A NESTED INFINITE RADICAL EXPRESSION FOR ODD NUMBERS

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Abstract

I am going to provide a Nested Infinite Radical Expression for all Odd Numbers greater than 1. Along with that, I will also prove the Proposition mentioned by various methods.

1 Introduction

Srinivasa Ramanujan proposed a nested infinite radical problem in JIMS (Journal of the Indian Mathematical Society). The problem (Q.289) says Find the value of :

$$\sqrt{1+2\sqrt{1+3\sqrt{1+...}}} \tag{1}$$

Ramanujan gave the solution of the above problem as 3. I have tried to create some similar series like this and later generalized it for all odd numbers greater than 1.

Proposition 1. *For all odd numbers (in the form 2k-1) greater than 1 we have*

$$2k-1=\sqrt{1+4\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}} \tag{2}$$

Later in the paper, I will also provide some strong proof using the generalization of nested expression given by Srinivasa Ramanujan, along with that I will also show that how any odd numbers can be transformed into a nested infinite expression and vice versa.

2 Generalization

I begin by proving Proposition 1,

$$2k-1=\sqrt{1+4\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}}$$

Proof. From the generalization of (1) given by Srinivasa Ramanujan, i.e.

$$\sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{...}}}=x+n+a \qquad (3)$$

if we set a=0, $x=k^2-k-1$, and n=1 we get,

$$\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}=k^2-k$$

After some algebric manupulation we get,

$$\sqrt{1+4\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}}=\sqrt{1+4(k^2-k)}$$

$$\Rightarrow \sqrt{1+4\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}}=\sqrt{(2k-1)^2}$$

$$\Rightarrow \sqrt{1+4\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}}=(2k-1) \qquad \square$$

and this completes our proof. Now I will provide some examples which would help us understand the need for the formula (2).

Example 2.1 Find a nested radical expression for : i) 131 ii) 729

Solution. i) To find the required expression we first set

$$2k-1=131 \Rightarrow k=66$$

Now, using (2) we get,

$$131=\sqrt{1+4\sqrt{1+4289\sqrt{1+4290\sqrt{1+...}}}}$$

ii)Again, to find the required expression we first set

$$2k-1=729 \Rightarrow k=365$$

Now, using (2) we get,

$$729=\sqrt{1+4\sqrt{1+132859\sqrt{1+132860\sqrt{1+...}}}}$$

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Example 2.2 Find the value of :

$$\sqrt{1+4\sqrt{1+65279\sqrt{1+65280\sqrt{1+...}}}}$$

Solution. Now to solve this we need to compare this with (2),

$$\sqrt{1+4\sqrt{1+\underbrace{65279}_{=k^2-k-1}\sqrt{1+65280\sqrt{1+...}}}}$$

$$So, k^2-k-1=65279$$

$$\Rightarrow k=256$$

or

$$\Rightarrow k=-255$$

Since k is positive so our final value for k is 256 and thus our final answer is 511. ◀

3 The Famous Problem

Now I will show how we can use the above-mentioned formula to solve the problem given by Srinivasa Ramanujan, i.e.

$$Let, S=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+...}}}}}$$

On comparing with (2) we get,

$$S=\sqrt{1+2\sqrt{1+3\underbrace{\sqrt{1+4\sqrt{1+5\sqrt{1+...}}}}_{=\sqrt{1+4\sqrt{1+(k^2-k-1)\sqrt{1+(k^2-k)\sqrt{1+...}}}}}}}$$

$$\Rightarrow k^2-k-1=5$$

$$\Rightarrow k=3$$

or

$$\Rightarrow k=-2$$

Since k is positive so our final value for k is 3, Now on moving back to our original equation we get,

$$\Rightarrow S = \sqrt{1 + 2\sqrt{1 + 3\underbrace{\sqrt{1 + 4\sqrt{1 + 5\sqrt{1 + ...}}}}_{=2(3)-1}}}$$

$$\Rightarrow S = \sqrt{1 + 2\sqrt{1 + 3 \cdot 5}} = \sqrt{1 + 2\sqrt{16}} = \sqrt{1 + 2 \cdot 4} = \sqrt{9} = 3 \quad \square$$

4 Acknowledgements

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References

[1] Nested radical - Wikipedia.

[2] The Problems Submitted by Ramanujan to the Journal of the Indian Mathematical Society by Bruce C. Berndt, Youn-Seo Choi, and Soon-Yi Kang