On Sums of Product of Powers of Generalized Binomial Coefficients and Arithmetic Progression

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1 Abstract

Let

$$S_j = \sum_{k=0}^{n} {\binom{n}{k}_m}^p (a + kd)^j \tag{1.1}$$

be the sums of product of powers of generalized binomial coefficients and arithmetic progression, where a, d, and p are any real or complex numbers, n and j are integers greater than or equal to zero. Let

$$\alpha_{(m,n,p)} = \sum_{k=0}^{n} {\binom{n}{k}_m}^p \tag{1.2}$$

be the sums of real or complex powers of generalized binomial coefficients. In this paper, we establish a relationship between S_1, S_2, S_3, and $\alpha_{(m,n,p)}$. Also, we establish a relationship between S_1, S_2, S_3, S_4, S_5, and $\alpha_{(m,n,p)}$. As a result, we establish a relationship between the sum of an arithmetic progression, sum of squares of an arithmetic progression, sum of cubes of an arithmetic progression, and the number of terms of an arithmetic progression. Also, we establish a relationship between the sum of an arithmetic progression, sum of squares of an arithmetic progression, sum of cubes of an arithmetic progression, sum of fourth powers of an arithmetic progression, sum of fifth powers of an arithmetic progression and the number of terms of an arithmetic progression. We also give two new different expressions for Franel numbers as well as the right-hand side of first Strehl identity.

Keywords: *sums of powers of binomial coefficients, arithmetic progression, Franel Numbers, first Strehl identity.*

2 Introduction

An arithmetic progression is a sequence of numbers such that the difference between the consecutive terms is constant. For example, the sequence $1, 4, 7, 10, 13, 16, 19 \cdots$ is an arithmetic progression with a common difference of 3. The first number known as the first term, the number of terms, the common difference, and the sum of n terms of an arithmetic progression are denoted by a, d, n, and S_n resprectively . Hence, the formula for finding the sum of n terms of an arithmetic progression is:

$$S_n = \frac{n}{2}(2a + (n-1)d).$$

The binomial expansion describes the expansion of $(x+y)^n$, for any positive integer n. The binomial expansion is denoted by

$$(x+y)^n = \sum_{k=0}^{n} x^{n-k} y^k. \tag{2.1}$$

For instance, if n is 4, then

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

The binomial coefficients are the positive integers that occur as coefficients in binomial expansion. For instance, the binomial coefficients of $(x+y)^4$ are $1, 4, 6, 4, 1$. A binomial coefficient is denoted by $\binom{n}{k}$, for $n \geq k \geq 0$, where n and k are integers. The formula for finding a binomial coefficient is:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!},$$

where $n! = 1 \cdot 2 \cdot 3 \cdots n$ is the factorial of n. The sum of all the coefficients of $(x+y)^n$ is 2^n, that is

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{2.2}$$

Many identities involving binomial coefficients have been discovered. For instance, G. Boros and V. Moll [1, 14–15] showed that sums of the form $\sum_{k=0}^{n} \binom{n}{k} k^r$ are given by:

$$\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}, \tag{2.3}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^2 = n(n+1)2^{n-2}, \tag{2.4}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^3 = n(n+3)2^{n-3}, \tag{2.5}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^4 = n(n+1)(n^2+5n-2)2^{n-4}, \tag{2.6}$$

$$\sum_{k=0}^{n} \binom{n}{k} k^5 = n^2(n^3+10n^2+15n-10)2^{n-5}, \tag{2.7}$$

and so on.

In 1894, Franel [2] showed that if

$$f_{(n,p)} = \sum_{k=0}^{n} \binom{n}{k}^p, \tag{2.8}$$

then

$$(n+1)^2 f_{(n+1,3)} = (7n^2+7n+2)f_{(n,3)} + 8n^2 f_{(n-1,3)}. \tag{2.9}$$

Also in 1985, Franel [3] showed that

$$(n+1)^3 f_{(n+1,4)} = 2(2n+1)(3n^2+3n+1)f_{(n,4)} + 4(4n-1)(4n+1)^2 f_{(n-1,4)}. \tag{2.10}$$

We should note that $f_{(n,3)}$ is called the nth Franel number. They arise in first Strehl identity. V. Strehl [4], in 1994 showed that

$$f_{(n,3)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}. \tag{2.11}$$

In this paper, we establish two relationships between the generalizations of (2.2), (2.3), (2.4), (2.5), (2.6) and (2.7). Consequently, we obtain some interesting results among which are finding two new different expressions for $f_{(n,3)}$. We present our main result in section three and list some important special cases of our main result in the same section.

3 Main Result

Let $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3} \cdots \binom{n}{n}$ be a sequence of binomial coefficients such that $\beta = \sum_{k=0}^{n} \binom{n}{k}$ is the sum of binomial coefficients, and $\binom{n}{k} = \binom{n}{n-k}$ holds for $n \geq k \geq 0$. We define $\binom{n}{0}_m, \binom{n}{1}_m, \binom{n}{2}_m, \binom{n}{3}_m \cdots \binom{n}{n}_m$ as a sequence of generalized binomial coefficients such that

$$\alpha_{(m,n,p)} = \sum_{k=0}^{n} \binom{n}{k}_m,$$

is the sum of generalized binomial coefficients, where $\binom{n}{k}_m = \binom{n}{n-k}_m$ holds for $n \geq k \geq 0$, and $\binom{n}{0}_m = \binom{n}{n}_m, \binom{n}{1}_m = \binom{n}{n-1}_m, \binom{n}{2}_m = \binom{n}{n-2}_m \cdots$ are any real or complex numbers. For instance, $2, -5, 7, 10, 7, -5, 2$ is a sequence of generalized binomial coefficients because the first number and the last number of the sequence are the same, the second number and the penultimate number of the sequence are the same, and so on. So, we may say $\binom{n}{0}_m = 2$, $\binom{n}{1}_m = -5$, $\binom{n}{2}_m = 7$, $\binom{n}{3}_m = 10$, $\binom{n}{4}_m = 7$, $\binom{n}{5}_m = -5$, and $\binom{n}{6}_m = 2$.

Theorem 1: Let $\alpha_{(m,n,p)}$ be the sums of powers of generalized binomial coefficients, that is

$$\alpha_{(m,n,p)} = \sum_{k=0}^{n} \binom{n}{k}_m^p.$$

If

$$A_p = \sum_{k=0}^{n} \binom{n}{k}_m^p (a + kd), \tag{3.1}$$

$$B_p = \sum_{k=0}^{n} \binom{n}{k}_m^p (a + kd)^2, \tag{3.2}$$

and

$$C_p = \sum_{k=0}^{n} \binom{n}{k}_m^p (a + kd)^3, \tag{3.3}$$

where a, d, p are real or complex numbers, then

$$C_p = \frac{A}{\alpha_{(m,n,p)}^2}(3B_p\alpha_{(m,n,p)} - 2{A_p}^2). \tag{3.4}$$

We should note that if $\binom{n}{k}_m = \binom{n}{n-k}_m$ for $n \geq k \geq 0$, then

$$\binom{n}{k}_m^p = \binom{n}{n-k}_m^p, \tag{3.5}$$

also holds for any real or complex p.

Proof. Let $S_j = \sum_{k=0}^{n} \binom{n}{k}_m^p (a+kd)^j$, where j is any integer greater than or equal to zero, we can see that for all real or complex a, d, p, S_j can be written as

$$S_j = \binom{n}{1}_m^p (a)^j + \binom{n}{2}_m^p (a+d)^j + \binom{n}{3}_m^p (a+2d)^j + \cdots + \binom{n}{n}_m^p (a+nd)^j. \quad (3.6)$$

Since $\binom{n}{k}_m^p = \binom{n}{n-k}_m^p$ is true for $n \geq k \geq 0$, we see that S_j can also be written as

$$S_j = \binom{n}{1}_m^p (a+nd)^j + \binom{n}{2}_m^p (a+(n-1)d)^j + \cdots + \binom{n}{n}_m^p (a)^j. \quad (3.7)$$

Adding (3.6) and (3.7), we have

$$2S_j = \sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^j + (a+(n-k)d)^j),$$

$$S_j = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^j + (a+(n-k)d)^j), \quad (3.8)$$

We know that $S_1 = A_p$, $S_2 = B_p$, $S_3 = C_p$. So, we have that

$$A_p = \frac{(2a+nd)}{2} \alpha_{(m,n,p)}. \quad (3.9)$$

Putting (3.9) in (3.4), we have

$$C_p = \frac{(2a+nd)}{2} \left(3B_p - \frac{(2a+nd)^2}{2} \alpha_{(m,n,p)} \right),$$

$$(2a+nd)^3 \alpha_{(m,n,p)} = 6(2a+nd)B_p - 4C_p. \quad (3.10)$$

$$B_p = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^2 + (a+(n-k)d)^2), \quad (3.11)$$

$$6(2a+nd)B_p = 3(2a+nd) \sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^2 + (a+(n-k)d)^2). \quad (3.12)$$

$$C_p = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^3 + (a+(n-k)d)^3), \quad (3.13)$$

$$4C_p = 2\sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^3 + (a+(n-k)d)^3). \tag{3.14}$$

Subtracting (3.14) from (3.12), we have

$$6(2a+nd)B_p - 4C_p = \sum_{k=0}^{n} \binom{n}{k}_m^p [3(2a+nd)[(a+kd)^2+(a+(n-k)d)^2]-2[(a+kd)^3+(a+(n-k)d)^3]].$$

$$6(2a+nd)B_p - 4C_p = (2a+nd)^3\alpha_{(m,n,p)},$$

which is the same as (3.10). Therefore, (3.4) is true.

□

Theorem 2: If

$$D_p = \sum_{k=0}^{n} \binom{n}{k}_m^p (a+kd)^4, \tag{3.15}$$

$$E_p = \sum_{k=0}^{n} \binom{n}{k}_m^p (a+kd)^5, \tag{3.16}$$

where a, d, p are any real or complex numbers, then

$$E_p = \frac{A_p}{\alpha^4_{(m,n,p)}}(5D_p\alpha^3_{(m,n,p)} - 4A_pC_p\alpha^2_{(m,n,p)} - 8A_p^2B_p\alpha_{(m,n,p)} + 8A_p{}^4). \tag{3.17}$$

Proof. Putting (3.9) in (3.17), we have

$$E_p = \frac{(2a+nd)}{2}\left(5D_p - 2(2a+nd)C_p - 2(2a+nd)^2B_p + \frac{(2a+nd)^4}{2}\alpha_{(m,n,p)}\right),$$

$$(2a+nd)^5\alpha_{(m,n,p)} = 4E_p + 4(2a+nd)^3B_p + 4(2a+nd)^2C_p - 10(2a+nd)D_p. \tag{3.18}$$

From (3.8), we know that $B_p = S_2$, $C_p = S_3$, $D_p = S_4$, $E_p = S_5$. So, we have

$$D_p = \frac{1}{2}\sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^4 + (a+(n-k)d)^4),$$

$$10(2a+nd)D_p = 5(2a+nd)\sum_{k=0}^{n} \binom{n}{k}_m^p ((a+kd)^4 + (a+(n-k)d)^4), \tag{3.19}$$

Also,

$$E_p = \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}_m^p ((a+kd)^5 + (a+(n-k)d)^5),$$

$$4E_p = 2\sum_{k=0}^{n}\binom{n}{k}_m^p ((a+kd)^5 + (a+(n-k)d)^5), \tag{3.20}$$

Let y be the difference of (3.20) and (3.19), we have

$$y = \sum_{k=0}^{n}\binom{n}{k}_m^p [2[(a+kd)^5+(a+(n-k)d)^5]-5(2a+nd)[(a+kd)^4+(a+(n-k)d)^4]] \tag{3.21}$$

Multiplying (3.11) by $4(2a+nd)^3$, we have

$$4(2a+nd)^3B_p = 2(2a+nd)^3\sum_{k=0}^{n}\binom{n}{k}_m^p ((a+kd)^2 + (a+(n-k)d)^2). \tag{3.22}$$

Multiplying (3.13) by $4(2a+nd)^2$, we have

$$4(2a+nd)^2C_p = 2(2a+nd)^2\sum_{k=0}^{n}\binom{n}{k}_m^p ((a+kd)^3 + (a+(n-k)d)^3). \tag{3.23}$$

Let y_1 be the sum of (3.22) and (3.23), we have

$$y_1 = 2(2a+nd)^2\sum_{k=0}^{n}\binom{n}{k}_m^p [(2a+nd)[(a+kd)^2+(a+(n-k)d)^2]+[(a+kd)^3+(a+(n-k)d)^3]] \tag{3.24}$$

Now, Adding (3.21) and (3.24), we see that

$$y + y_1 = (2a+nd)^5\alpha_{(m,n,p)}$$

$$4E_p + 4(2a+nd)^3B_p + 4(2a+nd)^2C_p - 10(2a+nd)D_p = (2a+nd)^5\alpha_{(m,n,p)}$$

This means that (3.18) is true. Therefore, (3.17) is true. □

Some Important Special Cases of the Main Result

If we let $a = 1$, $d = 1$, and $\alpha_{(m,n,p)}$ be the sums of real or complex powers of binomial coefficients, (3.4) becomes

$$\alpha_{(m,n,p)} = \sum_{k=0}^{n}\binom{n}{k}_m^p \left(6\left(\frac{k+1}{n+1}\right)^2 - 4\left(\frac{k+1}{n+1}\right)^3\right), \tag{3.25}$$

and (3.17) becomes

$$\alpha_{(m,n,p)} = \sum_{k=0}^{n} \binom{n}{k}_m^p \left(4\left(\frac{k+1}{n+1}\right)^2 + 4\left(\frac{k+1}{n+1}\right)^3 - 10\left(\frac{k+1}{n+1}\right)^4 + 4\left(\frac{k+1}{n+1}\right)^5\right). \tag{3.26}$$

We can see that (3.25) and (3.26) give two different expressions for the sums of real or complex powers of binomial coefficients. Letting $p = 3$ in (3.25) and (3.26) give another two different expressions for nth Franel number as well as first Strehl identity.

If we let $p = 0$ and subtract one from n, (3.4) becomes

$$C_0 = \frac{A_0}{n^2}(3nB_0 - 2A_0^2),, \tag{3.27}$$

and (3.17) becomes

$$E_0 = \frac{A_0}{n^4}(5D_0n^3 - 4A_0C_0n^2 - 8A_0^2B_0n + 8A_0^4), \tag{3.28}$$

where $A_0 = \sum_{k=0}^{n-1}(a+kd)$, $B_0 = \sum_{k=0}^{n-1}(a+kd)^2$, $C_0 = \sum_{k=0}^{n-1}(a+kd)^3$, $D_0 = \sum_{k=0}^{n-1}(a+kd)^4$, $E_0 = \sum_{k=0}^{n-1}(a+kd)^5$.
We can see that (3.27) establishes a relationship between the sum of an arithmetic progression, sum of squares of an arithmetic progression, sum of cubes of an arithmetic progression, and the number of terms of an arithmetic progression. Also, (3.28) establishes a relationship between the sum of an arithmetic progression, sum of squares of an arithmetic progression, sum of cubes of an arithmetic progression, sum of fourth powers of an arithmetic progression, sum of fifth powers of an arithmetic progression, and the number of terms of an arithmetic progression.

If we let $p = 1$, $\alpha_{(m,n,1)}$ be the sum of binomial coefficients, (3.4) becomes

$$C_1 = \frac{A_1}{2^{2n}}(3 \cdot 2^n B_1 - 2A_1^2),$$

and (3.17) becomes

$$E_1 = \frac{A_1}{2^{4n}}(5 \cdot 2^{3n} D_1 - 4 \cdot 2^{2n} A_1 C_1 - 8 \cdot 2^n A_1^2 B_1 + 8A_1^4),$$

where $A_1 = \sum_{k=0}^{n} \binom{n}{k}_m (a+kd)$, $B_1 = \sum_{k=0}^{n} \binom{n}{k}_m (a+kd)^2$, $C_1 = \sum_{k=0}^{n} \binom{n}{k}_m (a+kd)^3$, $D_1 = \sum_{k=0}^{n} \binom{n}{k}_m (a+kd)^4$, $E_1 = \sum_{k=0}^{n} \binom{n}{k}_m (a+kd)^5$.

If we let $p = 2$, $\alpha_{(m,n,2)}$ be the sum of squares of binomial coefficients, (3.4) becomes

$$C_2 = \frac{A_2}{\binom{2n}{n}^2}\left(3\binom{2n}{n}B_2 - 2A_2^2\right),$$

and (3.17) becomes

$$E_2 = \frac{A_2}{\binom{2n}{n}^4}\left(5\binom{2n}{n}^3 D_2 - 4\binom{2n}{n}^2 A_2C_2 - 8\binom{2n}{n}A_2^2B_2 + 8A_2^4\right),$$

where $A_2 = \sum_{k=0}^{n}\binom{n}{k}_m^2(a+kd)$, $B_2 = \sum_{k=0}^{n}\binom{n}{k}_m^2(a+kd)^2$, $C_2 = \sum_{k=0}^{n}\binom{n}{k}_m^2(a+kd)^3$, $D_2 = \sum_{k=0}^{n}\binom{n}{k}_m^2(a+kd)^4$, $E_2 = \sum_{k=0}^{n}\binom{n}{k}_m^2(a+kd)^5$.

4 Conclusion

In this paper, we have been able to establish two relationships between the generalizations of (2.2), (2.3) (2.4), (2.5), (2.6) and (2.7). As a result, some interesting formulas were derived. Two new different expressions for the sums of real or complex powers of binomial coefficients were also derived, thereby giving two new different expressions for Franel number as well as first Strehl identity. We also generalized a set of binomial coefficients to a set of real and complex numbers which behave like binomial coefficients as they exhibit the property $\binom{n}{k} = \binom{n}{n-k}$.

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