On Sums of Product of Powers of Palindromic Sequence and Arithmetic Progression

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Abstract

In this paper, we combine a real or complex palindromic sequence, (i.e, a sequence that remains the same when the sequence is reversed) with a sequence in arithmetic progression to produce the sums of product of powers of palindromic sequence and arithmetic progression. As a result, we establish a relationship between the sum of n terms of an arithmetic progression, the sum of their squares, the sum of their cubes, and the number of terms. Also, we establish a relationship between the sum of n terms of an arithmetic progression, the sum of their squares, the sum of their cubes, the sum of their fourth powers, the sum of their fifth powers and the number of terms. We also give two new different expressions for Franel numbers as well as the right-hand side of first Strehl identity.

Keywords: *palindromic sequence, binomial coefficients, arithmetic progression, Franel numbers, first Strehl identity.*

Introduction

An arithmetic progression is a sequence of numbers such that the difference between the consecutive terms is constant. For example, the sequence $1, 4, 7, 10, 13, 16, 19\ldots$ is an arithmetic progression with a common difference of 3. The first number known as the first term, the number of terms, the common difference, and the sum of n terms of an arithmetic progression are denoted by a, d, n, and S_n respectively . Hence, the formula for finding the sum of n terms of an arithmetic progression is:

$$S_n = \frac{n}{2}(2a + (n-1)d).$$

The binomial expansion describes the expansion of $(x+y)^n$, for any positive integer n. The binomial expansion is denoted by

$$(x+y)^n = \sum_{k=0}^{n} x^{n-k} y^k. \tag{2.1}$$

For instance, if n is 4, then

$$(x+y)^4 = x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4.$$

The binomial coefficients are the positive integers that occur as coefficients in binomial expansion. For instance, the binomial coefficients of $(x+y)^4$ are $1, 4, 6, 4, 1$. A binomial coefficient is denoted by $\binom{n}{k}$, for $n \geq k \geq 0$, where n and k are integers. The formula for finding a binomial coefficient is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ is the factorial of n. The sum of all the coefficients of $(x+y)^n$ is 2^n, i.e.,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{2.2}$$

Many identities involving binomial coefficients have been discovered. For instance, Boros and Moll [1, 14–15] showed that sums of the form $\sum_{k=0}^{n} \binom{n}{k} k^r$ are given by:

$$\sum_{k=0}^{n} \binom{n}{k} k = n 2^{n-1}, \tag{2.3}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1) 2^{n-2}, \tag{2.4}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^3 = n(n+3) 2^{n-3}, \tag{2.5}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^4 = n(n+1)(n^2+5n-2) 2^{n-4}, \tag{2.6}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^5 = n^2(n^3+10n^2+15n-10) 2^{n-5}, \tag{2.7}$$

and so on.

A palindromic sequence is a sequence that remains the same when the sequence is reversed. For example, $2, -5, 7, 7, -5, 2$ is a palindromic sequence because we have the same sequence when the numbers are reversed.

Franel [2] showed that if

$$f_{(n,p)} = \sum_{k=0}^{n} \binom{n}{k}^{p}, \tag{2.8}$$

then

$$(n+1)^2 f_{(n+1,3)} = (7n^2 + 7n + 2) f_{(n,3)} + 8n^2 f_{(n-1,3)}. \tag{2.9}$$

Also, Franel [3] showed that

$$(n+1)^3 f_{(n+1,4)} = 2(2n+1)(3n^2 + 3n + 1) f_{(n,4)} + 4(4n-1)(4n+1)^2 f_{(n-1,4)}. \tag{2.10}$$

We should note that $f_{(n,3)}$ is called the nth Franel number. They arise in first Strehl identity. Strehl [4] showed that

$$f_{(n,3)} = \sum_{k=0}^{n} \binom{n}{k}^{2} \binom{2k}{n}. \tag{2.11}$$

In this paper, we establish two relationships between the generalizations of (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7). Consequently, we obtain some interesting results among which are finding two new different expressions for $f_{(n,3)}$. We present our main results in section three and list some applications of our main results in the same section.

Main results

Let $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ be a sequence of binomial coefficients such that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ is the sum of binomial coefficients, and $\binom{n}{k} = \binom{n}{n-k}$ holds for $n \geq k \geq 0$. We define $\beta_{(n,0)}, \beta_{(n,1)}, \beta_{(n,2)}, \beta_{(n,3)}, \ldots, \beta_{(n,n)}$ as a palindromic sequence such that

$$\alpha_{(n,p)} = \sum_{k=0}^{n} \beta_{(n,k)}^{p},$$

is the sums of powers of a palindromic sequence, where $\beta_{(n,k)} = \beta_{(n,n-k)}$ holds for $n \geq k \geq 0$, and $\beta_{(n,0)} = \beta_{(n,n)}, \beta_{(n,1)} = \beta_{(n,n-1)}, \beta_{(n,2)} = \beta_{(n,n-2)} \ldots$ are any real or complex numbers.

Theorem 1. *Let $\alpha_{(n,p)}$ be the sums of powers of a palindromic sequence, i.e.,*

$$\alpha_{(n,p)} = \sum_{k=0}^{n} \beta^{p}_{(n,k)}.$$

If

$$A_{(n,p)} = \sum_{k=0}^{n} \beta^{p}_{(n,k)}(a+kd), \tag{3.1}$$

$$B_{(n,p)} = \sum_{k=0}^{n} \beta^{p}_{(n,k)}(a+kd)^2, \tag{3.2}$$

and

$$C_{(n,p)} = \sum_{k=0}^{n} \beta^{p}_{(n,k)}(a+kd)^3, \tag{3.3}$$

where a, d, p are real or complex numbers, then

$$C_{(n,p)} = \frac{A_{(n,p)}}{\alpha^2_{(n,p)}}(3B_{(n,p)}\alpha_{(n,p)} - 2A^2_{(n,p)}). \tag{3.4}$$

We should note that

$$\beta_{(n,k)} = \beta_{(n,n-k)}, \tag{3.5}$$

for $n \geq k \geq 0$.

Proof. Let $S_j = \sum_{k=0}^{n} \beta^{p}_{(n,k)}(a+kd)^j$, where j is any integer greater than or equal to zero, we can see that for all real or complex a, d, p, S_j can be written as

$$S_j = \beta^{p}_{(n,0)}(a)^j + \beta^{p}_{(n,1)}(a+d)^j + \beta^{p}_{(n,2)}(a+2d)^j + \cdots + \beta^{p}_{(n,n)}(a+nd)^j. \tag{3.6}$$

Since $\beta_{(n,k)} = \beta_{(n,n-k)}$ is true for $n \geq k \geq 0$, we see that S_j can also be written as

$$S_j = \beta^{p}_{(n,0)}(a+nd)^j + \beta^{p}_{(n,1)}(a+(n-1)d)^j + \beta^{p}_{(n,2)}(a+(n-2)d)^j + \cdots + \beta^{p}_{(n,n)}(a)^j. \tag{3.7}$$

Adding (3.6) and (3.7), we have

$$2S_j = \sum_{k=0}^{n} \beta^{p}_{(n,k)}((a+kd)^j + (a+(n-k)d)^j),$$

$$S_j = \frac{1}{2}\sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^j + (a+(n-k)d)^j), \tag{3.8}$$

We know that $S_1 = A_{(n,p)}$, $S_2 = B_{(n,p)}$, $S_3 = C_{(n,p)}$. So, we have that

$$A_{(n,p)} = \frac{(2a+nd)}{2}\alpha_{(n,p)}. \tag{3.9}$$

Putting (3.9) in (3.4), we have

$$C_{(n,p)} = \frac{(2a+nd)}{2}\left(3B_{(n,p)} - \frac{(2a+nd)^2}{2}\alpha_{(n,p)}\right),$$

$$(2a+nd)^3\alpha_{(n,p)} = 6(2a+nd)B_{(n,p)} - 4C_{(n,p)}. \tag{3.10}$$

$$B_{(n,p)} = \frac{1}{2}\sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^2 + (a+(n-k)d)^2), \tag{3.11}$$

$$6(2a+nd)B_{(n,p)} = 3(2a+nd)\sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^2 + (a+(n-k)d)^2). \tag{3.12}$$

$$C_{(n,p)} = \frac{1}{2}\sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^3 + (a+(n-k)d)^3), \tag{3.13}$$

$$4C_{(n,p)} = 2\sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^3 + (a+(n-k)d)^3). \tag{3.14}$$

Subtracting (3.14) from (3.12), we have

$$6(2a+nd)B_{(n,p)} - 4C_{(n,p)} = \sum_{k=0}^{n} \beta^p_{(n,k)}\big(3(2a+nd)\big((a+kd)^2+(a+(n-k)d)^2\big)-2\big)(a+kd)^3+(a+(n-k)d)^3\big)\big).$$

$$6(2a+nd)B_{(n,p)} - 4C_{(n,p)} = (2a+nd)^3\alpha_{(n,p)}. \tag{3.15}$$

we can see that (3.15) and (3.10) are the same. Therefore, (3.4) is true. $\square$

Theorem 2. *If*

$$D_{(n,p)} = \sum_{k=0}^{n} \beta^{p}_{(n,k)} (a+kd)^4, \tag{3.16}$$

$$E_{(n,p)} = \sum_{k=0}^{n} \beta^{p}_{(n,k)} (a+kd)^5, \tag{3.17}$$

where a, d are any real or complex numbers, then

$$E_{(n,p)} = \frac{A_{(n,p)}}{\alpha^4_{(n,p)}} (5D_{(n,p)}\alpha^3_{(n,p)} - 4A_{(n,p)}C_{(n,p)}\alpha^2_{(n,p)} - 8A^2_{(n,p)}B_{(n,p)}\alpha_{(n,p)} + 8A^4_{(n,p)}). \tag{3.18}$$

Proof. Putting (3.9) in (3.18), we have

$$E_{(n,p)} = \frac{(2a+nd)}{2} \left(5D_{(n,p)} - 2(2a+nd)C_{(n,p)} - 2(2a+nd)^2 B_{(n,p)} + \frac{(2a+nd)^4}{2}\alpha_{(n,p)} \right),$$

$$(2a+nd)^5 \alpha_{(n,p)} = 4E_{(n,p)} + 4(2a+nd)^3 B_{(n,p)} + 4(2a+nd)^2 C_{(n,p)} - 10(2a+nd)D_{(n,p)}. \tag{3.19}$$

From (3.8), we know that $B_{(n,p)} = S_2$, $C_{(n,p)} = S_3$, $D_{(n,p)} = S_4$, $E_{(n,p)} = S_5$. So, we have

$$D_{(n,p)} = \frac{1}{2} \sum_{k=0}^{n} \beta^{p}_{(n,k)} ((a+kd)^4 + (a+(n-k)d)^4),$$

$$10(2a+nd)D_{(n,p)} = 5(2a+nd) \sum_{k=0}^{n} \beta^{p}_{(n,k)} ((a+kd)^4 + (a+(n-k)d)^4), \tag{3.20}$$

Also,

$$E_{(n,p)} = \frac{1}{2} \sum_{k=0}^{n} \beta^{p}_{(n,k)} ((a+kd)^5 + (a+(n-k)d)^5),$$

$$4E_{(n,p)} = 2 \sum_{k=0}^{n} \beta^{p}_{(n,k)} ((a+kd)^5 + (a+(n-k)d)^5), \tag{3.21}$$

Let y be the difference of (3.20) and (3.20), we have

$$y = \sum_{k=0}^{n} \beta^{p}_{(n,k)} \big(2\big((a+kd)^5 + (a+(n-k)d)^5\big) - 5(2a+nd)\big((a+kd)^4 + (a+(n-k)d)^4\big)\big) \tag{3.22}$$

Multiplying (3.11) by $4(2a+nd)^3$, we have

$$4(2a+nd)^3 B_{(n,p)} = 2(2a+nd)^3 \sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^2 + (a+(n-k)d)^2). \tag{3.23}$$

Multiplying (3.13) by $4(2a+nd)^2$, we have

$$4(2a+nd)^2 C_{(n,p)} = 2(2a+nd)^2 \sum_{k=0}^{n} \beta^p_{(n,k)}((a+kd)^3 + (a+(n-k)d)^3). \tag{3.24}$$

Let y_1 be the sum of (3.23) and (3.24), we have

$$y_1 = 2(2a+nd)^2 \sum_{k=0}^{n} \beta^p_{(n,k)}\big((2a+nd)\big((a+kd)^2+(a+(n-k)d)^2\big)+\big((a+kd)^3+(a+(n-k)d)^3\big)\big) \tag{3.25}$$

Now, Adding (3.22) and (3.25), we see that

$$y + y_1 = (2a+nd)^5 \alpha_{(n,p)}$$

$$4E_{(n,p)} + 4(2a+nd)^3 B_{(n,p)} + 4(2a+nd)^2 C_{(n,p)} - 10(2a+nd) D_{(n,p)} = (2a+nd)^5 \alpha_{(n,p)}. \tag{3.26}$$

We can see that (3.19) and (3.26) are the same. Therefore, (3.18) is true.

□

Applications of the main results

We should note that a sequence of binomial coefficients on the nth row of Pascal's triangle is a special case of palindromic sequence.

Now, if we let $a=1$, $d=1$, and $\alpha_{(n,p)}$ be the sums of real or complex powers of binomial coefficients on the nth row of Pascal's triangle, (3.4) becomes

$$\alpha_{(n,p)} = \sum_{k=0}^{n} \binom{n}{k}^p \left(6\left(\frac{k+1}{n+1}\right)^2 - 4\left(\frac{k+1}{n+1}\right)^3\right), \tag{3.27}$$

and (3.18) becomes

$$\alpha_{(n,p)} = \sum_{k=0}^{n} \binom{n}{k}^p \left(4\left(\frac{k+1}{n+1}\right)^2 + 4\left(\frac{k+1}{n+1}\right)^3 - 10\left(\frac{k+1}{n+1}\right)^4 + 4\left(\frac{k+1}{n+1}\right)^5\right). \tag{3.28}$$

We can see that (3.27) and (3.28) give two different expressions for the sums of real or complex powers of binomial coefficients. Letting $p=3$ in (3.27) and (3.28) give another two different expressions for nth Franel number as well as first Strehl identity.

If we let $p = 0$ and subtract one from n, (3.4) becomes

$$C_{(n-1,0)} = \frac{A_{(n-1,0)}}{n^2}(3nB_{(n-1,0)} - 2A^2_{(n-1,0)}), \tag{3.29}$$

and (3.18) becomes

$$E_{(n,0)} = \frac{A_{(n,0)}}{n^4}(5D_{(n,0)}n^3 - 4A_{(n,0)}C_{(n,0)}n^2 - 8A^2_{(n,0)}B_{(n,0)}n + 8A^4_{(n,0)}), \tag{3.30}$$

where $A_{(n,0)} = \sum_{k=0}^{n-1}(a + kd)$, $B_{(n,0)} = \sum_{k=0}^{n-1}(a + kd)^2$, $C_{(n,0)} = \sum_{k=0}^{n-1}(a + kd)^3$, $D_{(n,0)} = \sum_{k=0}^{n-1}(a + kd)^4$, $E_{(n,0)} = \sum_{k=0}^{n-1}(a + kd)^5$.
We can see that (3.29) establishes a relationship between the sum of n terms of an arithmetic progression, the sum of their squares, the sum of their cubes, and the number of terms. This means that if three of the four variables in (3.29) are given, the fourth variable can be found easily using (3.29). Also, (3.30) establishes a relationship between the sum of n terms of an arithmetic progression, the sum of their squares, the sum of their cubes, the sum of their fourth powers, the sum of their fifth powers, and the number of terms.

If we let $p = 1$, $\alpha_{(n,1)}$ be the sum of binomial coefficients, (3.4) becomes

$$C_{(n,1)} = \frac{A_{(n,1)}}{2^{2n}}(3 \cdot 2^n B_{(n,1)} - 2A^2_{(n,1)}),$$

and (3.18) becomes

$$E_{(n,1)} = \frac{A_{(n,1)}}{2^{4n}}(5 \cdot 2^{3n} D_{(n,1)} - 4 \cdot 2^{2n} A_{(n,1)}C_{(n,1)} - 8 \cdot 2^n A^2_{(n,1)}B_{(n,1)} + 8A^4_{(n,1)}),$$

where $A_{(n,1)} = \sum_{k=0}^{n} \binom{n}{k}(a + kd)$, $B_{(n,1)} = \sum_{k=0}^{n} \binom{n}{k}(a + kd)^2$, $C_{(n,1)} = \sum_{k=0}^{n} \binom{n}{k}(a + kd)^3$, $D_{(n,1)} = \sum_{k=0}^{n} \binom{n}{k}(a + kd)^4$, $E_{(n,1)} = \sum_{k=0}^{n} \binom{n}{k}(a + kd)^5$.

If we let $p = 2$, $\alpha_{(n,2)}$ be the sum of squares of binomial coefficients, (3.4) becomes

$$C_{(n,2)} = \frac{A_{(n,2)}}{\binom{2n}{n}^2}\left(3\binom{2n}{n}B_{(n,2)} - 2A^2_{(n,2)}\right),$$

and (3.18) becomes

$$E_{(n,2)} = \frac{A_{(n,2)}}{\binom{2n}{n}^4}\left(5\binom{2n}{n}^3 D_{(n,2)} - 4\binom{2n}{n}^2 A_{(n,2)}C_{(n,2)} - 8\binom{2n}{n}A^2_{(n,2)}B_{(n,2)} + 8A^4_{(n,2)}\right),$$

where $A_{(n,2)} = \sum_{k=0}^{n} \binom{n}{k}^2(a+kd)$, $B_{(n,2)} = \sum_{k=0}^{n} \binom{n}{k}^2(a+kd)^2$, $C_{(n,2)} = \sum_{k=0}^{n} \binom{n}{k}^2(a+kd)^3$, $D_{(n,2)} = \sum_{k=0}^{n} \binom{n}{k}^2(a + kd)^4$, $E_{(n,2)} = \sum_{k=0}^{n} \binom{n}{k}^2(a + kd)^5$.

Problems

If n is any positive integer, Show that

$$\sum_{k=0}^{n}\binom{n}{k}^4 = 5\sum_{k=0}^{n-1}\binom{n}{k+1}^2\binom{n-1}{k}^2 - \frac{4n-1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}^4, \tag{13}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}^4 = 20\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n-1}{k}^3 - 6\frac{4n-1}{n}\sum_{k=0}^{n-1}\binom{n-1}{k}^4. \tag{14}$$

Conclusion

In this paper, we combined the powers of a palindromic sequence with the powers of an arithmetic progression to generalize (2.2), (2.3) (2.4), (2.5), (2.6) and (2.7), which in turn were used to establish two relationships. As a result, some interesting formulas were derived. Two new different expressions for the sums of real or complex powers of binomial coefficients were also derived, thereby giving two new different expressions for Franel number as well as first Strehl identity. A relationship between the sum of n terms of an arithmetic progression, the sum of their squares, the sum of their cubes, and the number of terms was established. Also, a relationship between the sum of n terms of an arithmetic progression, the sum of their squares, the sum of their cubes, the sum of their fourth powers, the sum of their fifth powers, and the number of terms was established.

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