Riemann's Functional Equation When $\zeta(s) = 0 = \zeta(1-s)$

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ABSTRACT

Riemann's Functional equation $\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ has values where $\zeta(s) = 0$ at negative even integers of s (-2,-4,-6...) when the function $\sin(\pi s/2)$ equals 0. This paper demonstrates that the only other case where $\zeta(s) = 0$ in Riemann's functional equation is when $\zeta(s) = \zeta(1-s)$ which is only true when the real part of s = 1/2.

SECTION I

Riemann's Functional equation $\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ can only have $\zeta(s) = 0$ if at least one of the following criteria are true:

$2^s\pi^{s-1} = 0$

$\sin(\pi s/2) = 0$

$\Gamma(1-s) = 0$

$\zeta(1-s) = 0$

The function $2^s\pi^{s-1}$ can never equal 0 as there is no value of s that would satisfy $2^s\pi^{s-1} = 0$. The gamma function $\Gamma(1-s)$ can also never equal 0.

The function $\sin(\pi s/2) = 0$ when $\pi s/2$ = whole number intervals of π. When s is a positive even integer, the product $\sin(\pi s/2)\Gamma(1-s)$ is non-zero because $\Gamma(1-s)$ has a simple pole, which cancels the simple zero of the sin function. The negative even integers of s (-2,-4,-6...) correspond to the trivial zeros of the Riemann zeta function where $\zeta(s) = 0$. There are no other cases where $\sin(\pi s/2) = 0$ for real, imaginary, or complex numbers except for when s equals even integers.

SECTION II

The only situation remaining that could make $\zeta(s) = 0$ in Riemann's functional equation is when $\zeta(1-s) = 0$. But if one considers the occurrence where $\zeta(1-s) = 0$, then it must also be true that $\zeta(s) = 0$. Secondly, if one considers the occurrence where $\zeta(s) = 0$, then it must be due to the fact that $\zeta(1-s) = 0$. This can only be true if $\zeta(1-s) = 0 = \zeta(s)$ and therefore $\zeta(1-s) = \zeta(s)$.

The only real number value that satisfies the real or complex equation $\zeta(1-s) = \zeta(s)$ is when the real part of s = 1/2, i.e. $\zeta(1-[1/2]) = \zeta(1/2)$. There are no other real values of s that satisfy the equation $\zeta(1-s) = 0 = \zeta(s)$. Only when the real part of s = 1/2 can this be true. Since s = 1/2 is the only real number that could be used to satisfy the real or complex equation $\zeta(s) = 0 = \zeta(1-s)$ then it must be true that $\zeta(s)$ can only equal 0 when the real part of s = 1/2.

CONCLUSION

Besides the trivial zeros resulting from Riemann's functional equation, the only occurrences where $\zeta(s) = 0$ is found to be when $\zeta(1-s) = 0 = \zeta(s)$. The equation $\zeta(1-s) = \zeta(s)$ can only be true when the real part of s = 1/2, thus $\zeta(s)$ can only equal 0 when the real part of s = 1/2. Therefore all non-trivial zeros of the Riemann Zeta function must lie on the critical line where real part of s = 1/2.

References

Riemann, Bernhard (1859), "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse", Monatsberichte der Berliner Akademie. In *Gesammelte Werke*, Teubner, Leipzig (1892), Reprinted by Dover, New York (1953).