Reckoning Dimensions

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Abstract. In this article, we seek an alternative avenue–in contrast to the conventional hypercube approach–to reckon physical or abstract dimensions from an information perspective alone. After briefly reviewing "bit" and "quantum of information–it", we propose a scheme to perceive higher dimensions using bits and concentric spherical shells that are intrinsically entangled.

1 Introduction

A bit is a binary digit, 0 or 1. Any binary set-up such as love-hatred, peace-war, true-false, up-down, high and low voltage levels of an electrical appliance, etc., can be effectively mapped to a bit. It is no surprise that bits are useful. Indeed, it's hard to imagine to be without bits today. Despite several breakthroughs, science is still plagued with many conceptual problems such as collapse of the wavefunction in quantum mechanics, space and time, etc. We can bypass such crises to far extent by digging out information inherent in the system of interest. Information (the processed data) is considered fundamental, and everything else flows and swims in it. But if information is that irreducible fundamental how come it is not able to identify those conceptual details, save technical ones, and paradoxes arise? May be we are superficial and digging not deep into the problem. Or, may be our ego and ignorance do not let us accept or see the solution. It is important to remark here that *information may not be just what we 'learn' about the world; it may be what 'makes' the world* [1]. John Archibald Wheeler, a great exponent of modern physics, in his search for links between information, physics, and quantum [2], summarizes everything in the catchphrase *it from bit*. That is, every physical quantity, every it, derives its ultimate significance from bits, binary yes-no questions. It should not be an exaggeration to claim that bit is the language or tongue of it (information). If we believe in Bohr, a qubit (quantum bit) that is a superposition of 0 and 1, and exhibits infinite possibilities, is not informative unless measured. And when measured, it collapses to a bit. What is the nature of information? Is it physical, engineered, material or non-material? The debate is still on. While some claim that information, much like temperature, is physical because it resides or manifests in a physical body, others insist on the idea that information exists independently of the physical system. Landauer discusses the physical nature of information in Ref. [3]. He argues that information is inevitably tied to a physical representation and therefore to restrictions and possibilities related to the laws of physics and the parts available in the universe. Notwithstanding, there is no problem with either claim, as long as information is a science. Recently, Foschini attempts to address the question "where the *it* from *bit* come from?" [4]. He argues on various aspects of information and suggests that the *it from bit* is a fine creation of the evolved mankind. Without doubt, it is clear that the knowledge of the material aspect only is not sufficient to understand the problem of information.

Driven almost by instinct alone, we know that our everyday world is three-dimensional, having length, width and altitude. That is, all objects in this universe can be described by giving these three coordinates. *The dimension of an object, geometric or abstract, is a measure of its descriptive complexity.* Roughly speaking, it is the minimal number of independent coordinates needed to specify a point on the object. To see how lower and higher dimensions relate to each other, take any geometric object (like a point, line, circle, plane, etc.), and drag or smear it in a direction perpendicular to its "embedding space" (drag a point to trace out a line, a line to trace out a rectangle, a circle to trace out a cylinder, a disk to a solid cylinder, etc.). The result is a one-dimension larger object than the previous one. Points are assumed to be zero-dimensional geometrical objects. When a point is smeared out it forms a one-dimensional object, line. Similarly, when a line is dragged at some non-zero angle, a two-dimensional object is created, and so on. [Dimension is formalized in mathematics as the intrinsic dimension of a topological space. This dimension is called the Lebesgue covering dimension (simply, the topological dimension). The archetypal example is Euclidean n-space R^n, which has topological dimension n.]

Objects and activities in our everyday world seem obvious, and therefore often go unnoticed and unquestioned. However, an inquisitive mind is infested with multitude of intriguing questions. Is our world really three-dimensional? Do higher dimensions exit? If yes, why does nature appear three-dimensional? How can we reckon higher dimensions? How does a dimension reveal itself? Is this revelation same to all species? What is the ultimate dimension of our universe? It is very much possible that there are certain limitations to our senses that we perceive only three dimensions, and hence are unable to access other realms of universal reality. According to ancient Hindu scriptures, our space is multidimensional, far greater than three. Exact figure is subject to analysis. Einstein, in his theory of relativity [5], combined space and time into four-dimensional spacetime. String theory [6], the most promising theory of the nature till date, claims that the universe is 10-dimensional. However, as the argument says, the extra dimensions cannot be detected because they are too small–just small enough to curl in on themselves, virtually invisible. The advantage of existence of higher dimensions is multifold. The laws of physics are simpler and elegant in higher dimensions, and physical laws appear to be unified. This sounds convincing as space and time together as space-time has done wonders in relativity. Another interesting observation is that an object, with access to higher dimension, if trapped or detained in lower dimensions can simply vanish leaving behind no clue of its escape. Kaku's HYPERSPACE [7] is a lucid, lively, and thought-provoking account on higher-dimensional physics.

In this article, we endeavor to perceive higher dimensions entirely from information theoretic perspective, using bit-strings alone. For this we devise a simple though interesting scheme: *consider arranging n-bit computational states (c-states) such that the nearest c-states differ by 1-bit Hamming distance* [8]. For n-bits there are 2^n c-states given as

$$\{\underbrace{00\cdots00}_{\text{n}},\underbrace{00\cdots01}_{\text{n}},\cdots,\underbrace{11\cdots11}_{\text{n}}\}.$$

d-bit Hamming distance is the number of places d in which two n-bit binary strings differ exactly. For example, Hamming distance between the binary strings 010 and 101 is 3, that between 001 and 100101 is 2. The Hamming distance of a binary string from itself is zero. At first sight, this seems trivial. Just put the c-states along the vertices of an "n-dimensional cube", and we are done! Though, paradigmatically perfectly right, there is a serious problem with this approach. Depiction and visualization becomes more and more obscure with increasing topological dimension $n > 3$. Here we propose an elegant approach to arrange the c-states as described earlier. This paradigm makes use of concentric (transparent!) spherical shells. For the sake of completeness and comparison we develop our approach along with the conventional n-cubic system upto $n = 4$.

2 Setup

In the conventional approach, the 2^n c-states are attached to the vertices of an n-cube such that the nearest c-states differ in 1-bit Hamming distance, and to go from one n-bit binary string to another which differ in d-bit Hamming distance, $1 \leq d \leq n$, one has to traverse d-segments/steps (say, of unit length).

In our approach, $n+1$ concentric spherical shells are drawn. The binary string of zeros $00 \cdots 00$ is associated with the innermost shell. Other binary strings which differ in d-bit Hamming distance are placed in d^{th}-shells, uniformly seated on the surface. This completes the desired arrangement. To illustrate, for $n = 3$, this reads as $\{000, \{001, 010, 100\}, \{011, 101, 110\}, 111\}$. That is, 000 is placed in the innermost shell, $\{001, 010, 100\}$ are placed in the next shell, $\{011, 101, 110\}$ are placed in the next to next shell, and 111 is placed in the outermost shell. (Note that there is nothing sacred about starting with the string $00 \cdots 00$ of zeros. One can associate any binary string to the innermost shell and continue placing other binary strings in remaining shells. However, one does not need to draw 2^n such diagrams, one for each binary string.) Now we ask: is this scheme consistent with the conventional one? How do we identify the binary strings which are d-bit Hamming distance away from a given string, say, the pivotal string? Note that, there are $\binom{n}{d}$ n-bit strings which are d-bit Hamming distance away from the pivotal string. This task can be accomplished if we adhere to the following counting scheme. The shells are labelled as S_k, $k = 0$ (for the innermost shell), $1, \cdots,$ n (for the outermost shell). If the pivotal string resides in the innermost (outermost) shell, one has to traverse radially outward (inward). However, if the pivotal string lies in the intermediate shell, then depending on the pivotal shell label k (even/odd) and Hamming distance d (even/odd), $\binom{n}{d}$ n-bit strings are located on either even or odd labelled shells k', within the reserved range

$$0 \leq [k - d] \leq k' \leq [k + d] \leq n, \tag{1}$$

as summarized in Table 1. Here, $[\cdots]$ reminds that k' must lie between 0 and n.

Pivotal shell (k)	Hamming distance (d)	Allowed shells (k')
0	d	d
even	even	even
even	odd	odd
odd	even	odd
odd	odd	even
n	d	$n - d$

Table 1: The scheme for placing $\binom{n}{d}$ n-bit strings which are d-bit Hamming distance away from given pivotal shell k to other shells k' in the reserved range $0 \leq [k - d] \leq k' \leq [k + d] \leq n$.

As already mentioned, there are $\binom{n}{d}$ n-bit strings which are d-bit Hamming distance away from the pivotal shell. Here, we wish to know the distribution

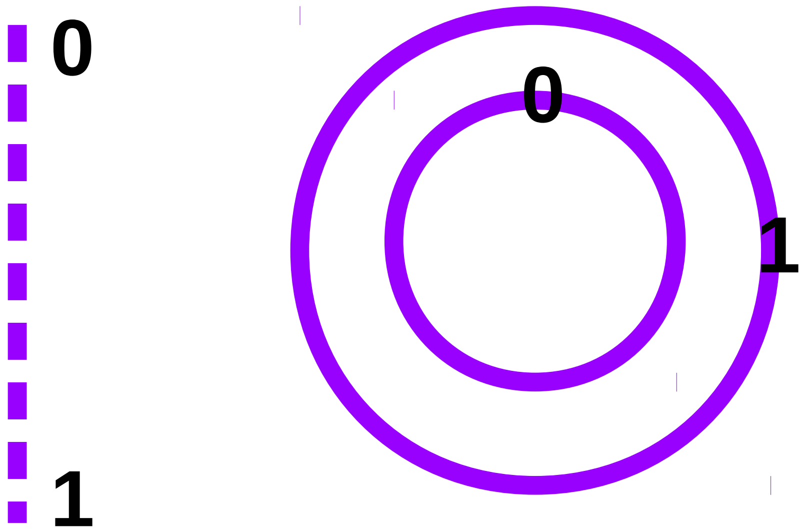

Figure 1: Single-bit strings, 0 and 1, on a line and concentric spherical shells.

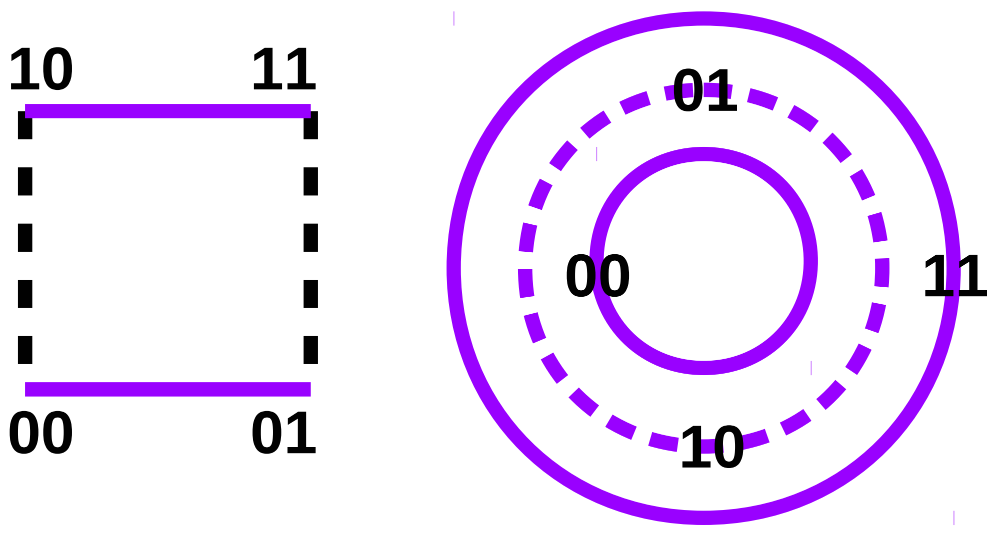

Figure 2: 2-bit strings on a rectangle and concentric spherical shells, such that the nearest strings differ by 1-bit Hamming distance.

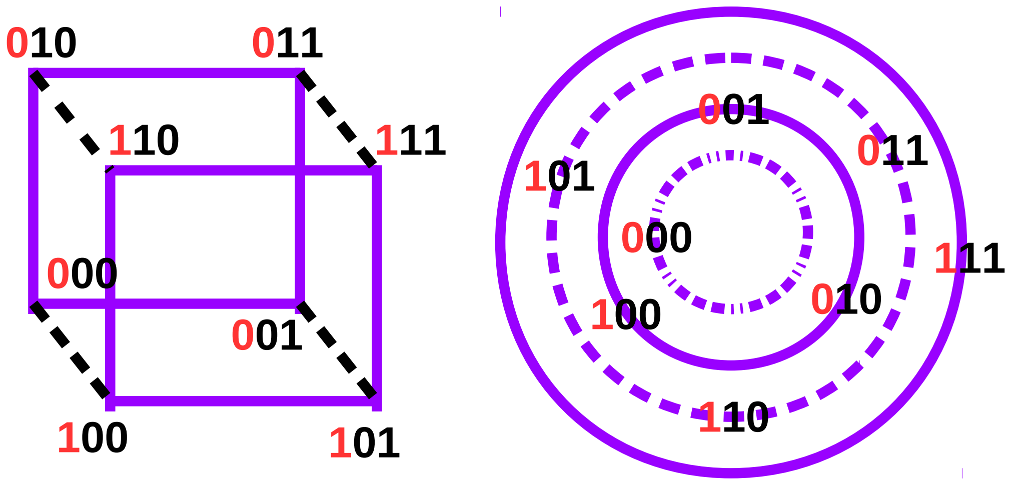

Figure 3: 3-bit strings on a cube and concentric spherical shells, such that the nearest strings differ by 1-bit Hamming distance.

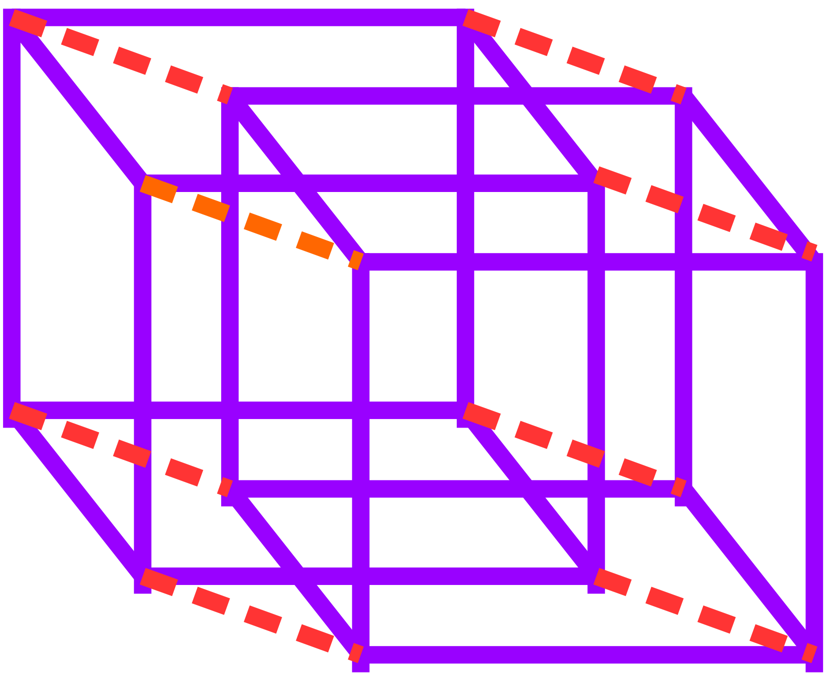

Figure 4: Identifying two cubes to reckon a 4-dimensional hypercube. Reckoning such hypercubes via conventional approach becomes more and more obscure with increasing topological dimension.

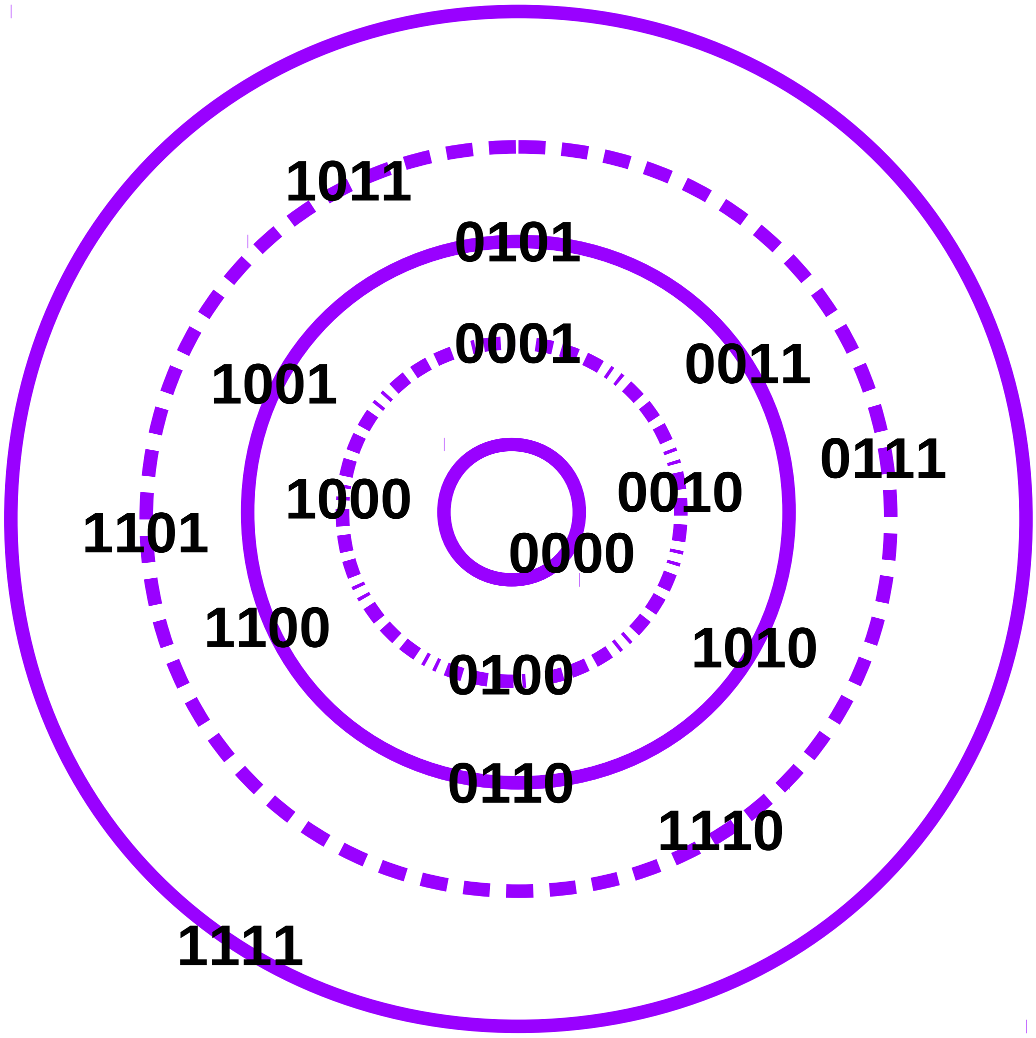

Figure 5: 4-bit strings on concentric spherical shells, such that the nearest strings differ by 1-bit Hamming distance.

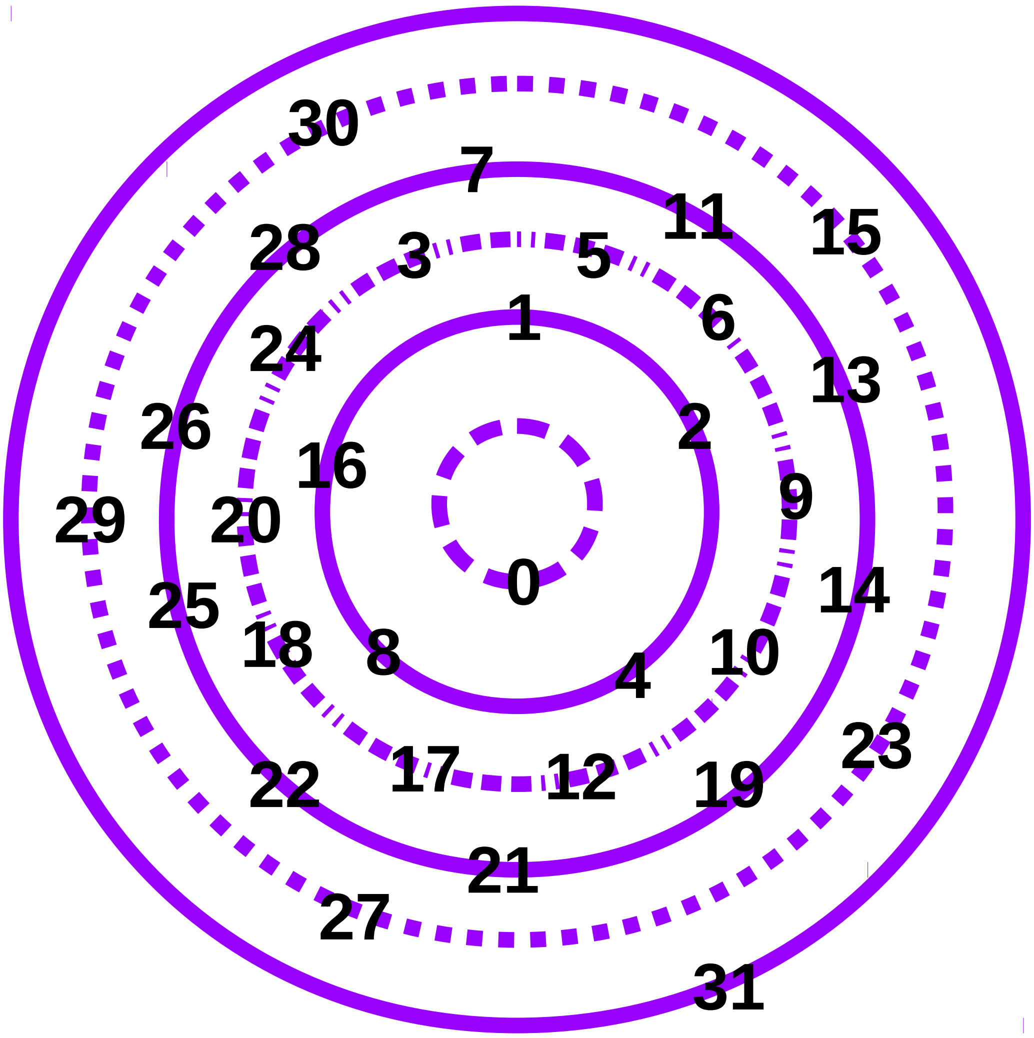

Figure 6: 5-bit strings on concentric spherical shells, such that the nearest strings differ by 1-bit Hamming distance. The binary strings are encrypted into decimal figures for convenience. For example, $31 \equiv 11111$, and so on.

of these $\binom{n}{d}$ strings. That is, how many of these are seated above and below the pivotal shell. Let d_{in} and d_{out} respectively be the (Hamming) distance of the innermost shell and the outermost shell from the pivotal shell, such that $d_{in} + d_{out} = n$. Then the number of n-bit strings below and above the pivotal shell, at Hamming distance d-bit away is (at least) $\binom{d_{in}}{d}$ and $\binom{d_{out}}{d}$ respectively. Sometimes these two numbers may not add up to $\binom{n}{d}$. The remaining strings in that situation, $\sum_r x_r$ in number (here the subscript r is the label of the shell in which the remaining strings reside), should be located on other allowed shells in the permissible range stated above in Eq. (1). Note that for the k^{th} pivotal shell, $r = k + d_{out} - d_{in}$ when $d = n$. This observation can be framed in the following equation

$$\binom{d_{in}}{d} + \binom{d_{out}}{d} + \sum_r x_r = \binom{n}{d}. \tag{2}$$

Illustrations are given in Table 2.

This idea of placing the binary strings in concentric spherical shells can be considered a naive proposal, but the merits of such a possibility is striking. *As our scheme is in "bijection" with the conventional n-dimensional cube approach, this can be seen as reckoning or viewing dimensions alternatively*. In this elegant scheme, it is relatively easier to view higher dimensions. Moreover, the dimensions seem to be interwoven and entangled. In transcending one dimension higher, it is simply not addition of an extra shell; rather it is an evolved process, as seen in Figs. 2 & 3. We will see how binary strings on intermediate shells lead naturally to an important class of entangled quantum states. As any dimension greater than three can be conceived manifestly in the concentric spherical shells model, it appears that there is no dimension greater than three.

Additional structures can be associated with this model. Imagine that all nodes corresponding to c-states are joined to each other with straight lines. This gives rise to the notion of vertices, edges and faces. Faces cut and cross each other. As all c-states are equivalent, they can be thought of rearranging their positions in time, with the only constraint that the Hamming distance amongst the c-states is not altered. Tesseract [7], a rotating four-dimensional hypercube, can be an illustration of the same.

3 Entanglement on Shells

We have seen what a bit is. It is a binary digit. Now, we learn about qubit. A qubit is an acronym for *quantum binary digit*. To understand the notion of qubit,

k	d_{in}	d_{out}	d	$\binom{d_{in}}{d} + \binom{d_{out}}{d} \stackrel{?}{=} \binom{n}{d}$	x_r
$n = 3$ (see Fig. 3)					
1	1	2	1	1+2=3	NA
			2	0+1 $\neq$ 3	$x_1 = 2$
			3	0+0 $\neq$ 1	$x_2 = 1$
2	2	1	1	2+1=3	NA
			2	1+0 $\neq$ 3	$x_2 = 2$
			3	0+0 $\neq$ 1	$x_1 = 1$
$n = 4$ (see Fig. 5)					
1	1	3	1	1+3=4	NA
			2	0+3 $\neq$ 6	$x_1 = 3$
			3	0+1 $\neq$ 4	$x_2 = 3$
			4	0+0 $\neq$ 1	$x_3 = 1$
2	2	2	1	2+2=4	NA
			2	1+1 $\neq$ 6	$x_2 = 4$
			3	0+0 $\neq$ 4	$x_1 = x_3 = 2$
			4	0+0 $\neq$ 1	$x_2 = 1$
3	3	1	1	3+1=4	NA
			2	3+0 $\neq$ 6	$x_3 = 3$
			3	1+0 $\neq$ 4	$x_2 = 3$
			4	0+0 $\neq$ 1	$x_1 = 1$

Table 2: Distribution of $\binom{n}{d}$ n-bit strings at d-bit Hamming distance from the k^{th}-pivotal shell as described in Table 1 under the constraint in Eq. (1). The subscript r in x_r denotes the allowed shell(s) in which the remaining strings are located when $\binom{d_{in}}{d} + \binom{d_{out}}{d} \neq \binom{n}{d}$. For the k^{th} pivotal shell, when Hamming distance d equals number of bits n in the binary strings, $r = k + d_{out} - d_{in}$. NA stands for "not applicable".

ask a person what he sees showing him a half-full glass of water. While an optimist will say glass is half-full, a pessimist will say glass is half-empty. What if the person is a quantum physicist? His answer that glass is both half-full and half-empty may drive you nuts! Similarly, a tossed-up coin while still in the air has both head and tail simultaneously. Interestingly, a quantum physicist opts for the holistic approach. He considers a linear "superposition" of all the possible (stationary) states of a physical system. In this spirit, a qubit, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ (the latter expression is a statement of the conservation of probability), is a generalized state of a two-level quantum system. Similarly, if a quantum system is multilevel, its generalized state is called "qudit". Two variables are said to be correlated when they cannot assume independent values in their allowed intervals, i.e., they are constrained. So, correlation is a constrained relation between two or more variables or quantities. Analogously, a two or more party quantum system is said to be entangled (entanglement is a kind of correlation) if its whole state cannot be "tensor factorized" into their individual quantum states. Yes, a whole can be different from its parts! Bell states [8], $|\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$ and $|\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$, are the best examples of two-qubit maximally entangled quantum states.

Imagine that each c-state is endowed with a "phase-clock". Then the aforementioned model can be used to describe entanglement on shells. Starting with c-state $|0\rangle^{\otimes n}$ as pivotal element, and labelling the concentric shells as $0, 1, \cdots, n$ (the innermost shell being the zeroth shell), the set of c-states residing in r^{th} shell naturally constitute the generalized Dicke state [9] $|gD^n_r\rangle = \sum c_{\mathcal{P}} \mathcal{P} \left(|0\rangle^{\otimes(n-r)} |1\rangle^{\otimes r}\right)$ with the normalization $\sum |c_{\mathcal{P}}|^2 = 1$, where the summation is over all permutations of $(n-r)$-$|0\rangle$s and r-$|1\rangle$s. All Dicke states but $|D^n_0\rangle = |0\rangle^{\otimes n}$ and $|D^n_n\rangle = |1\rangle^{\otimes n}$ are entangled. If the pivotal element is other than $|0\rangle^{\otimes n}$ the resulting state is again a generalized Dicke state upto local unitaries. Furthermore, two or more generalized Dicke states can be superposed together to give other entangled states. The most general state being $|\Psi\rangle = \sum_{r=0}^{n} \alpha_r |gD^n_r\rangle$, where $\sum_{r=0}^{n} |\alpha_r|^2 = 1$.

4 Conclusion

To sum up, bit is potent and it–the quantum of information–emerges from it. We have shown that how a naive arrangement of bit-strings on concentric spherical shells can help us to perceive higher topological dimensions greater than three in an elegant fashion. The dimensions appear intrinsically entangled. As a by-product of it, we see that such an arrangement gives rise to an important class of entangled quantum states.

Acknowledgements. AK acknowledges Rahul Kumar Singh, Udit Khanna and Mallesham K. for discussions and useful suggestions.

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