Delta v}{\Delta s} + 3\left[\frac{\partial^3 \vec{r}}{\partial v^2 \partial v}\right]_o \left(\frac{\Delta v}{\Delta s}\right)^2 \frac{\Delta u}{\Delta s} + \left[\frac{\partial^3 \vec{r}}{\partial v^3}\right]_o \left(\frac{\Delta u}{\Delta s}\right)^3\right] = 0 \; (25)$$

We may apply the same technique to prove that the subsequent Taylor series terms are zero for the expansion of the differential pertaining to the position vector as function of coordinates.

Appendix I

1.

$$\vec{r} = (u, v) \equiv (p, q) \text{ (26.1)}$$

$$\vec{r} = (u(p, q), v(p, q)) \text{ (26.2)}$$

We have,

$$\frac{du}{ds} = \frac{\partial u}{\partial p}\frac{dp}{ds} + \frac{\partial u}{\partial q}\frac{dq}{ds} \text{ (27.1)}$$

$$\frac{dv}{ds} = \frac{\partial v}{\partial p}\frac{dp}{ds} + \frac{\partial v}{\partial q}\frac{dq}{ds} \text{ (27.2)}$$

Vector

$$\left(\frac{du}{ds}, \frac{dv}{ds}\right) \leftrightarrow \left(\frac{dp}{ds}, \frac{dq}{ds}\right)$$

They are of the form

$$\frac{d\bar{x}^\mu}{ds} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{dx^\alpha}{ds}; x^\alpha = p, q; \bar{x}^\mu = u, v \text{ (28.1)}$$

Transformation elements, $\frac{\partial \bar{x}^\mu}{\partial x^\alpha}; \alpha = 1,2; \mu = 1,2$

$$M = \begin{bmatrix} \frac{\partial u}{\partial p} & \frac{\partial u}{\partial q} \\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q} \end{bmatrix} \text{ (28.2)}$$

Considering the preservation of length $MM^T = I$[Orthogonality condition;this has nothing to do with the system of coordinates being orthogonal or non orthogonal]

Now,

$$\frac{d^2\bar{x}^\mu}{ds^2} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{d^2x^\alpha}{ds^2} + \frac{\partial^2 \bar{x}^\mu}{\partial^2 x^\alpha}\frac{dx^\alpha}{ds} \text{ (29)}$$

$$\frac{d^2x^\alpha}{ds^2} \longrightarrow \frac{d^2\bar{x}^\mu}{ds^2}$$

is not a vector unless $\frac{\partial^2 \bar{x}^\mu}{\partial^2 x^\alpha} = 0 \Longrightarrow \frac{\partial \bar{x}^\mu}{\partial x^\alpha} = constant$ that is unless the transformations are of a linear nature.

Next we shall prove that $\left(\frac{\partial x}{\partial u},\frac{\partial x}{\partial v}\right) \leftrightarrow \left(\frac{\partial x}{\partial p},\frac{\partial x}{\partial q}\right)$ and $\left(\frac{\partial y}{\partial u},\frac{\partial y}{\partial v}\right) \leftrightarrow \left(\frac{\partial y}{\partial p},\frac{\partial y}{\partial q}\right)$

are vectors with respect to the transformations given by (21.2)

$$\frac{\partial x}{\partial u}=\frac{\partial x}{\partial p}\frac{\partial p}{\partial u}+\frac{\partial x}{\partial q}\frac{\partial q}{\partial u}$$

$$\frac{\partial x}{\partial v}=\frac{\partial x}{\partial p}\frac{\partial p}{\partial v}+\frac{\partial x}{\partial q}\frac{\partial q}{\partial v}$$

Therefore $\left(\frac{\partial x}{\partial u},\frac{\partial x}{\partial v}\right) \leftrightarrow \left(\frac{\partial x}{\partial p},\frac{\partial x}{\partial q}\right)$ is a vector with respect to the transformations M given by (21.2)

Again

$$\frac{\partial y}{\partial u}=\frac{\partial y}{\partial p}\frac{\partial p}{\partial u}+\frac{\partial y}{\partial q}\frac{\partial q}{\partial u}$$

$$\frac{\partial y}{\partial v}=\frac{\partial y}{\partial p}\frac{\partial p}{\partial v}+\frac{\partial y}{\partial q}\frac{\partial q}{\partial v}$$

Therefore $\left(\frac{\partial y}{\partial u},\frac{\partial y}{\partial v}\right) \leftrightarrow \left(\frac{\partial y}{\partial p},\frac{\partial y}{\partial q}\right)$ is a vector with respect to the transformations given by (28.2)

$$\left(\frac{\partial x}{\partial u}e_1+\frac{\partial x}{\partial v}e_2\right)\times\left(\frac{\partial y}{\partial u}e_1+\frac{\partial y}{\partial v}e_2\right)=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}e_1\times e_2+\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}e_2\times e_1$$

$$\left(\frac{\partial x}{\partial u}e_1+\frac{\partial x}{\partial v}e_2\right)\times\left(\frac{\partial y}{\partial u}e_1+\frac{\partial y}{\partial v}e_2\right)=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)e_1\times e_2 \ (29)$$

$$\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv}=\left(\frac{\partial x}{\partial u}e_1+\frac{\partial y}{\partial u}e_2\right)\times\left(\frac{\partial x}{\partial v}e_1+\frac{\partial y}{\partial v}e_2\right)=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}e_1\times e_2+\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}e_2\times e_1$$

$$\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv}=\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)e_1\times e_2 \ (30)$$

Therefore from (22) and (23) we obtain

$$\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{dv}=\left(\frac{\partial x}{\partial u}e_1+\frac{\partial x}{\partial v}e_2\right)\times\left(\frac{\partial y}{\partial u}e_1+\frac{\partial y}{\partial v}e_2\right) \ (31)$$

$\left(\frac{\partial x}{\partial u}e_1+\frac{\partial x}{\partial v}e_2\right)$ and $\left(\frac{\partial y}{\partial u}e_1+\frac{\partial y}{\partial v}e_2\right)$ being vectors the left side of (24) is a vector. Therefore the right side is also a vector: $\frac{d\vec{r}}{du}\times\frac{d\vec{r}}{du}$ is a vector

Thus the cross product transforms like a vector

One should take note of the formula[4] that

$$(Ma) \times (Mb) = \det[M](M^T)^{-1}\, a \times b$$

M is an invertible transformation matrix.For length preserving transformations [$|d\vec{r}|^2$ invariant:we have the orthogonality condition $MM^T = I \Longrightarrow M^T = M^{-1}$. Therefore $(M^T)^{-1} = (M^{-1})^{-1} = M$

Again $MM^T = I \Longrightarrow \det[MM^T] = 1 \Longrightarrow \det[M] det[M^T] = 1 \Longrightarrow [\det[M]]^2 = 1 \Longrightarrow \det[M] = \pm 1$

[The orthogonality condition $MM^T = I$ is not related to the system of coordinates being orthogonal or non orthogonal]

Thus considering the positive value for orthogonal transformations we do have,

$$(M\vec{a}) \times (M\vec{b}) = M(\vec{a} \times \vec{b}) \; (32.1)$$

$$\vec{a}' \times \vec{b}' = M(\vec{a} \times \vec{b}) \; (32.2)$$

[Prime in the above does not denote differentiation; it denote an objet in a different frame of reference]

Since the cross product is a vector we have from (32.2)

$$(\vec{a}' \times \vec{b}')^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} (\vec{a} \times \vec{b})^\alpha \; (33)$$

We also do have the following invariance,

$$g_{\alpha\beta} (\vec{a} \times \vec{b})^\alpha (\vec{a} \times \vec{b})^\beta = \bar{g}_{\mu\nu} (\vec{a}' \times \vec{b}')^\mu (\vec{a}' \times \vec{b}')^\nu \; (34)$$

The preservation of the inner product is related to the preservation of length---the orthogonality condition tht we took into account in obtaining (32.1)

That cross product is formally a vector rests heavily on the fact that $MM^T = I$

<u>Relevant Considerations 1</u>

$$\bar{A}^{\mu\nu\sigma} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\beta} \frac{\partial \bar{x}^\sigma}{\partial x^\gamma} A^{\alpha\beta\gamma} \; (35.1)$$

$$\bar{A}^{\mu\nu\sigma} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha} \frac{\partial \bar{x}^\nu}{\partial x^\beta} \left(\frac{\partial \bar{x}^\sigma}{\partial x^\gamma} A^{\alpha\beta\gamma} \right) (35.2)$$

$$\bar{A}^{\mu\nu\sigma} = \frac{\partial x^\sigma}{\partial \bar{x}^\varepsilon} \bar{A}^{\mu\nu\varepsilon} \; (35.3)$$

For fixed $\mu$and ν (35.3) exhibits the transformation of a tank one contravatiant tensor:

$$\bar{A}^{\mu\nu\varepsilon} \longrightarrow \bar{A}^{\mu\nu\sigma}$$

Combining (35.1) and (35.3) we obtain

$$\frac{\partial x^{\sigma}}{\partial \bar{x}^{\varepsilon}}\bar{A}^{\mu\nu\varepsilon} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\left(\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}}A^{\alpha\beta\gamma}\right) \quad (36)$$

Therefore from (36)

$$\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}}A^{\alpha\beta\gamma} \leftrightarrow \frac{\partial x^{\sigma}}{\partial \bar{x}^{\varepsilon}}\bar{A}^{\mu\nu\varepsilon}$$

$is$is a rank two contravariant tensor.Since $A^{\alpha\beta\gamma}$is a rank three contravariant tensor

$$\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} \leftrightarrow \frac{\partial x^{\sigma}}{\partial \bar{x}^{\varepsilon}}$$

is a rank one covariant tensor

Consequently we have,

$$\frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} = \frac{\partial \bar{x}^{\varepsilon}}{\partial x^{\gamma}}\frac{\partial x^{\sigma}}{\partial \bar{x}^{\varepsilon}} \quad (37)$$

$$\Rightarrow \frac{\partial \bar{x}^{\sigma}}{\partial x^{\gamma}} = \delta^{\sigma}{}_{\gamma}$$

$$\Rightarrow J = I \quad (38)$$

This ensures $JJ^T = I$ nevertheless it is too restrictive.

<u>Relevant Considerations 2</u>

If $A^{\alpha\beta}$ is symmetric[or antisymmetric] the $n^2 \times n^2$coeffiecient matrix effectively reduces to a singular matrix;else not.This stands independent of how J is related to K

Proof

We consider a general type of a matrix[irrespective of being symmetric or antiymmetric].Our transformation matrix is J

$$\bar{A}^{\mu\nu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}\frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\,A^{\alpha\beta} (46.1)$$

$$\bar{A}^{\nu\mu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}}A^{\alpha\beta} (46.2)$$

Let Let us inspect the $n^2 \times n^2$transformation matrix

$$[\bar{A}]_{n^2\times 1} = [T]_{n^2\times n^2}[A]_{n^2\times 1} \, (47)$$

The elements of the rows of T and A have to be arranged with a view to the sequence of the elements in $\bar{A}$[the elements of the column matrix $\bar{A}$ are the n^2 elements $\bar{A}^{\mu\nu}$ while the elements of A are the n^2 , $A^{\alpha\beta}$. The matrices $A$and $\bar{A}$ in (47) are not identical with those of (35.3).In (35.3) we have $n \times n$ matrices each $A$and $\bar{A}$ while in (47) $A$and $\bar{A}$ are column matrices. Equation (47) depicts (46.1) or equivalently (46.2).In the general case the rows in T against $\bar{A}^{\mu\nu}$ and $\bar{A}^{\nu\mu}$ are not identical.

Keeping the row against $\bar{A}^{\mu\nu}$ fixed if we reverse order of elements in the row against $\bar{A}^{\nu\mu}$, the two rows become identical and T becomes a singular matrix. This operation cannot be performed for the general case but it works perfectly well for symmetric and the antisymmetric matrices.

We recall (46.2)

$$\bar{A}^{\nu\mu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\alpha}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\beta}}A^{\beta\alpha}$$

α and β being dummy indices we interchange them:

$$\bar{A}^{\nu\mu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}A^{\beta\alpha} \, (48)$$

For the Symmetric case: $A^{\alpha\beta} = A^{\beta\alpha}$

$$\bar{A}^{\nu\mu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}A^{\alpha\beta}$$

We recall

$$\bar{A}^{\mu\nu} = -\frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}A^{\alpha\beta} \, (49)$$

The rows of T against $\bar{A}^{\mu\nu}$ and $\bar{A}^{\nu\mu}$are identical.Consequently T is a singular matrix.

For the antisymmetric case

$$A^{\alpha\beta} = -A^{\beta\alpha}$$

$$\bar{A}^{\nu\mu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}A^{\alpha\beta}$$

We recall

$$\bar{A}^{\mu\nu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}}A^{\alpha\beta}$$

The rows of T against $\bar{A}^{\mu\nu}$ and $\bar{A}^{\nu\mu}$are identical except for the factor (-1).Consequently T is a singular matrix.

One should keep in the mind that a singular matrix implies that either solution does not exist or there exists an infinitude of solutions

Next we consider the following two points

1. An arbitrary matrix can be decomposed into the sum of a symmetric and an antisymmetric matrix
2. A symmetric matrix[of rank two] transforms to a symmetric matrix and an anti symmetric matrix transforms to an antisymmetric matrix.

With that in view we consider an arbitrary matrix

Now an arbitrary matrix $C^{\alpha\beta} = A^{\alpha\beta} + B^{\alpha\beta}$ where A is a symmetric matrix and B an antisymmetric matrix

$$\bar{C}^{\mu\nu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} C^{\alpha\beta} \ (50.1)$$

$$\bar{A}^{\mu\nu} + \bar{B}^{\mu\nu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \left(A^{\alpha\beta} + B^{\alpha\beta}\right)$$

By point (2)

$$\bar{A}^{\mu\nu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} A^{\alpha\beta} \ (50.2)$$

$$\bar{B}^{\mu\nu} = \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}}\frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} B^{\alpha\beta} \ (50.3)$$

Now the transformation matrices of (50.2) and (50.3) are effectively singular matrices as we have shown earlier.Either solution does not exist or we have an infinitude of solution. Theory fails unless J=I .

Appendix II

$$\frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{ds} \ (51.1)$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\left[\frac{\partial \vec{r}}{\partial u}\frac{du}{ds} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{ds}\right] \ (52.2)$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial u}\right)\frac{du}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial v}\right)\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial u}\right) = \frac{\partial^2\vec{r}}{\partial u^2}\frac{du}{ds} + \frac{\partial^2\vec{r}}{\partial v\partial u}\frac{dv}{ds}$$

$$\frac{d}{ds}\left(\frac{\partial\vec{r}}{\partial v}\right) = \frac{\partial^2\vec{r}}{\partial v^2}\frac{dv}{ds} + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}$$

Therefore

$$\frac{d^2\vec{r}}{ds^2} = \frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + \frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2 + \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} + \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2} \quad (53)$$

We have by transposition,

$$\left[\frac{\partial^2\vec{r}}{\partial u^2}\left(\frac{du}{ds}\right)^2 + 2\frac{\partial^2\vec{r}}{\partial u\partial v}\frac{du}{ds}\frac{dv}{ds} + \frac{\partial^2\vec{r}}{\partial v^2}\left(\frac{dv}{ds}\right)^2\right] = \left[\frac{d^2\vec{r}}{ds^2} - \frac{\partial\vec{r}}{\partial u}\frac{d^2u}{ds^2} - \frac{\partial\vec{r}}{\partial v}\frac{d^2v}{ds^2}\right] \quad (54)$$

Conclusion

As asserted at the outset the egregema of Gauss has been derived by a novel process. We have arrived at an amazing result that the Gauss curvature is equal to the normal curvature of the surface at a point. This strange result leads to discrepancies like that the Gauss curvature of the sphere being zero. We have contradictions like an object being a vector and not a vector simultaneously. Finally we arrive at a shocking result as that normal curvature should be zero.

[Data sharing is not applicable to this article as no new data were created or analyzed in this study.]

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